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ON N-DIMENSIONAL ACHR-ALGEBRAS

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ABSTRACT.

We prove that a homomorphism φ from complete normed algebra A into an n-dimensional normed algebra B is automatically continuous. As a consequence, B is ACHR-algebra.

***KEYWORD**: Automatic continuity, Finite-dimension, Homomorphism, Normed algebra.*

INTRODUCTION.

A classical topic in the theory of automatic continuity is that of determining those normed algebras B which satisfy Property ACHR-algebra which is defined as follows:

If for each homomorphism φ :A $\to$B from a complete normed algebra (A,$\|.\|_A$) into a normed algebra (B,$\|.\|_B$) is continuous, then (B,$\|.\|_B$) is called (Automatic Continuity of Homomorphisms into the Right side of the arrow) algebra, in short ACHR-algebra.

Usually, property ACHR-algebra is considered in an associative context, so that the normed algebra B is assumed to be associative, and property ACHR-algebra for B means that, for every associative complete normed algebra A each homomorphism φ: A $\to$B is continuous. For an approach to results in this direction, the reader is referred to the survey paper of [Dales 1978]. In this paper we are interested in the natural non-associative meaning of property ACHR-algebra. Then the normed algebra B need not be associative, and, even if B is associative, property ACHR-algebra for B has the stronger sense that, for every possibly non- associative complete normed algebra A, each homomorphism φ : A $\to$B is continuous. In this new setting, we know that real or complex absolute-valued algebras, as well as complete normed complex algebras with no non-zero two-sided topological divisors of zero, have property ACHR-algebra (see[Rodriguez 2000]). It is known that a real or complex complete normed quadratic algebras have property ACHR-algebra if and only if has no isotropic element (see[Cedilnik 2013]).

As a main result, we show that an n-dimensional algebras with multiplication table (see Table 3) equipped with suitable norm have property ACHR if and only if

$$\alpha\delta\chi_1+\delta\chi_1\sum_{i=1}^{n-1}\beta_i\chi_i \neq 0$$

1.-TYPES OF ACHR-ALGEBRAS.

Algebras arising throughout this paper are not assumed to be associative . We present some facts about ACHR-algebras.

(i) If (B,$\|.\|_B$) is an ACHR-algebra and $C\subset B$ a subalgebra with a norm $\|.\|_C$ for which

$$\exists\omega>0 \quad \forall x\in C: \|x\|_C\le\omega\|x\|_B ,$$

then(C,$\|.\|_C$) is also ACHR-algebra.

If a normed algebra (B,$\|.\|$) has isotropic element x (i.e: $x\neq 0, x^2=0$), then it is not an ACHR-algebra.

(iii) Let a normed algebra $(B,\|.\|)$ be algebraic of the first order (any element generates a subalgebra of dimension $\le$ 1), then $(B,\|.\|)$ is an ACHR-algebra if and only if B={0} or B$\cong$F.

(iv) Any complex commutative associative semisimple Banach algebra is ACHR-algebra .

Let $(B,\|.\|)$ be a normed algebra, $e\notin B$, $D:=Fe\oplus B$. We make D unital algebra in the following way :

$$(\alpha e+x)(\beta e+y):=\alpha\beta e+(\alpha y+\beta x+xy)$$

and equip it with a suitable norm $\|.\|_D$. From (i) it follows that if

$$\exists\omega>0 \;\; \forall x\in B: \|x\|_B\le\omega\|x\|_D$$

and if $(D,\|.\|_D)$ is ACHR-algebra, then so is $(B,\|.\|_B)$. This statement has the opposite direction proposition(v).

(v) If (assuming previous definitions of $(B,\|.\|_B)$ *and* $(D,\|.\|_D)$) $(B,\|.\|_B)$ is ACHR-algebra, if

$\exists\omega>0\ \forall x\in B:\ \|x\|_D\le\omega\|x\|_B$

and if the direct sum $Fe\oplus B$ is topological, then $(D,\|.\|_D)$ is also ACHR-algebra.

Suppose that a normed algebra $(B,\|.\|)$ is a direct sum of two-sided ideals: $B=\underset{i\in I}{\oplus}B_i$. By(i), if $(B,\|.\|)$ is ACHR-algebra , so is each of $(B_i,\|.\|)$. This statement has the following reverse.

(vi) Let a normed algebra $(B,\|.\|)$ be a finite direct sum of two-sided ideals: $B=\overset{n}{\underset{i=1}{\oplus}}B_i$. If all $(B_i,\|.\|)$ are ACHR-algebra, then $(B,\|.\|)$ is too.

(vii) Suppose that a normed algebra $(B,\|.\|)$ is ACHR-algebra and that there exists another norm $\|\|.\|\|$ on B, such that $(B,\|\|.\|\|)$ is complete normed algebra. Then the topology of the norm $\|.\|$ is weaker than the one of $\|\|.\|\|$: $\exists\omega>0\ \forall x\in B:\ \|x\|\le\omega\|\|x\|\|$.

If $(B,\|.\|)$ is also complete, the topologies are homomorphic and $(B,\|\|.\|\|)$ is ACHR-algebra as well.

(viii) Let $(B,\|.\|_B)$ be a real normed algebra and $C:=B\otimes ₵$ its complexification with a norm $\|.\|_C$. If $(C,\|.\|_C)$ is ACHR-algebra, then also is $(B,\|.\|_B)$.

(ix) Let $(B,\|.\|)$ be a complex normed algebra and B_R the same algebra, viewed as a real algebra. If $(B_R,\|.\|)$ is ACHR-algebra, then also is $(B,\|.\|)$.

(x) Let $(B,\|.\|)$ be a real or complex complete normed quadratic algebra. Then the following statements are equivalent:

(a) B is ACHR-algebra;

(b) B has no isotropic elements;

(c) $B\cong ₵$ or $B\cong ₵\oplus ₵$ (=the direct sum of two one-dimensional ideals).

(xi) A two-dimensional algebra with the multiplication table

Table 1. Multiplication table

.	E	x
e	E	x
x	x	α e+ β x

equipped with a suitable norm is ACHR-algebra if and only if $\beta^2+4\alpha\neq 0$.

(xii) Any smooth algebra is ACHR-algebra.

2.- The MAIN RESULT.

Note that in facts (1-xi) a two-dimensional algebra is ACHR-algebra and we will show that a three-dimensional algebra is also ACHR-algebra. Finally, we generalize this result to an n-dimensional algebra to be ACHR-algebra.

Recall that from [Palmer 1994], a Point derivation of unital algebra A at γ (γ be a point of Gelfand space) is a linear functional δ : $A\to C$ satisfying: $\delta(ab)=\delta(a)\gamma(b)+\gamma(a)\gamma(b)$.

Proposition2.1. Let A be a unital normed algebra with a discontinuous point derivation δ at $\gamma\in\Gamma_A$ (*i.e* : Γ_A ={all non-zero algebra homomorphisms of A into C}). Let B be a unital normed algebra (such as the matrix algebra M_2) with some non-zero nilpotent element. Then there is a discontinuous unital homomorphism of A into B.

Proof. (see,[Palmer 1994]). □

Recall also that, the strong radical of an algebra A is the intersection of all maximal modular two-sided ideals of A and denoted by S-Rad(A). If S-Rad(A)={0}, then A is said to be strongly semi-simple.

Example2.1. Let A be any infinite-dimensional Banach space construct a Banach algebra by defining all products to be zero, let ω be a discontinuous linear functional on A. Then the map $\varphi: A\to M_2$ defined by

$$\varphi(a)=\begin{pmatrix}0 & \omega(a)\\ 0 & 0\end{pmatrix}$$

$\forall a\in A$

Is a discontinuous homomorphism into the finite-dimensional strongly semi-simple algebra M_2. Note that this construction depends on nothing except the fact that the matrix unit $e_{1,2}$ satisfies $(e_{1,2})^2=0$. Hence there is a discontinuous into any algebra with a non-zero nilpotent element (*i.e* ;a $\neq 0$,$a^n=o$). A non -zero element x of an algebra A is said to be isotropic (nilpotent) if $x^2=0$.

Corollary 2.1. Let φ be a homomorphism from a complete normed algebra A into a normed algebra B with isotropic element then φ is discontinuous .

Proof.(see, Example 2.1). □

Recall that from [Rodriguez 1983], for a vector space X we denote by L(X) the associative algebra of all linear mappings from X into X. For an element a in a nonassociative algebra A we denote by L_a (resp.: R_a) the element in L(A) defined by $L_a(x)=ax$ (resp.: $R_a(x)=xa$) for all x in A and we denote by L_a and R_a the sets $L_A=\{L_a : a\in A\}$, $R_A=\{R_a : a\in A\}$. A subalgebra A of an associative algebra B is called a full subalgebra of B if A contains the quasiinverses of its elements that are quasiregular in B. Let A be a nonassociative algebra. The full subalgebra of L(A) generated by $L_A \bigcup R_A$ will be called the full multiplication algebra of A and will be denoted by FM(A). Finally. If A is a nonassociative algebra and if C is any subalgebra of L(A) such that $L_A U R_A \subset$ C $\subset$ FM(A). Considered as the largest C-invariant subspace of A consisting of elements a such that L_a and R_a lie in the Jacobson radical of C. This subspace will be called the C-radical of A and denoted by C-Rad(A).The ultra-weak radical of A (uw-Rad(A)) is defined as the sum of all the C-radicals of A when C runs through the set of all subalgebras of L(A) satisfying $L_A \bigcup R_A \subset C \subset$ FM(A). Since the weak radical of A is a C-radical (take C = FM(A)) it follows that w-Rad(A) $\subset$ uw-Rad(A).

Proposition2.2. Let B be a complete normed algebra over F whose ultraweakradical is zero. Then all homomorphisms from complete normed algebras over F onto B are automatically continuous.

Proof.(see,[Rodriguez 1983]). □

Lemma2.1. Let B be a finite-dimensional algebra over F without isotropic elements. Then the ultra-weak radical of B is equal to zero.

Proof.(see,[Cedilnik and Rodriguez 2003]). □

Proposition2.3. Let ϕ be a homomorphism from a complete normed algebra A into a finite-dimension algebra B, then ϕ is continuous if and only if B has no isotropic element.

Proof. Let ϕ is continuous. If B has isotropic element by Corollary 2.1, we have contradiction.

Conversely, let B has no isotropic element and ϕ be a homomorphism from a complete normed algebra A into B . Since both the finite dimensionality and the absence of isotropic elements are inherited by all subalgebras of *B*, there is no loss of generality in assuming that ϕ is surjective . Then, since B has zero ultra-weak radical (by Lemma 2.1), the continuity of ϕ follows from Proposition 2.2. □

Proposition2.4. Let φ be a homomorphism from a complete normed algebra A into 3-dimension algebra B with multiplication table equipped in Table 2 with suitable norm. Then, φ is continuous if and only if $\alpha\delta a+\beta\delta^2 a+\gamma\delta ab\neq 0$.

Table 2. Multiplication table

.	e	a	b
e	e	a	b
a	a	δa	ab
b	b	ba	$\alpha e+\beta a+\gamma b$

Proof .(i) Suppose that φ is continuous we will prove that $\alpha\delta a+\beta\delta^2 a+\gamma\delta ab\neq 0$. If $\alpha\delta a+\beta\delta^2 a+\gamma\delta ab=0$, we have $(ab)^2=0$

(because $(ab)^2 = a^2b^2 = \delta a(\alpha e+\beta a+\gamma b)=\alpha\delta a+\beta\delta^2 a+\gamma\delta ab=0$).

Therefore, B has isotropic element. And by Corollary2.1, we get φ is not continuous, which is a contradiction.

Therefore, $\alpha\delta a+\beta\delta^2 a+\gamma\delta ab\neq 0$.

Let $\alpha\delta a+\beta\delta^2 a+\gamma\delta ab\neq 0$, we will prove that

φ is continuous:

from (i) note that B has no isotropic element and its finite-dimension and by Proposition 2.3, we have φ is continuous. □

Theorem2.1. Let φ be a homomorphism from a complete normed algebra A into an n-dimensional algebra B with multiplication table equipped in Table 3 with suitable norm and vectors $x(n-1)$ defined by:

Table 3 . Multiplication table

.	e	χ_1	χ_2	...	χ_{n-1}
e	e	χ_1	χ_2	...	χ_{n-1}
χ_1	χ_1	$\delta\chi_1$	$\chi_1\chi_2$	...	$\chi_1\chi_{n-1}$
χ_2	χ_2	$\chi_2\chi_1$	$\chi(2)$	...	$\chi_2\chi_{n-1}$
:	:	:	:	...	:
χ_{n-1}	χ_{n-1}	$\chi_{n-1}\chi_1$	$\chi_{n-1}\chi_2$	...	$\chi(n-1)$

$$x(n-1) = \alpha e + \sum_{i=1}^{n-1} \beta_i x_i \quad \text{for} \quad n \geq 3,$$

Then φ is continuous if and only if $\alpha\delta\chi_1 + \delta\chi_1 \sum_{i=1}^{n-1} \beta_i \chi_i \neq 0$.

Proof . We shall use the method of mathematical induction:

(i) If n=3 we get, the multiplication table as in the following form

.	e	χ_1	χ_2
e	e	χ_1	χ_2
χ_1	χ_1	$\delta\chi_1$	$\chi_1\chi_2$
χ_2	χ_2	$\chi_2\chi_1$	$\chi(2)$

where $x(2) = \alpha e + \sum_{i=1}^{2} \beta_i x_i = \alpha e + \beta_1 x_1 + \beta_2 x_2$.

Then φ is continuous if and only if $\alpha\delta\chi_1 + \beta_1\delta^2\chi_1 + \beta_2\delta\chi_1\chi_2 \neq 0$. This follows directly from proposition 2.4,

(ii) Suppose that the theorem is true when n=r

Then the multiplication table will be in the following form

.	e	χ_1	χ_2	...	χ_{r-1}
e	e	χ_1	χ_2	...	χ_{r-1}
χ_1	χ_1	$\delta\chi_1$	$\chi_1\chi_2$	...	$\chi_1\chi_{r-1}$
χ_2	χ_2	$\chi_2\chi_1$	$\chi(2)$	...	$\chi_2\chi_{r-1}$
:	:	:	:	...	:
χ_{r-1}	χ_{r-1}	$\chi_{r-1}\chi_1$	$\chi_{r-1}\chi_2$	...	$\chi(r-1)$

where $\chi(r-1) = \alpha e + \sum_{i=1}^{r-1} \beta_i \chi_i$. Then φ is continuous if and only if $\alpha\delta x_1 + \delta x_1 \sum_{i=1}^{r-1} \beta_i x_i \neq 0$.

Now, we shall prove that the theorem is true for $n = r + 1$. Consider the following table:

.	e	χ_1	χ_2	...	χ_r
e	e	χ_1	χ_2	...	χ_r
χ_1	χ_1	$\delta\chi_1$	$\chi_1\chi_2$	...	$\chi_1\chi_r$
χ_2	χ_2	$\chi_2\chi_1$	$\chi(2)$	...	$\chi_2\chi_r$
:	:	:	:	...	:
χ_r	χ_r	$\chi_r\chi_1$	$\chi_r\chi_2$	...	$x(r)$

where $\chi(r)=\alpha e+\sum_{i=1}^{r}\beta_i\chi_i$ for $r\geq 2$. Then, we shall prove that φ is continuous if and only if $\alpha\delta\chi_1+\delta\chi_1\sum_{i=1}^{r}\beta_i\chi_i\neq 0$. Let φ be continuous, we will prove that $\alpha\delta\chi_1+\delta\chi_1\sum_{i=1}^{r}\beta_i\chi_i\neq 0$. Now if $\alpha\delta\chi_1+\delta\chi_1\sum_{i=1}^{r}\beta_i\chi_i$ =0. Note that ($\chi_1\chi$ r)^2=0

(because: $(\chi_1\chi_r)^2=\chi_1^2\chi_r^2=\delta\chi_1(\alpha e+\sum_{i=1}^{r}\beta_i\chi_i)=\alpha\delta\chi_1+\delta\chi_1\sum_{i=1}^{r}\beta_i\chi_i=0$). Therefore, B has isotropic element ($\chi_1\chi_r\neq 0,(\chi_1\chi_r)^2=0)$ and by Corollary 2.1, we have φ is not continuous. Thus a contradiction. That is $\alpha\delta\chi_1+\delta\chi_1\sum_{i=1}^{r}\beta_i\chi_i\neq 0$. Conversely, let $\alpha\delta\chi_1+\delta\chi_1\sum_{i=1}^{r}\beta_i\chi_i\neq 0$, we will prove that φ is continuous. Note that B has no isotropic element (because $(\chi_1\chi_r)^2=\alpha\delta\chi_1+\delta\chi_1\sum_{i=1}^{r}\beta_i\chi_i\neq 0$) and finite-dimension. By proposition 2.3, we get φ is continuous. Consequently, the theorem is true for any n. This complete the proof. □

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حول الجبور من النمط ACHR ذا البعد n

المخلص.

اثبتنا ان التشاكل φ من جبر معياري كامل A الى جبر معياري B ذا بُعد n يكون مستمر تلقائياً. كنتيجة B يكون جبر من النمط ACHR.