ON S_p-CONNECTED SPACES

ALIAS BARAKAT KHALAF* AND HARDI ALI SHAREEF**
*Dept. of Mathematics, College of Science, University of Duhok, Kurdistan-Region, Iraq
**Dept. of Mathematics, Faculty of Science and Science Education, University of Sulaimani, Kurdistan-Region, Iraq
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ABSTRACT

In this paper we introduce a new concept of connectedness namely S_p-connected space. This class of spaces is strictly between semi-connectedness and connectedness. Several properties and characterizations of this concept are found.

Keyword: *semi-open sets, preclosed sets, semi-connected spaces, S_p-connected spaces.*

1. INTRODUCTION

A semi-open set was defined by Levine in [Levine, 1963] while Pipitone and Russo in [Pipitone et. al., 1975] used this set to introduced s-connectedness or semi-connectedness. By using the concept of preopen sets which introduced in [Mashhour et al, 1982], Popa defined the p-connected or preconnected [Popa**,** 1987]. Shareef in [Shareef, 2007] defined a new type of open sets called S_p-open sets .

Throughout this paper X and Y will always denote topological spaces on which no separation axioms are assumed unless explicitly stated. If U is a subset of X, then the closure of U and interior of U are denoted by $\text{cl}(U)$ and $\text{int}(U)$ respectively. The symbol $f: X \to Y$ represent a function from a space X into a space Y. Semi-closure of a set in any space was introduced by Crossley, and Hildebrand in [Crossley et. Al., 1971] which is the intersection of all semi-closed sets containing this set and denoted by scl, on the other hand S_p-closure in **[**Shareef, 2007**]** is defined by the intersection of all S_p-closed sets which contain it and denoted by $S_p\text{cl}$.

2. Preliminaries

In this section, we give definitions and results which are used in the next section.

Definition 2.1:

A subset A of a space X is said to be semi-open [Levine, 1963] (resp. preopen [Mashhour et. al., 1982], regular open, regular closed [Steen, 1970], β-open [Abd-El-Monsef, 1983], α-open [Njastad, 1965], δ-semiopen [Ekici, 2008] and γ-open [El-Atik, 1997] (equiv. sp-open [Dontchev, 1998] or b-open [Andrijevic, 1996])) set if $A \subseteq \text{cl}(\text{int}(A))$ (resp. $A \subseteq \text{int}(\text{cl}(A))$, $A = \text{int}(\text{cl}(A))$, $A = \text{cl}(\text{int}(A))$, $A \subseteq \text{cl}(\text{int}(\text{cl}(A)))$, $A \subseteq \text{int}(\text{cl}(\text{int}(A)))$, $A \subseteq \text{cl}(\text{int}_\delta(A))$ and $A \subseteq \text{cl}(\text{int}(A)) \cup \text{int}(\text{cl}(A))$). A semi-open set A of a space X is said to be S_p-open set if for each $x \in A$, there exists a preclosed set F such that $x \in F \subseteq A$ **[Mohammed, 2005]**.

The complement of semi-open (resp. preopen, β-open, α-open, δ-semiopen and γ-open (equiv. sp-open or b-open)) set in X is called semi-closed (resp. preclosed, β-closed, α-closed, δ-semiclosed, γ-closed (equiv. sp-closed or b-closed). The complement of S_p-open set is called S_p-closed sets and their families are denoted by $S_pO(X)$ and $S_pC(X)$ while the families of semi-open, preopen, α-open, β-open, γ-open, and δ-semiopen sets are denoted by $SO(X)$, $PO(X)$, $\alpha O(X)$, $\beta O(X)$, $\gamma O(X)$ and $\delta SO(X)$.

Lemma 2.2: Let Y be an open subspace of X. If F is a preclosed subset in a space X, then $F \cap Y$ is preclosed in Y.

Proof: Obvious.

Lemma 2.3: [Donchev, 1998] Let X be any space. If A is semi-open set in X and B is preopen set in X, then $A \cap B$ is semi-open set in B.

Lemma 2.4: [Shareef, 2007] Let Y be a regular closed subspace of the space X. If A is an S_p-open subset of Y, then A is S_p-open set in X.

Proposition 2.5: [Shareef, 2007] Let A, B be two subsets of a space X, then:

1. $S_p\text{cl}(A)$ is the smallest S_p-closed set which contains A.
2. A is S_p-closed if and only if $S_p\text{cl}(A) = A$.
3. $\text{scl}(A) \subseteq S_p\text{cl}(A)$.
4. If $A \subseteq B$, then $S_p\text{cl}(A) \subseteq S_p\text{cl}(B)$.

Definition 2.6: [Sarker, 1985] Two non-empty subsets A and B of a space X are said to be semi-separated sets if $A \cap \text{scl}(B) = \emptyset$ and $\text{scl}(A) \cap B = \emptyset$.

Remark 2.7: [Pipione, 1975] If B is the closure of an open set in a space X, then B and $X \backslash B$ are both semi-open sets in X.

Lemma 2.8: [Shareef, 2007] If A is a semi-open set in a space X, then $\mathrm{cl}(A)$ is S_p-open subset of X.

Definition 2.9: [Dontchev, 1998] A space X is said to be locally indiscrete if every open subset of X is closed.

Theorem 2.10: [Dontchev, 1998] For a space X the following conditions are equivalent:

1. X is locally indiscrete.
2. Every subset of X is preopen.
3. Every singleton in X is preopen.
4. Every closed subset of X is preopen.

Definition 2.11: [Sharma, 2011] A space X is said to be T_1-space if for each two distinct points x and y in X there exists two open sets U and V in X containing x and y, respectively, such that $y \notin U$ and $x \notin V$.

Proposition 2.12: If a space X is T_1-space, then $S_pO(X)$ =SO(X).

Proof: Obvious.

Theorem 2.13: [Khalaf, 2012] A space X is S_p-T_2 if and only if for each pair of distinct points $x, y \in X$, there exists a set U which is both S_p-open and S_p-closed containing one of them but not the other.

Lemma 2.14: [Pipitone, 1975] Let A be a subset of a space X, then A is semi-open set if and only if there exists an open set $G \subseteq A$ such that $\mathrm{cl}(A) = \mathrm{cl}(G)$.

Definition 2.15: A space X is said to be semi-connected [Sarker, 1985], if it cannot be expressed as the union of two semi-separated sets.

Equivalently, X is said to be semi-connected [Pipitone, 1975], if it cannot be written as a union of two non-empty disjoint semi-open sets. Otherwise we say that X is semi-disconnected.

Definition 2.16: A space X is said to be β-connected [Jafari, 2003] (resp., γ-connected [Duszynski, 2011], preconnected [Jafari, 2003], connected [Sharma, 2011] and δ-semiconnected [Ekici, 2008]) if X cannot be expressed as the union of two non-empty disjoint β-open (resp., γ-open, preopen, open and δ-semiopen) sets of X.

Lemma 2.17: [Ekici, 2008] For a space X, the following properties are equivalent:

1. $\mathrm{cl}(V) = X$ for every nonempty open set V of X,
2. $U \cap V \neq \emptyset$ for any nonempty semi-open sets U and V of X,
3. X is semi-connected,
4. X is δ-semiconnected.

Definition 2.18: [Noiri, 1980] A space X is said to be extremally disconnected space if the closure of each open set in X is open.

Corollary 2.19: [Shareef, 2007] If a space X is extremally disconnected, then every S_p-open subset of X is preopen subsets of X.

Definition 2.20: [Jafari, 2003] A space X is said to be *PS*-space if every preopen set in X is semi-open in X.

Corollary 2.21: [Jafari, 2003] If X is extremally disconnected *PS*-space, then β-connectedness, preconnectedness, semi-connectedness and connectedness are all equivalent.

Theorem 2.22: [Sharma, 2011] A space X is disconnected if and only if X is the union of two non-empty disjoint open sets.

Theorem 2.23: [Sharma, 2011] A space X is disconnected if and only if there exists a non-empty proper subset of X which is both open and closed.

The following definitions and results are from [Duszynski, 2011].

Lemma 2.24: If a space X is γ-connected, then it is β-connected.

A space X is said to be **B-SP-connected** (resp., **P-SP-connected**) if X cannot be written as a union of two non-empty disjoint sets S_1, S_2 of X such that $S_1 \in$ BO (X), $S_2 \in \beta$ O (X) (resp., $S_1 \in$ PO (X), $S_2 \in \beta$ O (X)) . A space X is said to **α-B-connected** (resp., **α -SP-connected**, **α -S-connected**) if X cannot be expressed as a union of two non-empty disjoint sets S_1; $S_2 \subset$ X such that $S_1 \in \alpha$O(X) and $S_2 \in$BO (X) (resp., $S_2 \in \beta$ O (X), $S_2 \in$ SO (X)).

Theorem 2.25: For every space X the following are equivalent:

1. X is β-connected space.
2. X is *B-SP*-connected space.
3. X is *P-SP*-connected space.

Theorem 2.26: For every space X the following are equivalent:

1. X is semi-connected space.
2. X is α-*S*-connected space.
3. X is α-*SP*-connected space.
4. X is α-*B*-connected space.

Corollary 2.27: [Duszynski, 2006] Connectedness and α-*P*-connectedness are equivalent notion for every space X.

Definition 2.28: A function $f: X \to Y$ is said to be S_p-continuous [Shareef, 2007] (resp. continuous [Sharma, 2011], irresolute [Crossley, 1972], s-continuous or (strongly semi-continuous) [Muhammed, 2005]) if the inverse image of every open (resp. open, semi-open, semi-open) set in Y is S_p-open (resp. open, semi-open, open) set in X.

Theorem 2.29: [Shareef, 2007] The following statements are equivalents for the function $f: X \to Y$:

(1) $f: X \to Y$ is S_p-continuous.

(2) The inverse image of every closed set in Y is S_p-closed set in X.

Theorem 2.30: [Sharma, 2011] A function $f: X \to Y$ is continuous if and only if the inverse image of every closed set in Y is closed in X.

3. S_p-Connected Space

Definition 3.1: Two non-empty subsets A and B of X are said to be S_p-separated sets if $S_p\text{cl}(A) \cap B = \emptyset$ and $A \cap S_p\text{cl}(B) = \emptyset$.

Example 3.2: Let $X = \{a, b, c\}$ and $\tau = \{\emptyset, X, \{a\}, \{b\}, \{a, b\}\}$. Then $\{a\}$ and {b} are S_p-separated sets in X because $S_p\text{cl}(\{a\}) \cap \{b\} = \{a\} \cap \{b\} = \emptyset$ and $\{a\} \cap S_p\text{cl}(\{b\}) = \{a\} \cap \{b\} = \emptyset$.

Proposition 3.3: Let Y be an open subspace of a space X and $A \in S_pO(X)$. Then $A \cap Y \in S_pO(Y)$.

Proof: Let Y be an open subspace of a space X and $A \in S_pO(X)$. Then $A \in SO(X)$ and $A = \bigcup_{\alpha \in I} F_\alpha$, where $F_\alpha \in PC(X)$ for each $\alpha \in I$. Now since Y is preopen set in X and A is semi-open set in X so by **[Lemma 2.3]** $A \cap Y \in SO(Y)$ and $A \cap Y = (\bigcup_{\alpha \in I} F_\alpha) \cap Y = \bigcup_{\alpha \in I}(F_\alpha \cap Y)$, but by **[Lemma 2.2]** $F_\alpha \cap Y \in PC(Y)$ for each $\alpha \in I$; therefore $A \cap Y \in S_pO(Y)$.

Lemma 3.4: Let Y be an open subspace of a space X and $A \subseteq Y$, then $S_p\text{cl}_Y(A) \subseteq S_p\text{cl}(A)$ where $S_p\text{cl}_Y$ denote the S_p-closure relative to the subspace Y.

Proof: Let $x \notin S_p\text{cl}(A)$ implies that there exists an S_p-open set U containing x such that $U \cap A = \emptyset$. Then $U \cap Y \cap A = \emptyset$, let $G = U \cap Y$. Since $U \in S_pO(X)$ and Y is open in X so by **[Lemma 3.3]** $G = U \cap Y \in S_pO(Y)$; therefore $G \cap A = \emptyset$ implies that $x \notin S_p\text{cl}_Y(A)$, hence $S_p\text{cl}_Y(A) \subseteq S_p\text{cl}(A)$.

Lemma 3.5: Let Y be a regular closed subspace of a space X and $A \subseteq Y$. Then $S_p\text{cl}(A) \subseteq S_p\text{cl}_Y(A)$.

Proof: Let $x \notin S_p\text{cl}_Y(A)$ implies that there exists an S_p-open set U in Y containing x such that $U \cap A = \emptyset$. Since Y is regular closed set in X then by **[Lemma 2.4]** U is S_p-open set in X implies that $x \notin S_p\text{cl}(A)$, so $S_p\text{cl}(A) \subseteq S_p\text{cl}_Y(A)$.

Theorem 3.6: Let (Y, τ_Y) be an open subspace of a space (X, τ) and let $A, B \subseteq Y$. If A and B are S_p-separated sets in X, then A and B are τ_Y-S_p-separated sets.

Proof: Let A and B be two τ-S_p-separated sets implies that $S_p\text{cl}(A) \cap B = \emptyset$ and $A \cap S_p\text{cl}(B) = \emptyset$. But since Y is open subspace of X so by **[Lemma 3.4]**, $S_p\text{cl}_Y(A) \subseteq S_p\text{cl}(A)$ and $S_p\text{cl}_Y(B) \subseteq S_p\text{cl}(B)$ implies that $S_p\text{cl}_Y(A) \cap B = \emptyset$ and $A \cap S_p\text{cl}_Y(B) = \emptyset$. Thus A and B are τ_Y-S_p-separated sets in Y.

Theorem 3.7: Let Y be a regular closed subset of a space (X, τ) and $A, B \subseteq Y$. If A and B are τ_Y-S_p-separated sets in Y, then they are τ-S_p-separated sets in X.

Proof: Let A and B be τ_Y-S_p-separated sets in Y. Then $S_p\text{cl}_Y(A) \cap B = \emptyset$ and $A \cap S_p\text{cl}_Y(B) = \emptyset$. Since Y is regular closed subspace of X, so by **[Lemma 3.5]**, $S_p\text{cl}(A) \subseteq S_p\text{cl}_Y(A)$ and $S_p\text{cl}(B) \subseteq S_p\text{cl}_Y(B)$ this implies that $S_p\text{cl}(A) \cap B = \emptyset$ and $A \cap S_p\text{cl}(B) = \emptyset$. Thus A and B are τ-S_p-separated sets in X.

Proposition 3.8: Two S_p-closed (S_p-open) subsets of a space X are S_p-separated if and only if they are disjoint.

Proof: Necessity. Let A and B be two disjoint S_p-closed sets in X. Then $A \cap B = \emptyset$ and since they are S_p-closed sets in X so by **[Proposition 2.5]**, $S_p\text{cl}(A) = A$ also $S_p\text{cl}(B) = B$ this implies that $S_p\text{cl}(A) \cap B = \emptyset$ and $A \cap S_p\text{cl}(B) = \emptyset$. Thus A and B are S_p-separated sets.

Sufficiency: Obvious.

Definition 3.9: A space X is said to be S_p-connected space if it cannot be expressed as a union of two non-empty proper S_p-separated sets of X.

Proposition 3.10: Every semi-connected space is S_p-connected.

Proof: Let X be semi-connected space. Then X cannot be expressed as a union of two semi-separated sets, to show X is S_p-connected space if possible suppose that X is not S_p-connected, then there exists two S_p-separated sets A and B such that $X = A \cup B$. Now $S_p\text{cl}(A) \cap B = \emptyset$ and $A \cap S_p\text{cl}(B) = \emptyset$, then by **[Proposition 2.5],** $\text{scl}(A) \cap B = \emptyset$ and $A \cap \text{scl}(B) = \emptyset$ this implies that by **[Definition 2.15]**, A and B are semi-separated sets. Therefore, X can be written as a union of semi-separated sets this implies that X is semi-connected which is a contradiction. Thus X is S_p-connected space.

The converse of [**Proposition 3.10]** is not true in general as it is shown in the following example:

Example 3.11: Let $X = \{a, b, c\}$ and $\tau = \{\emptyset, X, \{a\}, \{b\}, [a, b\}\}$. Then X is S_p-connected space but it is not semi-connected since

$X = \{a\} \cup \{b, c\}$, where $\{a\}$ and $\{b, c\}$ are semi-separated sets.

Corollary 3.12: Every β-connected space is S_p-connected.

Proof: Follows from **[Definition 2.16]** and **[Proposition 3.10]**.

Theorem 3.13: A space X is S_p-connected if and only if there is no non-empty proper subset of X which is both S_p-open and S_p-closed.

Proof: Let X be S_p-connected space and there exists a non-empty proper subset A of X which is both S_p-open and S_p-closed. Then $B = X\setminus A$ is also non-empty S_p-open and S_p-closed, but $S_p\mathrm{cl}(A) \cap B = A \cap B = \emptyset$ and $A \cap S_p\mathrm{cl}(B) = A \cap B = \emptyset$ this implies that A and B are S_p-separated set and $X = A \cup B$, then X is not S_p-connected space which is a contradiction. Thus there is no non-empty proper subset of X which is both S_p-open and S_p-closed.

Conversely: Let the hypothesis be satisfied, to show X is S_p-connected space. If possible suppose that X is not S_p-connected space, then there exists S_p-separated sets A and B such that $X = A \cup B$. Since $S_p\mathrm{cl}(A) \cap B = \emptyset$ implies that $A \cap B = \emptyset$, then $A = X\setminus B$ and now $S_p\mathrm{cl}(A) \subseteq X\setminus B = A$ so A is S_p-closed set, and since $S_p\mathrm{cl}(B) \cap A = \emptyset$ then $S_p\mathrm{cl}(B) \subseteq X\setminus A = B$ this implies that B is S_p-closed set. Now $X\setminus B$ is S_p-open set, but $A = X\setminus B$; therefore A is a non-empty proper subset of X which is both S_p-open and S_p-closed that is a contradiction. Hence X must be S_p-connected space.

Corollary 3.14: A space X is S_p-connected if and only if the only subsets of X which are both S_p-open and S_p-closed sets are $\emptyset$ and X.

Proof: Follows from **[Theorem 3.13]**.

Proposition 3.15: A space X is S_p-connected if and only if X cannot be expressed as the union of two non-empty disjoint S_p-open sets.

Proof: Let X be S_p-connected space and if possible suppose that X there exists two disjoint non-empty S_p-open sets A and B such that $X = A \cup B$. Then by **[Proposition 3.8]**, A and B are S_p-separated sets this implies that X is not S_p-connected space which is a contradiction. Thus X cannot be expressed as the union of two non-empty disjoint S_p-open sets.

Conversely: Let the hypothesis be satisfied and if possible suppose that X is not S_p-connected. Then there exist two S_p-separated sets A and B such that $X = A \cup B$, now since $S_p\mathrm{cl}(A) \cap B = \emptyset$ implies that $A \cap B = \emptyset$, but $S_p\mathrm{cl}(A) \subseteq X\setminus B = A$ this implies that A is S_p-closed set and by the same way B is also S_p-closed set, and then A and B are also S_p-open sets implies that A and B are disjoint non-empty S_p-open sets such that $X = A \cup B$ which is a contradiction. Thus X must be S_p-connected space.

Corollary 3.16: If a space X is S_p-connected T_1-space, then it is semi-connected.

Proof: Let X be an S_p-connected T_1-space, then by **[Theorem 3.15]**, X cannot expressed as the union of two non-empty disjoint S_p-open sets and since X is T_1-space, so by **[Proposition 2.12]** X cannot expressed as the union of two non-empty disjoint semi-open sets. This implies that X is a semi-connected space.

Remark 3.17: A space X is S_p-connected if and only if it cannot be written as a union of two non-empty disjoint S_p-closed sets.

The property of S_p-connectedness is not hereditary as shown by the following example:

Example 3.18:- Let $X = \{a, b, c, d\}$ and $\tau = \{\emptyset, X, \{a\}, \{a, b\}, \{a, c\}, \{a, b, c\}\}$. Then $S_pO(X) = \{\emptyset, X\}$, so the only non-empty subset of X which is both S_p-open and S_p-closed is X itself, therefore by **[Corollary 3.14],** X is S_p-connected space. Now let $Y = \{b, c\}$, then $\tau_Y = \{\emptyset, Y, \{b\}, \{c\}\}$ and $S_pO(Y) = \tau_Y$ implies that Y can be expressed as the union of two non-empty disjoint S_p-open sets in Y. Thus Y is not S_p-connected subspace.

Theorem 3.19: Let A be S_p-connected set in X and C, D be S_p-separated sets of X such that $A \subseteq C \cup D$. Then either $A \subseteq C$ or $A \subseteq D$.

Proof: Let A be S_p-connected set in X and C, D be S_p-separated sets of X such that $A \subseteq C \cup D$ and let $A \nsubseteq C$ and $A \nsubseteq D$. Now Suppose that $A \cap C \neq \emptyset$ and $A \cap D \neq \emptyset$, since $A \cap (C \cup D) = A$ implies that $A = (A \cap C) \cup (A \cap D)$. But since C and D are S_p-separated sets so $S_p\mathrm{cl}(C) \cap D = \emptyset$ and $C \cap S_p\mathrm{cl}(D) = \emptyset$. Now $(A \cap C) \cap S_p\mathrm{cl}(A \cap D) \subseteq (A \cap C) \cap S_p\mathrm{cl}(D) = A \cap (C \cap S_p\mathrm{cl}(D)) = \emptyset$ this implies that $(A \cap C) \cap S_p\mathrm{cl}(A \cap D) = \emptyset$. By the same way we can get $S_p\mathrm{cl}(A \cap C) \cap (A \cap D) = \emptyset$, so $A \cap C$ and $A \cap D$ are S_p-separated sets such that $A = (A \cap C) \cup (A \cap D)$ this implies that A is not S_p-connected set which is a contradiction. Thus either $A \subseteq C$ or $A \subseteq D$.

Theorem 3.20: Let X be a space such that any two elements x and y in X are contained in an S_p-connected subspace of X, then X is S_p-connected.

Proof: Suppose that X is not S_p-connected space, then X is the union of two non-empty S_p-separated sets A and B. Now since A and B are non-empty sets, so there exists $a \in A$ and $b \in B$ this implies that by hypothesis a and b are contained in some S_p-connected subspace Y of X, but $X = A \cup B$ implies that $Y \subseteq A \cup B$ and then by **[Theorem 3.19]**, either $Y \subseteq A$ or $Y \subseteq B$ this implies that either a, b are both in A or are both in B which is a contradiction. Hence X must be S_p-connected space.

Proposition 3.21: If U is an S_p-connected set in a space X, then $S_p\mathrm{cl}(U)$ is also S_p-connected set in X.

Proof: Let U be S_p-connected set in a space X and $S_p\mathrm{cl}(U)$ not S_p-connected in X. Then there exists two S_p-separated sets A and B in X such that $S_p\mathrm{cl}(U) = A \cup B$, but $U \subseteq S_p\mathrm{cl}(U)$ implies that $U \subseteq A \cup B$ and since U is S_p-connected set in X so by **[Theorem 3.19]** either $U \subseteq A$ or $U \subseteq B$. Now if $U \subseteq A$, then by **[Proposition 2.5]** $S_p\mathrm{cl}(U) \subseteq S_p\mathrm{cl}(A)$ and since $S_p\mathrm{cl}(A) \cap B = \emptyset$ implies that $S_p\mathrm{cl}(U) \cap B = B = \emptyset$ which is a contradiction. And if $U \subseteq B$, then by **[Proposition 2.5]** $S_p\mathrm{cl}(U) \subseteq S_p\mathrm{cl}(B)$ and $A \cap S_p\mathrm{cl}(B) = \emptyset$ implies that $A \cap S_p\mathrm{cl}(U) = A = \emptyset$ which is a contradiction. Then in both cases we get a contradiction. Hence $S_p\mathrm{cl}(U)$ is an S_p-connected set in X.

Theorem 3.22: Let U and V be two subsets of a space X. If U is S_p-connected in X such that $U \subseteq V \subseteq S_p\mathrm{cl}(U)$, then V is also S_p-connected set in X.

Proof: Let V be not S_p-connected set in X. Then there exists two S_p-separated sets A and B such that $V = A \cup B$, since $U \subseteq V$ this implies that $U \subseteq A \cup B$ and since U is S_p-connected set in X so by **[Theorem 3.19]** either $U \subseteq A$ or $U \subseteq B$. If $U \subseteq A$, then by **[Proposition 2.5]** $S_p\mathrm{cl}(U) \subseteq S_p\mathrm{cl}(A)$ and since A and B are S_p-separated sets so $S_p\mathrm{cl}(U) \cap B = \emptyset$, but $A \cup B = V \subseteq S_p\mathrm{cl}(U)$ this implies that $V \cap B = B = \emptyset$ which is a contradiction. By the same way if $U \subseteq B$ we get a contradiction. Thus V must be S_p-connected set in X.

Proposition 3.23: If for every non-empty S_p-open subset U of a space X, $S_p\mathrm{cl}(U) = X$, then X is S_p-connected .

Proof: Suppose that X is not S_p-connected space. Then by **[Proposition 3.15]** there exists two non-empty disjoint S_p-open sets U and V such that $X = U \cup V$, now since $U \cap V = \emptyset$ this implies that $U = X \backslash V$ and $V = X \backslash U$ and then they are also non-empty S_p-closed sets in X; therefore by **[Proposition 2.5]** $S_p\mathrm{cl}(U) = U \neq X$ and $S_p\mathrm{cl}(V) = V \neq X$ which is a contradiction to the hypothesis. Thus X is S_p-connected .

Remark 3.24: Let X be a δ-semi-connected space, then by **[Lemma 2.17]**, X is semi-connected space and by **[Proposition 3.10],** X is S_p-connected space.

Proposition 3.25: If a space X is extremally disconnected (or locally indiscrete) preconnected space, then X is S_p-connected.

Proof: Suppose that X is not S_p-connected space this implies that by **[Proposition 3.15],** there exist two non-empty disjoint S_p-open sets U and V such that $X = U \cup V$. Since X is extremally disconnected (locally indiscrete) space so by **[Corollary 2.19]** or(**[Theorem 2.10]**), U and V are preopen sets in X this implies that by **[Definition 2.16],** X is not preconnected which is a contradiction. Thus X must be S_p-connected space.

Corollary 3.26: Let X be extremally disconnected PS-space. If X is preconnected (resp. connected) space, then X is S_p-connected.

Proof: Follows from **[Proposition 3.25]** and **[Corollary 2.21]**.

Theorem 3.27: If a space X is disconnected, then X is not S_p-connected space.

Proof: Let X be disconnected space. Then by **[Theorem 2.23]** there exists a non-empty proper subset U of X which is both open and closed, and then $X \backslash U$ is open and closed set in X. But every clopen set is S_p-open set and $X = U \cup (X \backslash U)$ this implies that X is written as the union of two non-empty disjoint S_p-open sets so by **[Proposition 3.15]**, X is not S_p-connected space.

From the above theorem we get the following result.

Corollary 3.28: Every S_p-connected space is connected.

Lemma 3.29: Any S_p-T_2 space which contains at least two distinct points is not S_p-connected space.

Proof: Let X be S_p-T_2 space contains at least two distinct points. Then by **[Theorem 2.13]** there exists an S_p-clopen set U containing one of them but not the other this implies that X contains a non-empty proper set which is both S_p-open and S_p-closed set; therefore by **[Theorem 3.13]** X is not S_p-connected space.

Theorem 3.30: For a locally indiscrete space X the following statements are equivalent:

1. X is S_p-connected space.
2. $S_p\mathrm{cl}(U) = X$, for every non-empty S_p-open set U in X.
3. $U \cap V \neq \emptyset$, for any two non-empty S_p-open subsets U and V of X.

Proof: (1) → (2)

Let X be S_p-connected space and let there exists an non-empty S_p-open set U in X such that $S_p\text{cl}(U) \neq X$. Then there exists $y \in X$ such that $y \notin S_p\text{cl}(U)$ this implies that there exists an S_p-open set V containing y such that $U \cap V \neq \emptyset$, and since U is semi-open set so by **[Lemma 2.14]** there exists an open set $G \subseteq U$ in X such that $\text{cl}(G) = \text{cl}(U)$ and $G \subseteq U$ so by **[Remark 2.7]** $\text{cl}(U)$ and $X\backslash\text{cl}(U)$ are semi-open sets in X. Now by **[Lemma 2.8]** $\text{cl}(U)$ is S_p-open set also by **[Theorem 2.10]** $\text{cl}(U)$ is preopen set this implies that $X\backslash\text{cl}(U) \neq \emptyset$ is semi-open and preclosed set in X; therefore $X\backslash\text{cl}(U)$ is S_p-open set. But $X = \text{cl}(U) \cup (X\backslash\text{cl}(U))$ and $\text{cl}(U) \cap \left(X\backslash\text{cl}(U)\right) = \emptyset$ this implies that X is the union of two non-empty disjoint S_p-open sets, then by **[Proposition 3.15]** X is not S_p-connected which is a contradiction. Thus the condition (2) must be satisfied.

(2) → (1)

Follows from **[Proposition 3.23]**.

(2) → (3)

Suppose that there exists two non-empty S_p-open sets U and V in X such that $U \cap V = \emptyset$. Since $U \neq \emptyset$ and $V \neq \emptyset$ so $S_p\text{cl}(U) \neq X$ this contradicts condition (2). Thus $U \cap V \neq \emptyset$, for any two non-empty S_p-open subsets U and V of X.

(3) → (2)

Suppose that there exists a non-empty S_p-open set U such that $S_p\text{cl}(U) \neq X$. Then there exists $y \in X$ such that $y \notin S_p\text{cl}(U)$, so there exists an S_p-open set V containing y such that $U \cap V = \emptyset$ which contradicts condition (3). Hence the proof is complete.

Corollary 3.31: Every γ- connected space is S_p-connected.

Proof: Follows from **[Lemma 2.24]** and **[Corollary 3.12]**.

Corollary 3.32: Every B-SP-connected (resp. P-SP-connected) space is S_p-connected.

Proof: Follows from **[Theorem 2.25]** and **[Corollary 3.12]**.

Corollary 3.33: Every α-S-connected (resp. α-SP-connected, α-B-connected) space is S_p-connected space.

Proof: Follows from **[Theorem 2.26]** and **[Proposition 3.10]**.

Corollary 3.34: Every S_p-connected space is α-P-connected.

Proof: Let X be S_p-connected space. Then by **[Corollary 3.28]** X is connected and by **[Corollary 2.27]** X is α-P-connected.

Proposition 3.35: If Y is a regular closed subspace of a locally indiscrete S_p-connected space X, then Y is S_p-connected subspace.

Proof: Let Y be a regular closed subspace of a locally indiscrete S_p-connected space. To show Y is S_p-connected subspace, let U be a non-empty S_p-open set in Y, since Y is regular closed in X then by **[Lemma 2.4]**, U is S_p-open in X and since X is locally indiscrete S_p-connected space so by **[Theorem 3.30]**, $S_p\text{cl}(U) = X$. But Y is regular closed in X, then by **[Lemma 3.5]**, $S_p\text{cl}_Y(U) = X$ this implies that by **[Theorem 3.30]** Y is S_p-connected subspace.

Theorem 3.36: A space X is S_p-connected if there exists a locally indiscrete S_p-connected subspace such that Y is open and $S_p\text{cl}(Y) = X$

Proof: Let Y be a locally indiscrete S_p-connected subspace of a space X such that $S_p\text{cl}(Y) = X$ and Y be open in X. Now let A and B be two non-empty S_p-open sets in X, since $S_p\text{cl}(Y) = X$ and Y is open set in X , so by **[Proposition 3.3],** $A \cap Y$ and $B \cap Y$ are S_p-open sets in Y and they are non-empty also. But since Y is locally indiscrete S_p-connected subspace, so by **[Theorem 3.30]**, $\emptyset \neq (A \cap Y) \cap (B \cap Y) \subseteq A \cap B$ this implies that by **[Theorem 3.30],** X is S_p-connected space.

Theorem 3.37: Let X be a space and let $\{C_\alpha : \alpha \in \Delta\}$ be a collection of S_p-connected sets in X such that $\bigcap_{\alpha\in\Delta} C_\alpha \neq \emptyset$. Then $\bigcup_{\alpha\in\Delta} C_\alpha$ is S_p-connected set in X.

Proof: Suppose that $\bigcup_{\alpha\in\Delta} C_\alpha$ be not S_p-connected in X, then $\bigcup_{\alpha\in\Delta} C_\alpha$ can be expressed as the union of two S_p-separated sets A and B this implies that $\bigcup_{\alpha\in\Delta} C_\alpha = A \cup B$. Now since for all $\alpha \in \Delta$, $C_\alpha \subseteq \bigcup_{\alpha\in\Delta} C_\alpha$ implies that $C_\alpha \subseteq A \cup B$ and since C_α is S_p-connected set in X for each $\alpha \in \Delta$, then by **[Theorem 3.19]** either $C_\alpha \subseteq A$ or $C_\alpha \subseteq B$ for all $\alpha \in \Delta$. If $C_\alpha \subseteq A$ for all $\alpha \in \Delta$, then $\bigcup_{\alpha\in\Delta} C_\alpha \subseteq A$ which is a contradiction of the assumption that A and B are S_p-separated of $\bigcup_{\alpha\in\Delta} C_\alpha$. By the same way if $C_\alpha \subseteq B$ we get a contradiction. Thus $\bigcup_{\alpha\in\Delta} C_\alpha$ must be S_p-connected set in X.

Proposition 3.38: A space X is S_p-connected if and only if each S_p-continuous function from X into a discrete two point space $\{a.b\}$ is constant.

Proof: Let X be S_p-connected space and $f: X \to \{a, b\}$ be S_p-continuous function, where Y is a discrete space of at least two points. Now since f is S_p-continuous so by **[Theorem 2.29]** for each $y \in f(X) \subseteq \{a, b\}$, $f^{-1}(\{y\})$ is S_p-open, S_p-closed and non-empty set in X. But since X is S_p-connected space, so by **[Corollary 3.14]**,

$f^{-1}(\{y\}) = X$ this implies that $f(x) = y$ for all $x \in X$, then f is a constant function.

Conversely: Let the hypothesis be satisfied and suppose that X is not S_p-connected. Then by **[Theorem 3.13]**, there exists a proper subset A of X which is both S_p-open and S_p-closed in X. This implies that $X\backslash A$ is also non-empty proper subset of X which is both S_p-open and S_p-closed in X. Now define a function $f: X \to \{a, b\}$ by setting $f(x) = a$ if $a \in A$ and $f(x) = b$ if $x \in X\backslash A$, since $\{a, b\}$ is discrete and $A \cap (X\backslash A) = \emptyset$, then the definition of f shows that $f^{-1}(\emptyset) = \emptyset$, $f^{-1}(\{a, b\}) = X$. Also $f^{-1}(\{a\}) = A$ and $f^{-1}(\{b\}) = X\backslash A$. Thus we have shown that the inverse image of every open set in $\{a, b\}$ is S_p-open in X, then by **[Definition 2.28]**, f is S_p-continuous function, but f is not constant which is a contradiction. Hence X must be S_p-connected space.

Theorem 3.39: Let $f: X \to Y$ be a surjective S_p-continuous function. If X is an S_p-connected space, then Y is connected.

Proof: Let X be S_p-connected and suppose that Y is disconnected, then by **[Theorem 2.22],** Y is the union of two non-empty disjoint open sets U and V of Y. Since f is S_p-continuous function so by **[Definition 2.28]** $f^{-1}(U)$ and $f^{-1}(V)$ are non-empty disjoint S_p-open sets in X, but $f(X) = Y = U \cup V$ this implies that $X = f^{-1}(U) \cup f^{-1}(V)$, and then X is the union of two non-empty disjoint S_p-open sets which implies that X is not S_p-connected this is a contradiction. Thus Y is connected.

Theorem 3.40: Let $f: X \to Y$ be a surjective irresolute function. If X is an semi-connected space, then Y is S_p-connected.

Proof: Let X be s-connected and Y is not an S_p-connected space. Then by **[Proposition 3.15],** there exist two disjoint non-empty S_p-open sets U and V such that $Y = U \cup V$, and since f is irresolute and U, V are semi-open in Y sets, so by **[Definition 2.28],** $f^{-1}(U)$ and $f^{-1}(V)$ are also non-empty disjoint semi-open sets in X. Now $f(X) = Y = U \cup V$ this implies that $X = f^{-1}(U) \cup f^{-1}(V)$; therefore X is the union of two non-empty disjoint semi-open sets, and then by **[Definition 2.15]**, X is not semi-connected space which is a contradiction. Thus Y must be S_p-connected space.

Theorem 3.41: Let $f: X \to Y$ be a surjective open continuous function. If X is S_p-connected space, then Y is also S_p-connected.

Proof: Let Y be not S_p-connected space. Then by **[Proposition 3.15],** Y can be expressed as the union of two non-empty disjoint S_p-open sets U and V in Y, but since f is continuous and open function so by **[Proposition 2.30],** $f^{-1}(U)$ and $f^{-1}(V)$ are non-empty disjoint S_p-open sets in X. And since $f(X) = Y = U \cup V$ implies that $X = f^{-1}(U) \cup f^{-1}(V)$, then by **[Proposition 3.15]**, X is not S_p-connected which is a contradiction. Thus Y must be S_p-connected space.

Theorem 3.42: Let $f: X \to Y$ be a surjective s-continuous function. If X is connected space, then Y is S_p-connected space.

Proof: Let Y be not S_p-connected space. Then by **[Proposition 3.15],** there exists two disjoint non-empty S_p-open sets U and V in Y such that $Y = U \cup V$. Since f is s-continuous and U, V are semi-open sets in Y, so from **[Definition 2.28]** we have $f^{-1}(U)$ and $f^{-1}(V)$ are non-empty disjoint open sets in X, but $f(X) = Y = U \cup V$ implies that $X = f^{-1}(U) \cup f^{-1}(V)$ and then by **[Theorem 2.22]** X is disconnected which is a contradiction. Thus Y must be S_p-connected space.

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حول الفضاءات المتصلة من النمط S_p

الملخص

في هذا البحث ادخلنا مفهوما جديدا للفضاءات المتصلة سميت بالفضاءات المتصلة من النمط S_p . هذا الصنف من الفضاءات يقع بين الفضاءات الشبه متصلة والفضاءات المتصلة. الكثير من خواص وصفات هذا المفهوم وجدت.

ل دۆر ڤالاهییێن پیکڤه ژ جورێ S_p

كورتى

دڤێ ڤەكولينێدا مه جورەكى نوى ژ ڤالاهییێن پێكڤه دا نیاسین بناڤێ ڤالاهییێن پیكڤه ژ جورێ S_p . ئەڤ جوره د كەفته دناڤ بەرا ڤالاهییێن شتى- بێكڤه و ڤالاهییێن پێكڤه. گەلك ساخلەتين وسيفاتێن ڤى جورى هاتينه ديتن.