ABSTRACT:
This article discusses two new subclasses of the bi-univalent functions category \( \Sigma \) in the open unit disk \( U \). The primary goal of the article is to obtain estimations of the coefficients \([a_2]\) and \([a_3]\) for the functions that are within these two new subclasses.

KEYWORDS: Taylor–Maclaurin Series, Univalent Function, Coefficient Bounds, Bi-univalent Function.

1. INTRODUCTION AND PRELIMINARIES
Assume that \( U = \{z \in \mathbb{C} \text{ such that } |z| < 1\} \) is an open disk which is called unit disk in \( \mathbb{C} \) and \( \mathcal{A} \) represents the category of functions \( f \) that are holomorphic in \( U \) with the Taylor-Maclaurin series:

\[
f(z) = z + \sum_{n=2}^{\infty} a_n z^n, \quad z \in U, \quad (1.1)
\]
and normalization \( f(0) = 0, \ f'(0) = 1 \). The subclass \( \mathcal{D} \) of \( \mathcal{A} \) is the category which contains of univalent functions in \( U \) (to learn more, see (Duren, 2001) and also the most recent works (Breaz, Breaz, & Srivastava, 2009; Srivastava & Eker, 2008). The thoroughly investigation subcategory of univalent functions category \( \mathcal{S} \) are the category of starlike functions of the order \( \xi (0 \leq \xi < 1) \), denoted with the aid of using \( \mathcal{S}^*(\xi) \), and the category of convex functions of the order \( \eta \) denoted with the aid of using \( \mathcal{C}(\eta) \) in \( U \). These two classes are defined as

\[
\mathcal{S}^*(\xi) := \left\{ f \in \mathcal{S}; \Re \left( \frac{zf'(z)}{f(z)} \right) > \xi \quad (z \in U; \quad 0 \leq \xi < 1) \right\} \quad (1.2)
\]
and

\[
\mathcal{C}(\eta) := \left\{ f \in \mathcal{S}; \Re \left( 1 + \frac{zf''(z)}{f'(z)} \right) > \xi \quad (z \in U; \quad 0 \leq \xi < 1) \right\} \quad (1.3)
\]

From the definitions (1.2) and (1.3) the following can be easily deduced:

\[
f(z) \in \mathcal{O}(\xi) \Leftrightarrow zf'(z) \in \mathcal{S}^*(\xi) \quad (1.4)
\]
This is commonly known that if \( f \) is a univalent function, holomorphic from the domain \( D_2 \) onto the domain \( D_2 \), then the inverse function \( g \) which is determined by

\[
g(f(z)) = z; \quad (z \in D_2)
\]
is a univalent and holomorphic function from \( D_2 \) to \( D_2 \). Additionally, we know that the image of \( U \) under any function \( f \in \mathcal{S} \) includes a disk of radius \( \frac{1}{2} \) due to the well-known Koebe One-Quarter Theorem (see (Srivastava & Eker, 2008)). So, each univalent function \( f \in U \) has an inverse \( f^{-1} \) that follows the following criteria:

\[
f^{-1}(f(z)) = z \quad (z \in U),
\]
and

\[
f^{-1}(w) = w \quad (|w| < r_0(f); \quad r_0(f) \geq \frac{1}{2}).
\]
(1.5)
The series expansion of the inverse function \( f \) in a disk about the origin is of the form:

\[
f^{-1}(w) = w + \delta_2 w^2 + \delta_3 w^3 + \ldots.
\]
(1.6)
The Koebe function’s inverse offers the tightest restriction on all \([a_k] \) in (1.6) (see (Ding, Ling & Bao, 1995; Lewin, 1967)). According to the previous steps it is clear that the univalent function \( f(z) \) and its inverse function \( f^{-1}(w) \) satisfy the condition (1.5) in the neighborhood of the origin or equivalently

\[
w = f^{-1}(w) + a_2[f^{-1}(w)]^2 + a_3[f^{-1}(w)]^3 + \ldots.
\]
Then,

\[
f^{-1}(w) = w - a_2[f^{-1}(w)]^2 - a_3[f^{-1}(w)]^3 - \ldots.
\]
(1.7)
By using (1.1) and (1.6) in (1.7), a simple computation gives

\[
g(w) = g^{-1}(w) = w - a_2 w^2 + (2a_2^2 - a_2)w^3 - (5a_2^3 - 5a_2a_3 + a_4)w^4 + \ldots.
\]
(1.8)
Ding, Ling & Bao (Ding, Ling & Bao, 1995) presented the following category $Q_3(\beta)$ of holomorphic functions defined as follows:

$$Q_3(\beta) = \left\{ f \in A; \Re \left((1-\lambda) \frac{f''(z)}{f'(z)} + \beta f'(z) \right) > \beta, \beta < 1, \lambda \geq 0 \right\}.$$  

It is obvious that $Q_3(\beta) \subset Q_3(\beta)$ for $\lambda_1 > \lambda_2 \geq 0$. Thus, for $\lambda \geq 1, 0 \leq \beta < 1$, $Q_3(\beta) \subset Q_3(\beta) = \{ f \in A; \Re f'(z) > \beta, 0 \leq \beta < 1 \}$ and hence $Q_3(\beta)$ is univalent category (see (Chichri, 1977)).

Another exciting work is done by Çağlar and Deniz ( Çağlar & Deniz, 2017). They obtained the coefficient estimates for a new subclass $T_\mu(\gamma, \varphi)$ of analytic and bi-univalent functions in the open unit disk $U$ defined by Salagean differential operator. Also Srivastava, Bulut, Çağlar and Yağmur (Srivastava, Bulut, Çağlar & Yağmur, 2013) introduced and investigated an interesting subclass $N_\mu^h(\lambda, \mu)$ of analytic and bi-univalent functions in the open unit disk $U$. For functions belonging to the class $N_\mu^h(\lambda, \mu)$, they obtained estimates on the first two Taylor-Maclaurin coefficients $a_2$ and $a_3$. In addition, Çağlar, Orhan and Yağmur, (Çaglar, Orhan & Yağmur, 2013) considered two new subclasses $N_\mu^h(\alpha, \lambda)$ and $N_\mu^h(\beta, \lambda)$ of bi-univalent functions defined in $U$ and they presented some interesting results.

In 1967, Lewin (Lewin, 1967) investigated the category $\Sigma$ and demonstrated that $|a_2| \leq 1.51$. Later, Brannan, David & Clunie (Brannan, David & Clunie, 1980) proved that $|a_2| \leq \sqrt{2}$. Besides that, Netanyaú and analysis, (1969).

Awhile later in 1981, Wright and Styer (Styer & Wright, 1981) showed that there exist a function $f(z) \in \Sigma$ with $|a_2| > \frac{4}{3}$. The most excellent known gauge for functions it has been acquired in 1984 by Tan (Tan, 1984), that is $|a_2| \leq 1.485$. The coefficient gauge issue including the bound of $|a_2| (n \in N(1,2))$ for every $f \in \Sigma$ given by (1.1) is unresolved. Taha and Brannan (Brannan & Taha, 1988) showed certain subcategory of the bi-univalent functions category $\Sigma$ comparative to the commonplace subcategory $C(\alpha)$ and $S^*(\alpha)$ of convex and starlike function of order $\alpha (0 \leq \alpha < 1)$, (see (Brannan, Clunie & Kirwan, 1970)). Thus, according to Taha and Brannan (Brannan & Taha, 1988), a function $f \in \Sigma$ belongs to the category $S_\Sigma^x(\alpha)$ of strongly starlike functions of the order $\alpha (0 < \alpha \leq 1)$ if the conditions below are met:

$$f \in \Sigma$$

and

$$\left| \arg \left( \frac{zg'(w)}{g(w)} \right) \right| < \frac{\alpha \pi}{2}, \quad 0 < \alpha \leq 1, \quad w \in U$$

and

$$\left| \arg \left( \frac{zf'(z)}{f(z)} \right) \right| < \frac{\alpha \pi}{2}, \quad 0 < \alpha \leq 1, \quad z \in U,$$

where $g$ is the expansion of $f^{-1}$ on $U$. The categories $C^x_\Sigma(\alpha)$ and $S_\Sigma^x(\alpha)$ of bi-convex function of order $\alpha$ and bi-starlike functions of the order $\alpha$, comparing (individually) to the work categories $C(\alpha)$ and $S^*(\alpha)$, were moreover presented similarly. For every of the function categories $C^x_\Sigma(\alpha)$ and $S_\Sigma^x(\alpha)$, they discovered non-sharp gauges on the primary two Taylor–Maclaurin coefficients $|a_2|$ and $|a_3|$ (see (Brannan & Taha, 1988)).

Using the method of convolution on category of holomorphic functions $A$, Ruscheweyh (Ruscheweyh, 1975) considered the operator $R$ as follows:

$$R^n f(z) = f(z) + \sum_{k=2}^{\infty} \sigma(n,k) a_k z^k,$$

where

$$\sigma(n,k) = \frac{\Gamma((n + k)/2)}{(n - 1)!} \Gamma((n + 1)/2).$$

The expression $R^n f(z)$ is said to be the Ruscheweyh derivative of order $n$ of $f$ and the sign $\ast$ represents the Hadamard product. We can deduce that (Hussain, Khan, Zaighum, Darus, & Shareef, 2017)

$$R^n f(z) = z + \sum_{k=2}^{\infty} \sigma(n,k) a_k z^k,$$

where

$$\sigma(n,k) = \frac{\Gamma((n + k)/2)}{(n - 1)!} \Gamma((n + 1)/2).$$

The aim of this article is to define two new categories of functions in the class $\Sigma$, depending on the Ruscheweyh operator, and obtain estimates the coefficients $|a_2|$ and $|a_3|$ for all elements of these categories by using techniques that are used previously by Srivastava, Mishra, & Gochhayat. (Srivastava, Mishra, & Gochhayat, 2010), see also, (Frasin & Aouf, 2011; Xu et al., 2012).

**Definition 1.1.** A function $f$ which is given by (1.1) is called in the category $H_\Sigma(n, \gamma, \alpha)$ if $f \in \Sigma$ and

$$\left| \arg \left( \frac{R^n f(z)}{f(z)} \right) + \gamma z (R^n f(z))' \right| < \frac{\pi}{n}, \quad z \in U,$$  

and

$$\left| \arg \left( \frac{R^n g(w)}{g(w)} \right) + \gamma w (R^n g(w))' \right| < \frac{\pi}{n}, \quad w \in U,$$

where $n \in N_0 = N \cup \{0\}$, $\gamma > 0$, $0 < \alpha \leq 1,$

$$2(1-\alpha) \sum_{i=1}^{\infty} \frac{(-1)^{i-1}}{\gamma i + 1} \leq 1$$

and the function $g(z)$ is of the form (1.8).

**Definition 1.2.** A function $f$ which is of the form (1.1) is called in the category $H_\Sigma(n, \gamma, \beta)$ if $f \in \Sigma$ and

$$\left| \arg \left( \frac{R^n f(z)}{f(z)} \right) + \gamma z (R^n f(z))' \right| > \beta, \quad z \in U,$$  

and

$$\left| \arg \left( \frac{R^n g(w)}{g(w)} \right) + \gamma w (R^n g(w))' \right| > \beta, \quad w \in U.$$  

In this article, we have two main results that are expressed in the form of two theorems, and to prove these results, we must remind the next lemma.

**Lemma 1.3.** (Pommerenke & Rapoche, 1975) Assume that $h$ belongs to $P$ then $|c_i| \leq 2$ for every $i \geq 1,$ in which $P$ is the collection of all holomorphic functions $h$ in $U$, $\Re(h(z)) > 0$ and

$$h(z) = 1 + c_1 z + c_2 z^2 + c_3 z^3 + \cdots; \quad z \in U.$$

**2. BOUNDS OF THE COEFFICIENTS OF THE FUNCTION SUBCLASS $H_\Sigma(n, \gamma, \alpha)$**

In this section, we obtained the estimations of the Taylor coefficients $|a_2|$ and $|a_3|$ of the functions which is belong to this category $H_\Sigma(n, \gamma, \alpha)$.

**Theorem 2.1.** If $f$ is given by (1.1) is in the category $H_\Sigma(n, \gamma, \alpha)$, in which $n \in N_0$, $\gamma > 0$, $0 < \alpha < 1,$ and

$$2(1-\alpha) \sum_{i=1}^{\infty} \frac{(-1)^{i-1}}{\gamma i + 1} \leq 1.$$  

then

$$|a_2| \leq \frac{2a}{\sqrt{3a(1+2\gamma)(n+1)(n+2)-(4\alpha-1)(\gamma+1)^2(n+1)^2)}},$$

and

$$|a_3| \leq \frac{4a}{(\gamma+1)^2(n+1)^2}.$$

**Proof.** From (1.9) and (1.10), it follows that

$$\left(R^n f(z) \right)' + \gamma z (R^n f(z))'' = [h_i(z)]^\ast; \quad z \in U.$$  

(2.1)
\[ R^n g(w) = \frac{a^2(c_2 + d_2)}{6a(1 + 2y)(\sigma(n, 3) - 4(\alpha - 1)(1 + y)^2(\sigma(n, 3))^2)} \]

By using lemma (1.3), we have

\[ |a_2|^2 \leq \frac{3a(1 + 2y)(n + 1)(n + 2) - 4(\alpha - 1)(1 + y)^2(n + 1)^2}{4a^2} \]

or

\[ |a_2|^2 \leq \frac{3a(1 + 2y)(n + 1)(n + 2) - 4(\alpha - 1)(1 + y)^2(n + 1)^2}{4a^2} \]

Hence, we get

\[ |a_2|^2 \leq \frac{2a}{(1 + 2y)(n + 1)(n + 2) - 4(\alpha - 1)(1 + y)^2(\sigma(n, 3))^2} \]

Finally, we obtained the required bound for \( |a_2| \).

Next, we can obtain the bound for \( |a_3| \) by simple calculation so,

\[ 6(1 + 2y)\sigma(n, 3) a_3 - 6(1 + 2y)\sigma(n, 3) a_3^2 = a(c_2 - d_2) + \frac{\alpha(a^{-1})}{2} (c_1^2 - d_1^2) \]

By using (2.15) and (2.16), we obtain

\[ 6(1 + 2y)\sigma(n, 3) a_3 = a(c_2 - d_2) + 6(1 + 2y)\sigma(n, 3)(a_3^2) \]

Then

\[ a_3 = \frac{a(c_2 - d_2) + \alpha(a^{-1})}{6(1 + 2y)\sigma(n, 3) + 4(1 + y)^2(\sigma(n, 2))^2} \]

Obtaining this bound concludes the proof of the theorem.

Putting \( y = 1 \) in Theorem 2.1, we obtain the next outcome:

**Corollary 2.2.** If \( f \) is given by (1.1) in the category \( H_n^2(n, 1, \alpha) \), where \( n \in \mathbb{N} \), \( a < \alpha \leq 1 \), and \( 2(1 - \alpha) \sum_{n=1}^{\infty} \frac{(-1)^n}{\gamma n} \leq 1 \). Then

\[ |a_2| \leq \frac{2a}{\sqrt{9a(n + 1)(n + 2) - 16(\alpha - 1)(n + 1)^2}} \]

and

\[ |a_3| \leq \frac{4a}{9a(n + 1)(n + 2) + 4a^2} \]

### 3. Coefficient Bounds for the Function Subclass \( H_n^2(n, \gamma, \beta) \)

Here, we discussed another subclass of \( \mathcal{H}_n^2(n, \gamma, \beta) \), and we obtained the estimation of the Taylor coefficients \( |a_2| \) and \( |a_3| \) of the functions which is belong to this category.

**Theorem 3.1.** If \( f \) is given by (1.1) in the category \( \mathcal{H}_n^2(n, \gamma, \beta) \), where \( n \in \mathbb{N}, \gamma > 0, 0 < \beta < 1 \) and \( 2(1 - \alpha) \sum_{n=1}^{\infty} \frac{(-1)^n}{\gamma n} \leq 1 \). Then

\[ |a_2| \leq \sqrt{\frac{2(1 - \beta)}{8(1 + y^2)(n + 1)(n + 2)}} \]

and

\[ |a_3| \leq \frac{(1 - \beta)^2}{4a^2} \]

**Proof.** From (1.11) and (1.12), it follows that there are two functions \( h_1 \) and \( h_2 \) in \( \mathcal{P} \) such that

\[ R^n g(z) = \beta + (1 - \beta)h_1(z); z \in \mathbb{U}. \]

(3.1) and

\[ \mathcal{R}^n g(w) = \frac{a^2(c_2 + d_2)}{a^2(c_2 + d_2)}. \]

Therefore,

\[ (1 + (1 - \beta)h_2(w); w \in \mathbb{U}. \]

(3.2)

where \( h_1 \) and \( h_2 \) are in \( \mathcal{P} \) which have the forms (2.3) and (2.4) respectively.

So,

\[ \beta + (1 - \beta)h_1(z) = 1 + (1 - \beta)\gamma w + (1 - \beta)\gamma^2 z^2 + \cdots. \]

(3.3)
and
\[ \beta + (1 - \beta)h_{2}(w) = 1 + (1 - \beta)d_{1}w + (1 - \beta)d_{2}w^{2} + \ldots. \] (3.4)

Now, equating the equations (2.5) and (3.3), we get
\[ 1 + 2(1 + \gamma)(n, 2)a_{3}z^{2} + 3(1 + 2\gamma)(n, 3)a_{4}z^{3} + \ldots = 1 + (1 - \beta)cz_{2} + (1 - \beta)cz_{2}^{2} + \ldots. \] (3.5)

Again, equating the equations (2.7) and (3.4), we get
\[ 1 - 2(1 + \gamma)(n, 2)d_{1}w + 3(1 + 2\gamma)(n, 3)(2a_{4} - a_{5})w^{2} + \ldots = 1 + (1 - \beta)d_{1}w + (1 - \beta)d_{2}w^{2} + \ldots. \] (3.6)

In this step, we will equate the coefficients in (3.5) and (3.6)
\[ 2(1 + \gamma)(n, 2)a_{2} = (1 - \beta)c_{1}, \] (3.7)
\[ 3(1 + 2\gamma)(n, 3) - a_{2} = (1 - \beta)c_{2}, \] (3.8)
and
\[ -2(1 + \gamma)(n, 2)a_{2} = (1 - \beta)d_{1}. \] (3.9)
\[ 3(1 + 2\gamma)(n, 3)(2a_{2} - a_{3}) = (1 - \beta)d_{2}. \] (3.10)

Putting (3.11) in (3.10), we get
\[ c_{1} = -d_{1}. \] (3.11)
and
\[ c_{2}^{2} + d_{1}^{2} = \frac{3(1 + 2\gamma)(n, 2)^{2}}{(1 - \beta)^{2}}a_{2}^{2}. \] (3.12)

By using (3.8) and (3.10), we get
\[ a_{2}^{2} = \frac{(1 - \beta)(c_{2} + d_{2})}{6(1 + 2\gamma)(n, 3)}. \] (3.13)

By using lemma (1.3), we get
\[ |a_{2}| \leq \frac{2\sqrt{1 - \beta}}{\sqrt{3(1 + 2\gamma)(n + 1)(n + 2)}}, \] which is the bound for \(|a_{2}|\) as given in the theorem.

To obtain the bound for \(|a_{3}|\), substituting (3.10) from (3.8), we get
\[ a_{3} = a_{2}^{2} + \frac{(1 - \beta)(c_{2} - d_{2})}{6(1 + 2\gamma)(n, 3)}. \]

Now, using (3.11) and (3.12)
\[ a_{3} = \frac{(1 - \beta)^{2}c_{2}^{2} + (1 - \beta)(c_{2} - d_{2})}{4(1 + \gamma)^{2}(n, 2)^{2} + 6(1 + 2\gamma)(n, 3)}. \]

Applying lemma (1.3), we get
\[ |a_{3}| \leq \frac{1 - \beta}{2} \frac{(1 + \gamma)^{2}(n + 1)^{2}}{2} + \frac{4(1 - \beta)}{3(1 + \gamma)(n + 1)(n + 2)}. \]

which is the bound for \(|a_{3}|\) as mentioned in the text of the theorem. ■

Putting \(\gamma = 1\) in the Theorem 3.1, we obtain the next outcome:

**Corollary 3.2.** If \(f\) is given by (1.1) in the category \(H_{\infty}(n, 1, 0, \beta)\), where \(n \in N_{0}, 0 \leq \beta < 1\) and \((2 - \beta)\sum_{i=1}^{n-3} \frac{i^{i-1}}{i+1} \leq 1\). Then
\[ |a_{2}| \leq \frac{2\sqrt{1 - \beta}}{3 \sqrt{(n + 1)(n + 2)}}, \]
and\[ |a_{3}| \leq \frac{(1 - \beta)^{2}}{4(n + 1)^{2}} + \frac{4(1 - \beta)}{9(n + 1)(n + 2)}. \][star]

### 4. CONCLUSION

In this article, our investigation is due to the fact that we can find interesting and useful applications of special functions and especially bi-univalent functions. The new bi-univalent function subclasses \(H_{\infty}(n, \gamma, \alpha)\) and \(H_{\infty}(n, \gamma, \beta)\) in the open disk \(U\), was examined in this paper. We discovered the coefficients of \(|a_{2}|\) and \(|a_{3}|\) in the Taylor series of them. Additionally, we discovered some corollaries and implications of the primary findings. Furthermore, the provided bounds improve and extend some previous results.

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