q-TOPOLOGICAL OPERATORS IN TOPOLOGICAL SPACES
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ABSTRACT
Operation Approach on a new class of topological spaces with δP-open subsets was called a q-open set. In this article, we examined some basic properties of open sets such as closure, Closureq, limit points, derived sets, neighbourhood, interior, exterior, boundary, frontier and saturated set of already existing q-open sets in topological spaces.

KEYWORDS: q-open set, Closureq, q-derived sets, q-neighbourhood, interiorq, q-exterior, q-boundary, q-frontier and q-saturated.

1. INTRODUCTION


2. PRELIMINARIES

Let Y be a set and τ be topology then (Y, τ) be the topological space. Here P be a subset of the topological space (Y, τ). Definition 2.1 [8]. A subset P is called semi-open if \( P \subseteq Cl(\text{Int}(P)) \).

The complement is called a semi-closed set.

Definition 2.2 [12].

\( a. \) A δ-preopen subset P of a space Y is called a \( \delta P \)-open set if for all \( p \in P, \exists \) a semi-closed set \( F \) such that \( p \in F \subseteq P \).

The collection of all \( \delta P \)-open sets is denoted by \( \delta P(Y) \).

The point \( p \) in Y is called \( \delta P \)-closure of P if \( p \in F \land F \neq \emptyset \), for every \( \delta P \)-open set \( H \) containing \( p \). It is denoted by \( \delta P Cl(P) \).

Definition 2.2 [11]. An operation \( q \) on \( \delta P(Y) \) is a mapping \( q: \delta P(Y) \rightarrow \delta P(Y) \) such that \( H \subseteq H^q \) for every \( \delta P \)-open \( X \), where \( \delta P(Y) \) is the power set of \( Y \) and \( H^q \) is the value of \( H \) under \( q \).

Definition 2.3 [11]. Let \((Y, \tau)\) be a topological space and \( q : \delta P(Y) \rightarrow \delta P(Y) \) be an operation on \( \delta P(Y) \). A nonempty set \( P \) of \( Y \) is called q-open if for all \( x \in P, \exists \) a \( \delta P \)-open set \( H \) such that \( x \in H \) and \( H^q \subseteq P \).

Proposition 2.4 [11]. Every q-open set is a \( \delta P \)-open set.

Proposition 2.5 [11]. The union of any class of q-open sets in \( Y \) is q-open.

Definition 2.5 [11]. Let \((Y, \tau)\) be any topological space. An operation \( q \) on \( \delta P(Y) \) is said to be regular.q-operation if for each \( p \in Y \) and for every pair of \( \delta P \)-open sets \( H_1 \) and \( H_2 \) such that both containing \( p \), \( \exists \) a \( \delta P \)-open set \( F \) containing \( p \) such that \( F^q \subseteq H_1^q \cap H_2^q \).

Definition 2.6 [11]. A topological space \((Y, \tau)\) with an operation \( q \) on \( \delta P(Y) \) is said to be q-regular if for given \( p \in Y \) and for each \( \delta P \)-open set \( H \) containing \( p \), \( \exists \) a \( \delta P \)-open set \( F \) containing \( p \) such that \( F^q \subseteq H \).

3. TOPOLOGICAL PROPERTIES OF q-OPEN SETS

3.1. CLOSUREq AND q-CLOSURE OF A SET

Definition 3.1. Let \( P \) be a subset of a topological space \((Y, \tau)\) and \( q \) be an operation on \( \delta P(Y) \). A point \( p \in Y \) is called closureq point of the set \( P \) if for all \( \delta P \)-open \( H \) containing \( p \), \( H^q \cap P \neq \emptyset \). The family of closureq points of \( P \) is called closureq of \( P \) and is denoted by \( Cl_q(P) \).

Proposition 3.1.2. Let \( P \) be a subset of a topological space \((Y, \tau)\) and \( q \) be an operation on \( \delta P(Y) \), then \( \delta P \) \( Cl(P) \subseteq Cl_q(P) \).

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Proof. Consider let \( p \in \delta P \cap \{ p \cap q \} \) then by Definition 2.2 P ∩ H = Φ. For every \( \delta P \) open set containing x. Always H ∩ H̅, which gives \( P \cap H = \emptyset \subseteq P \cap H \neq \emptyset \). By Definition 3.1.1, \( p \in \mathcal{C}(P) \).

Thus, \( \delta P \cap \{ P \cap q \} \subseteq \mathcal{C}(P) \).

**Definition 3.1.3.** Let P be a subset of a topological space \((Y, τ)\) and \( q \) be an operation on \( \delta P \cap \{ q \} \). The \( q \)-closure of P is defined as the intersection of all \( q \)-closed sets of \( Y \) containing \( P \) and it is denoted by \( q \mathcal{C}(P) \).

\( q \mathcal{C}(P) = \{ E \cap P \subseteq E; Y - E \text{is a}\-q\text{-open set in } Y \} \).

**Proposition 3.1.4.** Let \( P \) and \( Q \) be subsets of a topological space \((Y, τ)\) and \( q \) be an operation on \( \delta P \cap \{ q \} \), then the following statements are true:

(a) \( P \subseteq \mathcal{C}(P) \subseteq \mathcal{C}(P \cup Q) \).

(b) \( P \subseteq \delta P \cap \{ P \cap Q \} \subseteq \mathcal{C}(P \cup Q) \).

(c) \( \mathcal{C}(P) \) is a \( q \)-closed set in \( Y \) and it is the smallest \( q \)-closed set containing \( P \).

(d) \( \mathcal{C}(q) = \mathcal{C}(P) \cup \emptyset \) and \( \mathcal{C}(Y) = \mathcal{C}(Y) = Y \).

(e) \( P \) is a \( q \)-closed set iff \( \mathcal{C}(P) = P \).

(f) \( P \) is a \( q \)-closed set iff \( \mathcal{C}(P) \subseteq \{ P \} \).

(g) \( P \subseteq Q \), then \( \mathcal{C}(P) \subseteq \mathcal{C}(Q) \), and \( \mathcal{C}(P) \subseteq \mathcal{C}(Q) \).

(h) \( \mathcal{C}(P \cap Q) \subseteq \mathcal{C}(P) \cap \mathcal{C}(Q) \).

(i) \( \mathcal{C}(P \cup Q) \subseteq \mathcal{C}(P \cup Q) \).

(j) \( \mathcal{C}(P) \cup \mathcal{C}(Q) \subseteq \mathcal{C}(P) \cup \mathcal{C}(Q) \).

(k) \( \mathcal{C}(P) \cap \mathcal{C}(Q) \subseteq \mathcal{C}(P \cup Q) \).

(l) \( \mathcal{C}(G \cap \{ P \}) = \mathcal{C}(P) \).

**Proposition 3.1.5.** If \((Y, τ)\) is a \( q \)-regular, space, then \( \delta P \cap \{ q \} \subseteq \mathcal{C}(P) \).

**Proof.** \( \delta P \cap \{ q \} \subseteq \mathcal{C}(P) \) is proved in Proposition 3.1.2.

**Example 3.2.3.** Consider the space \( Y = \{ p, q, r, s \} \) and \( τ = \{ \emptyset, Y, \{ p \}, \{ p, q \}, \delta \mathcal{S}(O) = (\emptyset, Y, (\{ p \}, \{ p, q \}), \{ q \}, \{ q, r \}, \{ q, r, s \}, \{ q, r, s, s \} \) \).

Then \( \delta \mathcal{S}(O) = (\emptyset, Y) \). An operation \( q \): \( \delta \mathcal{S}(O) \to \mathcal{P}(Y) \) is defined as follows, for every \( H \in \delta \mathcal{S}(O) \).

\( H^q = \{ \emptyset, H, \} \cap \{ H \} = \emptyset \).

Yields, \( p \notin \mathcal{C}(P \cup Q) \).

Thus, \( \mathcal{C}(P \cup Q) \subseteq \{ \{ p \} \cup \emptyset \} \).

(b) From Proposition 3.1.4 (k)

\( \mathcal{C}(P) \cap \mathcal{C}(Q) \subseteq \mathcal{C}(P \cup Q) \) so it is enough to obtain that \( \mathcal{C}(P \cup Q) \subseteq \mathcal{C}(P \cup Q) \). Consider \( \mathcal{C}(P) \cap \mathcal{C}(Q) \). Then there exist two \( q \)-open sets \( G_1 \) and \( G_2 \) containing \( P \cap G_1 = \emptyset \) and \( G_2 \cap G_2 = \emptyset \). As \( q \) is a \( q \)-regular operation on \( \delta \mathcal{S}(O) \) then by Proposition 3.1.2 \( G_1 \cap G_2 \) is a \( q \)-open set in \( Y \) so \( (A \cup B) \cap G_1 \cap G_2 = \emptyset \). Hence the reverse inequality of (c) is not true in general.
Corollary 3.2.4. Let \( P \) be a subset of a topological space \((Y, \tau)\) and \( q \) be an operation on \( \delta \mathcal{P}_2(O(Y)) \). Then \( q \) \( \delta \mathcal{P}_2(O(Y)) \subseteq \mathcal{C}(Y) \).

Proof. Let \( p \in q \mathcal{D}(P) \) from Definition 3.2.1. for every \( q \)-open set \( G \) containing \( p \) satisfying
\[ G \cap (P - \{p\}) \neq \emptyset \Rightarrow G \cap P \neq \emptyset, \]
It follows that \( p \in q \mathcal{C}(O(Y)) \).

So, the corollary is proved.

3.3. \( q \)-NEIGHBOURHOOD OF A POINT AND A SET

Definition 3.3.1. A subset \( N \) of a topological space \((Y, \tau)\) is called a \( q \)-neighbourhood of a point \( p \in Y \) if \( \exists q \)-open set \( G_p \)
in \( Y \) such that \( p \in G \subseteq N \).

The collection of all \( q \)-neighbourhood is denoted by \( q \mathcal{N}(p) \).

Definition 3.3.2. A subset \( N \) of a topological space \((Y, \tau)\) is called a \( q \)-neighbourhood of a set \( P \) if \( \exists q \)-open set \( G \)
in \( Y \) such that \( P \subseteq G \subseteq N \).

Remark 3.3.3. Let \( U \subseteq (Y, \tau) \) be a \( q \)-open set if it is a \( q \)-neighbourhood of each of its points.

Proposition 3.3.4. If \( E \) is a \( q \)-closed subset of a topological space \((Y, \tau)\) and \( p \in Y - E \), then \( \exists \) a \( q \)-neighbourhood \( N \) of \( p \) such that \( N \cap E = \emptyset \).

Proof. Let \( E \) be \( q \)-closed subset of a topological space \( Y \), then \( Y - E \) is \( q \)-open set. Let \( p \in Y - E \). By Remark 3.3.3, \( Y - E \) is a \( q \)-neighbourhood of each of its points. Finally \( \exists \) a \( q \)-neighbourhood \( N \) of \( p \) such that \( N \subseteq Y - E \) which gives that \( N \cap E = \emptyset \).

Proposition 3.3.5. For a topological space \((Y, \tau)\) the following results of \( q \)-neighbourhood are true for all \( p \) in \( Y \):

a. \( q \mathcal{N}(p) \neq \emptyset \).
b. If \( p \in q \mathcal{N}(p) \), then \( p \in P \).
c. If \( p \in q \mathcal{N}(p) \) and \( P \subseteq \mathcal{Q} \), then \( q \subseteq q \mathcal{N}(p) \).
d. If \( q \subseteq q \mathcal{N}(p) \) and \( q \subseteq \mathcal{Q} \), then \( P \cup q \subseteq q \mathcal{N}(p) \).
e. If \( p \in q \mathcal{N}(p) \) then there exist \( q \subseteq q \mathcal{N}(p) \) such that \( P \subseteq Q \) and \( Q \subseteq q \mathcal{N}(q) \).

Proof. (a) By definition of \( q \)-neighbourhood, \( p \in Y \) which is a \( q \)-open set of \( Y \) such that \( p \in Y \cap Y \) implies \( p \) is in \( q \mathcal{N}(p) \) for all \( p \). Therefore \( q \mathcal{N}(p) \neq \emptyset \).

(b) Let \( p \in q \mathcal{N}(p) \) implies \( q \)-neighbourhood of \( p \), then by definition \( p \in \emptyset \).

(c) Let \( p \in q \mathcal{N}(p) \) and \( P \subseteq \mathcal{Q} \). Since \( P \subseteq q \mathcal{N}(p) \) exists a \( q \)-open set \( G \) such that \( P \subseteq G \subseteq \mathcal{Q} \), then \( G \subseteq q \mathcal{N}(p) \).

d. Let \( p \in q \mathcal{N}(p) \) and \( q \subseteq \mathcal{Q} \), then there exists two \( q \)-open sets \( G_1 \) and \( G_2 \) such that \( p \in G_1 \subseteq P \) and \( p \in G_2 \subseteq \mathcal{Q} \). Then \( p \in G_1 \cup G_2 \subseteq P \cup \mathcal{Q} \).

Since \( G_1 \cup G_2 \) is a \( q \)-open set we get \( P \subseteq q \mathcal{N}(p) \).

(e) Let \( p \in q \mathcal{N}(p) \), then there exist a \( q \)-open set \( G \) such that \( p \in G \subseteq \mathcal{Q} \). Since \( G \) is a \( q \)-open set, it is \( q \)-neighbourhood of each of its points by Remark 3.3.3. Thus, there exist a \( q \)-open set \( Q \subseteq \mathcal{Q} \) such that \( Q \cap P \subseteq q \mathcal{N}(p) \).

3.4. \( q \)-INTERIOR

Definition 3.4.1. Let \( P \) be a subset of a topological space \((Y, \tau)\) and \( q \) be an operation on \( \delta \mathcal{P}_2(O(Y)) \). A point \( p \in P \) is called \( q \)-interior point of \( P \) if \( \exists q \)-open set \( H \) containing \( p \), such that
\[ H \subseteq P \] and we denote the family of such points by \( Int_q(P) \).

Proposition 3.4.2. Let \( q \) be an operation on \( \delta \mathcal{P}_2(O(Y)) \), then for a subset \( P \) of \( Y \)
\[ Int_q(P) = \{ q \in O(Y) \text{ and } G \subseteq P \}. \]
containing $p$ such that $H^c \subseteq Y - P$. Gives, $H^c \cap P = \emptyset$, so $p \notin Cl_p(P)$ by Definition 3.1.1.
Finally, $p \notin Cl_p(Y - P) \Rightarrow Y - Cl_p(Y - P).
Reversing the process we get, $Int_p(Y - P) \supseteq Y - Cl_p(Y - P)$
So (a) holds.
(b) Let $p \notin Cl_p(Y - P)$, then $\exists$ a $\partial P_2$-open set $H$ containing $p$, we get $H^c \cap (Y - P) = \emptyset$ gives, $H^c \subseteq Y - P$ so, $p \notin Int_p(Y - P)$, finally $p \notin Y - Int_Y(P)$. So $Cl_p(Y - P) \supseteq Y - Int_Y(P)$.
Reversing the process we get, $Cl_p(Y - P) \subseteq Y - Int_Y(P)$
So (b) holds.
(c) Follows from (a) by replacing $P$ by $Y - P$.
(d) Follows from (b) by replacing $P$ by $Y - P$.

Example 3.4.7. Consider the space $Y = \{p,q,r,s\}$ and $\tau = \{\emptyset, Y, \{p\}, \{p,q\}, \{p,q,r\}, \{p,q,r,s\}\}$.
$\partial P_2O(Y) = \{\emptyset, Y, \{p\}, \{q\}, \{r\}, \{q,r\}, \{p,q,r\}, \{q,r,s\}\}$.
An operation $g : \partial P_2O(Y) \rightarrow P(Y)$ is defined as follows, for each $H \in \partial P_2O(Y)$
\[H^g = \begin{cases} H, \text{ if } H \neq \{q,r\} \\
Y, \text{ if } H = \{q,r\}\end{cases}\]
Then $g$ is not a regular-operation.

$q O(Y) = \{\emptyset, Y, \{r\}, \{p,q\}, \{p,q,r\}, \{q,r\}, \{q,r,s\}\}$.
For a subset $P = \{p,q\}$ and $Q = \{q,r\}$
$Int_Y(P) \cap Int_Y(Q) = Int_Y((p,q)) \cap Int_Y((q,r)) = \{p,q\} \cap \{q,r\} = \{q\} \neq Int_Y(\{q\}) = \emptyset$.

3.5. $q$-EXTERIOR

Definition 3.5.1. Let $P$ be a subset of a topological space $(Y, \tau)$ and $q$ be an operation on $\partial P_2O(Y)$. The $q$-interior of $P$ is defined as the $q$-interior of $Y - P$.
That is $Ext_q(P) = Int_q(Y - P)$.

Proposition 3.5.2. Let $P$ be a subset of a topological space $(Y, \tau)$, and $q$ be an operation on $\partial P_2O(Y)$.
Then the following results are true:

a. $Ext_q(Y) = \emptyset$.
b. $Ext_q(\emptyset) = Y$.
c. $Ext_q(P) = Y - Cl_Y(P)$.
d. If $P \subseteq Q$ then $Ext_q(P) \supseteq Ext_q(Q)$.
e. $Ext_q(P \cup Q) \subseteq Ext_q(P) \cup Ext_q(Q)$.
f. $Ext_q(P) \cap \partial P_2O(Y) \subseteq Ext_q(P) \cap Ext_q(Y)$.
g. $Ext_q(Ext_q(P)) = Int_q(Cl_Y(P))$.
h. $Ext_q(Ext_q(Y - Ext_q(P)))$.
i. $Int_q(P) \subseteq Ext_q(Int_q(P))$.
Proof. The proof is obvious.

The following example shows that the reverse inequality of (e), (f) and (i) is not true in general.

Example 3.5.3. Consider the space $Y = \{p,q,r,s\} = \{\emptyset, Y, \{p\}, \{q\}, \{r\}, \{s\}, \{r,s\}\}$.
$\partial P_2O(Y) = \{\emptyset, Y, \{p\}, \{q\}, \{r\}, \{q,r\}, \{p,q,r\}, \{q,r,s\}\}$.
An operation $g : \partial P_2O(Y) \rightarrow P(Y)$ is defined as follows, for each $H \in \partial P_2O(Y)$
\[H^g = \begin{cases} \{\emptyset, Y, \{q\}, \{r\}\}, \text{ if } H \neq \{q,r\} \\
\emptyset, \text{ if } H = \{q,r\}\end{cases}\]
Hence, $Ext_g(Ext_g(\{s\}) = Ext_g(Int_g(Y - \{s\}) = Ext_g(\emptyset) = Y$.
Thus, $Int_g(\{s\}) = \{s\} \notin Y$.
Hence the reverse inequality of (i) is not true.

Corollary 3.5.4. $Ext_q(P \cap Q) \supseteq Ext_q(P) \cup Ext_q(Q)$.
Proof. $Ext_q(P \cap Q) = Y - (Int_q(P \cap Q)) = Int_q((Y - P) \cup (Y - Q)) \supseteq Int_q((Y - P)) \cup Int_q((Y - Q)) = Ext_q(P) \cup Ext_q(Q)$.

Proposition 3.5.5. $Ext_q(P \cup Q) \subseteq Ext_q(P) \cap Ext_q(Q)$, the equality holds if $q$ is a regular-operation.
Proof. $Ext_q(P \cup Q) \subseteq Ext_q(Y - (P \cup Q)) = Ext_q((Y - P) \cap (Y - Q)) \subseteq Ext_q(P) \cap Ext_q(Q)$, by Proposition 3.4.5 as $q$ is regular-operation the equality holds.

3.6. $q$-BOUNDARY

Definition 3.6.1. Let $P$ be a subset of a topological space $(Y, \tau)$ and $q$ be an operation on $O(Y)$, the $q$-boundary of $P$ is defined as the family of points which do not belong to $q$-interior or $q$-exterior of $P$. It is denoted by $Bd_q(P)$.

Remark 3.6.2. Let $P$ be a subset of a topological space $(Y, \tau)$ and $q$ be an operation on $\partial P_2O(Y)$, then
$Y - Ext_q(P) = Int_q(P) \cup Bd_q(P)$.
Proof. Follows from Definition 3.6.1.

Proposition 3.6.3. Let $q$ be an operation on $\partial P_2O(Y)$. The following conditions are equivalent for a $q$-closed subset:
$Y - Bd_q(P) = Int_q(P) \cup Ext_q(P)$.
$Cl_q(P) = Int_q(P) \cup Bd_q(P)$.
$Bd_q(P) = Cl_q(P) - Int_q(P) = Cl_q(P) \cap Cl_Y(Y - P)$.
Proof. (a) $\Rightarrow$ (b) Remark 3.6.2. Y - $Ext_q(P) = Int_q(P) \cup Bd_q(P)$ gives $Y - (Int_q(Y - P)) = Int_q(P) \cup Bd_q(P) \Rightarrow Cl_Y(P) = Int_q(P) \cup Bd_q(P)$.
By Proposition 3.6.4 (d) (b) $\Rightarrow$ (c) We know from the definition 3.6.1 that,
$Bd_q(P) = \left(Y - Int_q(P)\right) \cap \left(Y - Ext_q(P)\right)$.
$Bd_q(P) = Y - (\{Int_q(P) \cup Ext_q(P)\}$.
$Y - Bd_q(P) = Int_q(P) \cup Ext_q(P) = Int_q(P) \cup Int_q(Y - P) = Y - (Cl_q(Y - P)) \cup Y - Cl_q(Y - P)$.
[by Proposition 3.4.6 (a)]
Taking complement,
$Bd_q(P) = Cl_q(Y - P) \cap Cl_q(Y) \quad \text{---------} (*)$
$= \left(Y - (Int_q(P)) \cap Cl_q(Y)\right)$ [by Proposition 3.4.6 (b)]
$Bd_q(P) = Cl_q(P) - Int_q(P)$ [by Proposition 3.6.4 (b)]
(c) $\Rightarrow$ (a) We have, $Bd_q(P) = q Cl(Y) - Int_q(Y)$
Consider
$Int_q(P) \cup Int_q(Y - P) = \left(Y - (Int_q(P)) \cap (Y - Ext_q(Y - P))\right)$
$= \left(Y - (Int_q(P)) \cap (Y - Ext_q(Y - P))\right)$
$= \left(Y - (Cl_q(Y - P)) \cap Cl_q(Y - P)\right) [by \ Proposition \ 3.4.6 \ (b)] = Y - Bd_q(P)$ from (*).

Corollary 3.6.4. $Bd_q(P) = Bd_q(Y - P)$ for part of a subset $P$ of $Y$.
Proof. From (*) in Proposition 3.6.3, we get

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Proof. (a) Let $P$ be a $q$-open set, then $Y - P$ is $q$-closed (i.e.) $Y - P = \text{Cl}_q(Y - P)$ for, 
$P \cap \text{Bd}_d = P \cap \text{Cl}_q(Y - P) = P \cap (Y - P) = \emptyset \cap \text{Cl}_q(Y - P) = \emptyset.$
On the other hand, Suppose $P \cap \text{Bd}_d = \emptyset$, 
\begin{align*}
P \cap \text{Cl}_q(Y - P) &= \emptyset \Rightarrow \text{Cl}_q(Y - P) = \emptyset ,
\end{align*}
P is $q$-closed $P \subseteq \text{Bd}_d$.
(b) Let $P$ be a $q$-closed set, then $P = \text{Cl}_q(P)$.
\begin{align*}
\text{Cl}_q(Y - P) \\ \subseteq (\text{Cl}_q(P) \cap Y) \\ \subseteq (P \cap Y) \\ = P.
\end{align*}
Hence, $P \subseteq \text{Bd}_d$.

Proposition 3.6.6. Let $P$ be a subset of a topological space $(Y, \tau)$ and $q$ be an operation on $\delta P_S(O(Y))$. Then the following properties hold:
a. $P$ is $q$-open if and only if $P \cap \text{Bd}_d = \emptyset$.
b. $P$ is $q$-closed if and only if $P \subseteq \text{Bd}_d$.
Proof. (a) Let $P$ be a $q$-open set, then $Y - P$ is $q$-closed (i.e.) $Y - P = \text{Cl}_q(Y - P)$ for,
$P \cap \text{Bd}_d = P \cap \text{Cl}_q(Y - P) = P \cap (Y - P) = \emptyset \cap \text{Cl}_q(Y - P) = \emptyset.$
On the other hand, Suppose $P \cap \text{Bd}_d = \emptyset$, 
\begin{align*}
P \cap \text{Cl}_q(Y - P) &= \emptyset \Rightarrow \text{Cl}_q(Y - P) = \emptyset ,
\end{align*}
P is $q$-closed $P \subseteq \text{Bd}_d$.
(b) Let $P$ be a $q$-closed set, then $P = \text{Cl}_q(P)$.

Proposition 3.6.7. Let $q$ be an operation on $\delta P_S(O(Y))$. Then for any two subsets $P$ and $Q$ of a topological space, if $q$ is $q$-regular then the following equality holds:
a. $\text{Bd}_d(P \cup Q) \subseteq [\text{Bd}_d(P) \cap \text{Cl}_q(Y - Q)] \cup [\text{Bd}_d(Q) \cap \text{Cl}_q(Y - P)]$.
b. $\text{Bd}_d(P \cap Q) \subseteq [\text{Bd}_d(P) \cap \text{Cl}_q(Q)] \cup [\text{Bd}_d(Q) \cap \text{Cl}_q(Y - P)]$.
Proof. (a) Consider $\text{Bd}_d(P \cup Q) = \text{Cl}_q(Y - P) \cap \text{Cl}_q(Y - Q)$.
Then $\text{Cl}_q(P \cup Q) \subseteq (\text{Cl}_q(Y - P) \cap \text{Cl}_q(Y - Q)) \cap \text{Cl}_q(P \cup Q)$.
\begin{align*}
\subseteq \text{Cl}_q(Y - A) \\ \subseteq \text{Cl}_q(Y - Q) \\ \subseteq \text{Bd}_d(Q) \cap \text{Cl}_q(Y - Q).
\end{align*}
\begin{align*}
\text{Cl}_q(P \cap Q) \subseteq (\text{Cl}_q(Y - P) \cap \text{Cl}_q(Y - Q)) \cap \text{Cl}_q(P \cap Q)
\subseteq \text{Cl}_q(Y - Q) \\ \subseteq \text{Bd}_d(Q) \cap \text{Cl}_q(Y - Q).
\end{align*}
(b) Consider $\text{Bd}_d = \text{Cl}_q(Y - Q) \cap \text{Bd}_d(Q) \cap \text{Cl}_q(Y - P)$.

3.8. $q$-SATURATED

Definition 3.8.1. Let $P$ be a subset of a topological space $(Y, \tau)$ and $q$ be an operation on $\delta P_S(O(Y))$. $P$ is called $q$-saturated if $\text{qSat}(p) \subseteq P$, for every point $p$ in $P$.
The family of $q$-saturated sets in $(Y, \tau)$ is defined as $q$ Sat $(Y)$.
Remark 3.8.2. In \((Y, \tau)\) if \(P\) is saturated set \(Y - P\) is also a saturated set. But if \(A \in q\, Sat(Y)\) then \((Y - P)\) need not to be a \(q\, Sat(Y)\).

Proposition 3.8.3. Every \(q\)-closed set is a \(q\)-saturated set.
Proof. Let \(P\) be a \(q\)-closed set and let \(x \in P\). Then \(\{p\} \subseteq P \Rightarrow q\, Cl(\{p\}) \subseteq q\, Cl(P) = P\), as \(P\) is a \(q\)-closed set \(\Rightarrow q\, Cl(\{p\}) \subseteq P \Rightarrow P\) is a \(q\)-saturated set.

The converse is not true. That is if \(P\) is \(q\)-saturated set then \(P\) is need not to be \(q\)-closed set.

Example 3.8.4. Consider the space \(Y = \{p, q, r, s\}\), \(\tau = \{\emptyset, Y, \{p\}\}\) and
\[
\delta P_0(Y) = \{\emptyset, Y, \{q\}, \{r\}, \{s\}, \{q, r\}, \{q, s\}, \{r, s\}, \{q, r, s\}\}.
\]
An operation \(q: \delta P_0(Y) \to \mathcal{P}(Y)\) is defined as follows, for every \(H \in \delta P_0(Y)\)
\[
H^q = \begin{cases} 
Cl(H), & \text{if } q \in H \\
H, & \text{if } q \notin H 
\end{cases}
\]
Implies, \(q\, O(Y) = \{\emptyset, Y, \{r\}, \{q, r\}, \{q, s\}\}\) and
\[
q\, C(Y) = \{\emptyset, Y, \{p\}, \{p, q\}, \{p, r\}, \{p, s\}\}.
\]
\(q\, Sat(Y) = \{\emptyset, Y, \{p\}, \{p, q\}, \{p, r\}, \{p, s\}\}\).

Here \(\{p, q\} \notin q\, Sat(Y)\) but \(\{p, q\} \notin q\, C(Y)\).

In conclusion, the various properties of \(q\)-topological operators were studied with examples.

REFERENCES


