Available online at sjuoz.uoz.edu.krd

journals.uoz.edu.krd

Science Journal of University of Zakho

Vol. 11, No. 3, pp. 317– 322, July-September, 2023

p-ISSN: 2663-628X
e-ISSN: 2663-6298

ϱ- TOPOLOGICAL OPERATORS IN TOPOLOGICAL SPACES

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Received: 4 Feb. 2023 / Accepted: 29 May 2023 / Published: 9 July 2023 *https://doi.org/10.25271/sjuoz.2023.11.3.1117*

ABSTRACT

Operation Approach on a new class of topological spaces with δP_S-open subsets was called a ϱ-open set. In this article, we examined some new classes of sets via ϱ-open sets on an operation ϱ on δP_S-open sets in topological spaces such as closure, Closure_ϱ, limit points, derived sets, neighbourhood, interior, exterior, boundary and frontier. Some properties of these topological properties are investigated. Moreover, a new class of set via the operation ϱ on δP_S-open sets called saturated set are defined. Finally, some relationships among these classes of sets are given and some examples are illustrated.

KEYWORDS: ϱ-open set,Closure_ϱ, ϱ-derived sets, ϱ-neighbourhood, interior_ϱ, ϱ-exterior, ϱ-boundary, ϱ-frontier and ϱ-saturated.

1. INTRODUCTION

Semi-open sets were initially proposed by Levine [8] in 1963. In 1968,Velicko [13] was the first to allude to the class of δ -open subsets of a topological space. In 1979,Kasahara [7] developed the concept of α-closed graphs of functions and examined the idea of an operation on an open set τ. In 1982 Mashhour, Abd El-Monsef, and El-Deeb [9] developed the idea of preopen sets. Ogata [5] changed the designation of the operation α to operation γ on τ in 1991. In 1993, Raychaudhuri and Mukherjee [10] discovered and explored a class of sets called δ-preopen. Khalaf and Asaad [1] in 2009 introduced a new concept P_S-open sets in topological spaces. Khalaf and Ameen [2] in 2010 introduced the new class of open sets called sc-open sets in topological spaces. Assad [3] introduced Operation approaches on P_S-open sets and their separation axioms in 2016.Jayashree and Sivakamasundari [6] in 2018 initiated the Operation approaches on δ-open sets. An operation γ on the collection of gα-open subsets of a topological space was introduced by Asaad and Ameen [4] in 2019. Ameen, Asaad, and Muhammed [14] in 2019 introduced the super class of δ-open sets. In 2020, Vidhyapriya, Shanmugapriya and Sivakamsundari [12] developed a new type of open set called δP_S-open sets by combining the ideas of δ-preopen and P_S-open sets. Shanmugapriya, Vidhyapriya and Sivakamsundari [11] in 2021 introduced a ϱ-open set as a novel class of operational open sets in topological spaces. Using ϱ-open sets, here we examined fundamental properties of topological operators: the ϱ-closure, closure_ϱ, ϱ-limitpoints,ϱ-derivedsets,ϱ-neighborhood, ϱ-interior,ϱ-exterior, ϱ-boundary,ϱ-frontier of a set and ϱ-saturated sets.

2. PRELIMINARIES

Let Y be a set and τ be topology then (Y, τ) be the topological space. Here P be a subset of the topological space (Y, τ).

Definition 2.1[8]. A subset P is called semi-open if $P \subseteq \text{Cl}(\text{Int}(P))$.

The complement is called a semi-closed set.

Definition 2.2[12].

a. A δ-preopen subset P of a space Y is called a δP_S -open set if for all $p \in P$, ∃ a semi-closed set F such that $p \in F \subseteq P$.The collection of all δP_S-open sets is denoted by $\delta P_S O(Y)$.

b. The point p in Y is called δP_S-closure of P iff $P \cap H \neq \emptyset$, for every δP_S-open set H containing p.It is denoted by $\delta P_S Cl(P)$.

Definition 2.2[11]. An operation ϱ on $\delta P_S O(Y)$ is a mapping $\varrho: \delta P_S O(Y) \to \mathcal{P}(Y)$ such that $H \subseteq H^\varrho$ for every ∈ $\delta P_S O(X)$, where $\mathcal{P}(Y)$ is the power set of Y and H^ϱ is the value of H under ϱ.

Definition 2.3[11]. Let (Y, τ) be a topological space and $\varrho: \delta P_S O(Y) \to \mathcal{P}(Y)$ be an operation on $\delta P_S O(Y)$. A nonempty set P of Y is called a ϱ-open set if for all $x \in P$, ∃ a δP_S -open set H such that $x \in H$ and $H^\varrho \subseteq P$.

Proposition 2.4[11]. Every ϱ-open set is a δP_S-open set.

Proposition 2.3[11]. The union of any class of ϱ-open sets in Y is ϱ-open.

Definition 2.5[11]. Let (Y, τ) be any topological space. An operation ϱ on $\delta P_S O(Y)$ is said to be regular_ϱ-operation if for each $p \in Y$ and for every pair of δP_S-open sets H_1 and H_2 such that both containing p, ∃ a δP_S- open set F containing p such that $F^\varrho \subseteq {H_1}^\varrho \cap {H_2}^\varrho$.

Definition 2.6[11]. A topological space (Y, τ) with an operation ϱ on $\delta P_S O(Y)$ is said to be ϱ- regular space if for given $p \in Y$ and for each δP_S -open set H containing p, ∃ a δP_S-open set F containing p such that $F^\varrho \subseteq H$.

3. TOPOLOGICAL PROPERTIES OF ϱ-OPEN SETS

3.1. CLOSURE_ϱ AND ϱ-CLOSURE OF A SET

Definition 3.1.1. Let P be a subset of a topological space (Y, τ) and ϱ be an operation on $\delta P_S O(Y)$. A point $p \in Y$ is called closure_ϱ point of the set P if for all δP_S-open set H containing p, $H^\varrho \cap P \neq \emptyset$ .The family of closure_ϱ points of P is called closure_ϱ of P and is denoted by $Cl_\varrho(P)$.

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Proposition 3.1.2. Let P be a subset of a topological space (Y, τ) and ϱ be an operation on $\delta P_S O(Y)$, then $\delta P_S\, Cl(P) \subseteq Cl_\varrho(P)$.

Proof. Consider let $p \in \delta P_S\, Cl(P)$ then by Definition 2.2 $P \cap H \neq \emptyset$, for every δP_S-open set H containing x. Always $H \subseteq H^\varrho$, Which gives $P \cap H \neq \emptyset \subseteq P \cap H^\varrho \neq \emptyset$. By Definition 3.1.1, $p \in Cl_\varrho(P)$.

Thus, $\delta P_S\, Cl(P) \subseteq Cl_\varrho(P)$.

Definition 3.1.3. Let P be a subset of a topological space (Y, τ) and ϱ be an operation on $\delta P_S O(Y)$. The ϱ-closure of P is defined as the intersection of all ϱ -closed sets of Y containing P and it is denoted by $\varrho\, Cl(P)$.

$\varrho\, Cl(P) = \cap \{ E \mid P \subseteq E;\ Y - E$ is ϱ-open set in Y$\}$.

Proposition 3.1.4. Let P and Q be subsets of a topological space (Y, τ) and ϱ be an operation on $\delta P_S O(Y)$, then the following statements are true:

- **a.** $P \subseteq Cl_\varrho(P) \subseteq \varrho\, Cl(P)$.
- **b.** $P \subseteq \delta P_S\, Cl(P) \subseteq \varrho\, Cl(P)$.
- **c.** $\varrho\, Cl(P)$ is a ϱ-closed set in Y and it is the smallest ϱ-closed set containing P.
- **d.** $Cl_\varrho(\emptyset) = \varrho\, Cl(\emptyset) = \emptyset$ and $Cl_\varrho(Y) = \varrho\, Cl(Y) = Y$.
- **e.** P is a ϱ-closed set iff $\varrho\, Cl(P) = P$.
- **f.** P is a ϱ-closed set iff $Cl_\varrho(P) = P$.
- **g.** If $P \subseteq Q$, then $\varrho\, Cl(P) \subseteq \varrho\, Cl(Q)$ and $Cl_\varrho(P) \subseteq Cl_\varrho(Q)$.
- **h.** $Cl_\varrho(P \cap Q) \subseteq Cl_\varrho(P) \cap Cl_\varrho(Q)$.
- **i.** $\varrho\, Cl(P \cap Q) \subseteq \varrho\, Cl(P) \cap \varrho\, Cl(Q)$.
- **j.** $Cl_\varrho(P) \cup Cl_\varrho(Q) \subseteq Cl_\varrho(P \cup Q)$.
- **k.** $\varrho\, Cl(P) \cup \varrho\, Cl(Q) \subseteq \varrho\, Cl(P \cup Q)$.
- **l.** $\varrho\, Cl\,(\varrho\, Cl(P)) = \varrho\, Cl(P)$.

Proposition 3.1.5. If (Y, τ) is a ϱ-regular space, then $\delta P_S Cl(P) = Cl_\varrho(P)$.

Proof. $\delta P_S Cl(P) \subseteq Cl_\varrho(P)$ is proved in Proposition 3.1.2.

Let $p \notin \delta P_S Cl(P)$, then $\exists$ a δP_S-open set H containing p such that $P \cap H = \emptyset$. As (Y, τ) is a ϱ-regular space then by Definition 3.1.4, for all $p \in Y$ and for all δP_S-open set F containing p such that $F^\varrho \subseteq H$, so $P \cap F^\varrho = \emptyset$. Hence$p \notin Cl_\varrho(P)$.Therefore $\delta P_S Cl(P) = Cl_\varrho(P)$.

Proposition 3.1.6. If ϱ is a $regular_\varrho$-operation on $\delta P_S O(Y)$, then

- **a.** $Cl_\varrho(P) \cup Cl_\varrho(Q) = Cl_\varrho(P \cup Q)$
- **b.** $\varrho\, Cl(P) \cup \varrho\, Cl(Q) = \varrho\, Cl(P \cup Q)$.

Proof. (a) From Proposition 3.1.4 (f) we have, $Cl_\varrho(P) \cup Cl_\varrho(Q) \subseteq Cl_\varrho(P \cup Q)$.

On the other side, consider $p \notin Cl_\varrho(P) \cup Cl_\varrho(Q)$ $\exists$ a pair of δP_S-open sets H_1 and H_2 such that both containing p, $H_1{}^\varrho \cap P = \emptyset$ and $H_2{}^\varrho \cap Q = \emptyset$. Now ϱ is a $regular_\varrho$-operation on $\delta P_S\ O(Y)$ then for all $p \in Y$ and $\exists$ a δP_S -open set F containing p such that that $F^\varrho \subseteq H_1{}^\varrho \cap H_2{}^\varrho$.

So,$(P \cup Q) \cap F^\varrho \subseteq (P \cup Q) \cap H_1{}^\varrho \cap H_2{}^\varrho$

but $(P \cup Q) \cap H_1{}^\varrho \cap H_2{}^\varrho =$

$[(P \cup Q) \cap H_1{}^\varrho] \cap H_2{}^\varrho = \emptyset \cap H_2{}^\varrho = \emptyset$.Which gives, $(P \cup Q) \cap F^\varrho = \emptyset$.

Yields, $p \notin Cl_\varrho(P \cup Q)$.

So, $Cl_\varrho(P \cup Q) \subseteq Cl_\varrho(P) \cup Cl_\varrho(Q)$.

Thus, $Cl_\varrho(P) \cup Cl_\varrho\, Cl(Q) = Cl_\varrho(P \cup Q)$.

(b) From Proposition 3.1.4 (k)

$\varrho\, Cl(P) \cup \varrho\, Cl(Q) \subseteq \varrho\, Cl(P \cup Q)$ so it is enough to obtain that$\varrho\, Cl(P \cup Q) \subseteq \varrho\, Cl(P) \cup \varrho\, Cl(Q)$.

Consider$p \notin \varrho\, Cl(P) \cup \varrho\, Cl(Q)$, then there exist two ϱ-open sets $G_1$and G_2 containing p such that $P \cap G_1 = \emptyset$ and $Q \cap G_2 = \emptyset$. As ϱ is a $regular_\varrho$-operation on $\delta P_S O(Y)$ then by Proposition 3.1.2 $G_1 \cap G_2$ is a ϱ-open set in Y so $(A \cup B) \cap G_1 \cap G_2 = \emptyset$. Finally $p \notin \varrho\, Cl(P \cup Q)$, thus

$\varrho\, Cl(P \cup Q) \subseteq \varrho\, Cl(P) \cup \varrho\, Cl(Q)$.

So $\varrho\, Cl(P) \cup \varrho\, Cl(Q) = \varrho\, Cl(P \cup Q)$.

Proposition 3.1.7. Let P be a subset of a topological space (Y, τ) and ϱ be an operation on $\delta P_S O(Y)$. Then $p \in \varrho\, Cl(P)$ iff $P \cap G \neq \emptyset$, for every ϱ-open set G of Y containing p.

Proof. Consider $p \in \varrho\, Cl(P)$ and suppose $P \cap G = \emptyset$ for some ϱ-open set G of Y containing p.

Then $P \subseteq Y - G$ and $Y - G$ is a ϱ-closed set in Y. Then, $\varrho\, Cl(P) \subseteq Y - G$. Thus, $p \in (Y - G)$, this is a contradiction. So $P \cap G \neq \emptyset$ for every ϱ- open set G of Y containing p.

To prove the contrary, if $p \notin \varrho\, Cl(P)$ then $\exists$ a ϱ-closed set E such that $P \subseteq E$ and $p \notin P$ so $p \notin E$.

Then $Y - E$ is a ϱ-open set such that $p \in (Y - E)$ and $P \cap (Y - E) = \emptyset$ contradicting our hypothesis.

Thus, $p \in \varrho\, Cl(P)$.

3.2. ϱ -LIMIT POINT

Definition 3.2.1. Let P be a subset of a topological space (Y, τ) and ϱ be an operation on $\delta P_S O(Y)$. A point $p \in Y$ is called ϱ -limit point of P if for every ϱ-open set G containing p, $G \cap (P - \{p\}) \neq \emptyset$.

The family of ϱ- limit points of P is called a ϱ-derived set of P and it is denoted by $\varrho\, D(P)$.

Some properties of ϱ-derived set are mentioned in the following propositions.

Proposition 3.2.2. The following properties hold for any sets P and Q in a topological space (Y, τ) with an operation ϱ on $\delta P_S O(Y)$.

a. $\varrho\, D(\emptyset) = \emptyset$.

b. If $P \subseteq Q$, then $\varrho\, D(P) \subseteq \varrho\, D(Q)$.

c. $\varrho\, D(P \cap Q) \subseteq \varrho\, D(P) \cap \varrho\, D(Q)$.

d. $\varrho\, D(P \cup Q) \supseteq \varrho\, D(P) \cup \varrho\, D(Q)$.

e. $\varrho\, D(\varrho\, D(P)) - P \subseteq \varrho\, D(P)$.

f. $\varrho\, D(P \cup \varrho\, D(P)) \subseteq P \cup \varrho\, D(P)$.

Proof. Proof of (a) is obvious.

(b)We have $P \subseteq Q$, $P - \{x\} \subseteq B - \{x\} \Rightarrow G \cap (A - \{x\}) \subseteq G \cap (B - \{x\})$.Then

$G \cap (A - \{x\}) \neq \emptyset \Rightarrow G \cap (B - \{x\}) \neq \emptyset$.

Hence $x \in \varrho\, D(A) \Rightarrow x \in \varrho\, D(B)$.

Thus $\varrho\, D(A) \subseteq \varrho\, D(B)$.

This example proves that the reverse inequality of (c),(d), (e) and (f) is not true in general.

Example 3.2.3.Consider the space $Y = \{p, q, r, s\}$and $\tau = \{\emptyset, Y, \{p\}, \{p, q\}$and $\delta P_S O(Y) = \{\emptyset, Y, \{q\}, \{r\}, \{s\}, \{q, r\}, \{q, s\}, \{r, s\}, \{q, r, s\}\}$

$\delta C(Y) = \{ \emptyset, Y\}$.An operation $\varrho : \delta P_S O(Y) \to \mathcal{P}(Y)$ is defined as follows, for every $H \in \delta P_S O(Y)$.

$$H^\varrho = \begin{cases} \delta Cl\ H, & \text{if } q \notin H \\ H, & \text{if } q \in H \end{cases}$$

Implies $\varrho\, O(Y) = \{ \emptyset, Y, \{q\}, \{q, r\}, \{q, s\}, \{q, r, s\}\}$.

For a subset $P = \{p, r\}$ and $Q = \{p, s\}$ therefore, $\varrho\, D(\{p, r\} \cap \{p, s\}) = \varrho\, D(\{p\}) = \emptyset \not\supseteq$

$\varrho\, D(\{p, r\}) \cap \varrho\, D(\{p, s\}) = \{p\} \cap \{p\} = \{p\}$.

Hence the reverse inequality of (c) is not true.

For a subset $P = \{q\}$ and $Q = \{p, r, s\}$ therefore, $\varrho\, D(\{q\} \cup \{p, r, s\}) = \varrho\, D(X) = X \not\subseteq$

$\varrho\, D(\{q\}) \cup \varrho\, D(\{p, r, s\}) = \{p, r, s\} \cup \{p\} = \{p, r, s\}$.

Hence the reverse inequality of (d) is not true.

For a subset $P = \{q, r\}$ therefore,

$\varrho\, D(\varrho\, D(\{q, r\}) - \{q, r\} = \{p\} \not\supseteq \varrho\, D(\{q, r\}) = \{p, r, s\}$.

Hence the reverse inequality of (e) is not true.

For a subset $P = \{r, s\}$ therefore,

$\varrho\, D(\{r, s\} \cup \varrho\, D(\{r, s\}) = \{p\} \not\supseteq \{r, s\} \cup \varrho\, D(\{r, s\}) = \{p, r, s\}$.

Hence the reverse inequality of (f) is not true.

Corollary 3.2.4. Let P be a subset of a topological space (Y, τ) and ϱ be an operation on δPs O(Y). Then $\varrho\, D(P) \subseteq \varrho\, Cl(P)$.
Proof. Let $p \in \varrho D(P)$ from Definition 3.2.1, for every ϱ-open set G containing p satisfying
$G \cap (P - \{p\}) \neq \emptyset \Rightarrow G \cap P \neq \emptyset$, it follows that $p \in \varrho\, Cl(P)$. So, the corollary is proved.

3.3. ϱ-NEIGHBOURHOOD OF A POINT AND A SET

Definition 3.3.1. A subset N of a topological space(Y, τ) is called a ϱ-neighbourhood of a point $p \in Y$, if ∃ a ϱ-open set G in Y such that $p \in G \subseteq N$.
The collection of all ϱ-neighbourhood is denoted by $\varrho\, N(p)$.

Definition 3.3.2. A subset N of a topological space(Y, τ) is called a ϱ-neighbourhood of a set P if ∃ a ϱ-open set G in Y such that $P \subseteq G \subseteq N$.

Remark 3.3.3. Let $U \subseteq (Y, \tau)$ be a ϱ-open set iff it is a ϱ-neighbourhood of each of its points.

Proposition 3.3.4. If E is a ϱ-closed subset of a topological space Y and $p \in Y - E$, then ∃ a ϱ-neighbourhood N of p such that $N \cap E = \emptyset$.
Proof. Let E be ϱ-closed subset of a topological space Y, then $Y - E$ is ϱ-open set. Let $p \in Y - E$. By Remark 3.3.3, $Y - E$ is a ϱ-neighbourhood of each of its points. Finally ∃ a ϱ-neighbourhood N of p such that
$N \subseteq Y - E$ which gives that $N \cap E = \emptyset$.

Proposition 3.3.5. For a topological space (Y, τ) the following results of ϱ-neighbourhood are true for all p in Y:
a. $\varrho\, N(p) \neq \emptyset$.
b. If $P \in \varrho\, N(p)$, then $p \in P$.
c. If $P \in \varrho\, N(p)$and $P \subseteq Q$, then $Q \in \varrho\, N(p)$.
d. If $P \in \varrho\, N(p)$and $Q \in \varrho\, N(p)$, then $P \cup Q \in \varrho\, N(p)$.
e. If $P \in \varrho\, N(p)$then there exist $Q \in \varrho\, N(p)$ such that $Q \subseteq P$ and $Q \in \varrho\, N(q)$ for all $q \in Q$.
Proof. (a) Since by definition of ϱ-neighbourhood,
$p \in Y$ which is a ϱ-open set of Y such that $p \in Y \subseteq Y$ implies Y is in $\varrho\, N(p)$ for all p .Therefore $\varrho\, N(p) \neq \emptyset$.
(b) Let $P \in \varrho\, N(p)$implies P is ϱ-neighbourhood of p, then by definition $p \in P$.
(c) Let $P \in \varrho\, N(p)$and $P \subseteq Q$.Since $P \in \varrho\, N(p)$ there exists a ϱ-open set G such that $p \in G \subseteq P$, then
$p \in G \subseteq P \subseteq Q$, implies Q is a ϱ-neighbourhood of p, that is $Q \in \varrho\, N(p)$.
(d) Let $P \in \varrho\, N(p)$ and $Q \in \varrho\, N(p)$, then there exists two ϱ-open sets G_1 and G_2such that
$p \in G_1 \subseteq P$ and $p \in G_2 \subseteq Q$. Then $p \in G_1 \cup G_2 \subseteq P \cup Q$. Since $G_1 \cup G_2$ is a ϱ-open set we get $P \cup Q$ is a ϱ-neighbourhood of p. Thus, $\cup\, Q \in \varrho\, N(p)$.
(e) Let $P \in \varrho\, N(p)$, then there exist a ϱ-open set G such that $p \in G \subseteq P$.Since Q is a ϱ-open set, it is ϱ-neighbourhood of each of its points by Remark 3.3.3.Thus, there exist a ϱ-open set $Q = G$ such that $Q \subseteq P$ and $Q \in \varrho\, N(q)$ for all $q \in Q$.

3.4. ϱ-INTERIOR

Definition 3.4.1. Let P be a subset of a topological space(Y, τ) and ϱ be an operation on $\delta P_S O(Y)$. A point $p \in P$ is called ϱ-interior point of P if ∃ a δP_S-open set H containing p, such that $H^\varrho \subseteq P$ and we denote the family of such points by $Int_\varrho(P)$.

Proposition 3.4.2. Let ϱ be an operation on $\delta P_S O(Y)$, then for a subset P of Y
$Int_\varrho(P) = \cup \{G \mid G \in \varrho\, O(Y) \text{ and } G \subseteq P\}$.
Proof. Let $p \in Int_\varrho(P)$, then by Definition 3.4.1, ∃ a δP_S-open set H_p such that $p \in H_p$ and ${H_p}^\varrho \subseteq A$. $Int_\varrho(A) = \cup_{p \in Int_\varrho(A)}\{p\} \subseteq \cup H_p \subseteq \cup {H_p}^\varrho \subseteq A$.
Let $\cup {H_p}^\varrho = U$, which is a ϱ-open set by Proposition 2.3 and Proposition 2.5. Finally, $p \in Int_\varrho(P) = U \subseteq P$.
So $p \in \cup \{G \mid G \in \varrho\, O(Y) \text{and } G \subseteq P\}$.

Proposition 3.4.3. Let ϱ be an operation on $\delta P_S O(Y)$, then the following properties holds:
a. $Int_\varrho(\emptyset) = \emptyset$ and $Int_\varrho(Y) = Y$.
b. $Int_\varrho(A)$ is the largest ϱ-open set contained in P.
c. P is ϱ -open set iff $Int_\varrho(P) = P$.
d. If $P \subseteq Q$, then $Int_\varrho(P) \subseteq Int_\varrho(Q)$.
e. $Int_\varrho(P \cap Q) \subseteq Int_\varrho(P) \cap Int_\varrho(Q)$.
f. $Int_\varrho(P) \cup Int_\varrho(Q) \subseteq Int_\varrho(P \cup Q)$.
g. $Int_\varrho\left(Int_\varrho(P)\right) = Int_\varrho(P)$.
Proof. Proof of (a), (b), (c) and (g) are obvious from existing results and definitions.
The following example shows that the reverse inequality of (d), (e) and (f) is not true in general.

Example 3.4.4.Consider the space $Y = \{p, q, r, s\}$ and $\tau = \{\emptyset, Y, \{p, q\}\}$ $\delta P_S O(Y) = \{\emptyset, Y, \{r\}, \{s\}, \{r, s\}\}$. An operation $\varrho : \delta P_S O(Y) \to \mathcal{P}(Y)$ is defined as follows, for every $H \in \delta P_S O(Y)$

$$H^\varrho = \begin{cases} ClH, & \text{if } r \in H \\ H, & \text{if } r \notin H \end{cases}$$

Implies $\varrho\, O(Y) = \{\emptyset, Y, \{s\}, \{r, s\}\}$.
For a subset $P = \{q, s\}$ and $B = \{q, r, s\}$ and $P \subseteq Q$. Hence $Int_\varrho(\{q, s\}) \not\supseteq Int_\varrho(\{q, r, s\}) = \{d\} \not\supseteq \{r, s\}$.
Hence the reverse inequality of (d) is not true.
For a subset $P = \{p, q, s\}$ and $Q = \{p, r, s\}$, implies $P \cap Q = \{p, s\}$.
Hence,
$Int_\varrho(\{p, q, s\} \cap \{p, r, s\}) = Int_\varrho(\{p, s\}) = \{s\} \not\supseteq Int_\varrho(\{p, q, s\}) \cap Int_\varrho(\{p, r, s\}) = \{s\}$.
Thus, the reverse inequality of (e) is not true.
For a subset $P = \{p, q, s\}$ and $Q = \{p, r, s\}$, implies $P \cup Q = Y$.
Hence,$Int_\varrho(\{p, q, s\} \cup \{p, r, s\}) = Int_\varrho(Y) = Y \not\subseteq Int_\varrho(\{p, q, s\}) \cup Int_\varrho(\{p, r, s\}) = \{r, s\}$.
Hence the reverse inequality of (f) is not true.

Proposition 3.4.5. If ϱ is a $regular_\varrho$-operation on $\delta P_S O(Y)$, then for part of subsets P, Q of a space Y, $Int_\varrho(P) \cap Int_\varrho(Q) = Int_\varrho(P \cap Q)$.
Proof. We have $Int_\varrho(P \cap Q) \subseteq Int_\varrho(P) \cap Int_\varrho(Q)$ by Proposition 3.4.3(f).
On the other hand, let $p \in Int_\varrho(P) \cap Int_\varrho(Q)$ implies that, $p \in Int_\varrho(P)$ and $Int_\varrho(Q)$.Then there exists δP_S-open sets U and V containing p such that $U^\varrho \subseteq P$ and $V^\varrho \subseteq Q$, so $U^\varrho \cap V^\varrho \subseteq P \cap Q$.Since ϱ is a$regular_\varrho$-operation, there exists a δP_S-open set W containing p such that $W^\varrho \subseteq U^\varrho \cap V^\varrho \Rightarrow W^\varrho \subseteq P \cap Q$, hence $p \in Int_\varrho(P \cap Q)$.
Therefore, $Int_\varrho(P) \cap Int_\varrho(Q) = Int_\varrho(P \cap Q)$.

Proposition 3.4.6. Let P be a subset of a topological space (Y, τ) and ϱ be an operation on $\delta P_S O(Y)$, then the following conditions are hold:
a. $Int_\varrho(Y - P) = Y - Cl_\varrho(P)$.
b. $Cl_\varrho(Y - P) = Y - Int_\varrho(P)$.
c. $Y - \left(Cl_\varrho(Y - P)\right) = Int_\varrho(P)$.

d. $Y-\left(\mathrm{Int}_{\varrho}(Y-P)\right)=\mathrm{Cl}_{\varrho}(P)$.

Proof. (a) Let $p\in Int_{\varrho}(Y-P)$, then $\exists$ a δP_S-open set H containing p such that $H^{\varrho}\subseteq Y-P$. Gives, $H^{\varrho}\cap P=\emptyset$, so $p\notin Cl_{\varrho}(P)$ by Definition 3.1.1.
Finally, $p\in Cl_{\varrho}(P)$, so $Int_{\varrho}(Y-P)\subseteq Y-Cl_{\varrho}(P)$.
Reversing the process we get, $Int_{\varrho}(Y-P)\supseteq Y-Cl_{\varrho}(P)$
So (a) holds.
(b) Let $p\notin Cl_{\varrho}(Y-P)$, then $\exists$ a δP_S-open set H containing p, we get $H^{\varrho}\cap(Y-P)=\emptyset$ gives, $H^{\varrho}\subseteq P$ so, $p\in Int_{\varrho}(P)$, finally $p\notin Y-Int_{\varrho}(P)$. So $Cl_{\varrho}(Y-P)\supseteq Y-Int_{\varrho}(P)$.
Reversing the process we get, $Cl_{\varrho}(Y-P)\subseteq Y-Int_{\varrho}(P)$
So (b) holds.
(c) Follows from (a) by replacing P by $Y-P$.
(d) Follows from (b) by replacing P by $Y-P$.

Example 3.4.7. Consider the space $Y=\{p,q,r,s\}$ and $\tau=\{\emptyset,Y,\{p\},\{r\},\{p,q\},\{p,r\},\{p,q,r\},\{p,r,s\}\}$
$\delta P_SO(Y)=\{\emptyset,Y,\{q\},\{r\},\{p,q\},\{q,r\},\{p,q,r\},\{q,r,s\}\}$.
An operation ϱ: $\delta P_SO(Y)\to\mathcal{P}(Y)$ is defined as follows, for every $H\in\delta P_SO(Y)$

$$H^{\varrho}=\begin{cases}H, & if\ H\neq\{q,r\}\\ Y, & if\ H=\{q\} and\ H=\{q,r\}\end{cases}$$

Then ϱ is not a $regular_{\varrho}$-operation.
$\varrho\ O(Y)=\{\emptyset,Y,\{r\},\{p,q\},\{p,q,r\},\{q,r,s\}\}$.For a subset $P=\{p,q\}$and $Q=\{q,r\}$
$Int_{\varrho}(P)\cap Int_{\varrho}(Q)=Int_{\varrho}(\{p,q\})\cap Int_{\varrho}(\{q,r\})=\{p,q\}\cap\{q,r\}=\{q\}\neq Int_{\varrho}(\{q\})=\emptyset$.

3.5. ϱ -EXTERIOR

Definition 3.5.1. Let P be a subset of a topological space (Y,τ) and ϱ be an operation on $\delta PsO(Y)$.The
ϱ -exterior of P is defined as the ϱ-interior of $Y-P$.
That is $Ext_{\varrho}(P)=Int_{\varrho}(Y-P)$.

Proposition 3.5.2. Let P be a subset of a topological space(Y,τ) and ϱ be an operation on $\delta PsO(Y)$. Then the following results are true:

a. $Ext_{\varrho}(Y)=\emptyset$.
b. $Ext_{\varrho}(\emptyset)=Y$.
c. $Ext_{\varrho}(P)=Y-Cl_{\varrho}(P)$.
d. If $P\subseteq Q$ then $Ext_{\varrho}(P)\supseteq Ext_{\varrho}(Q)$.
e. $Ext_{\varrho}(P\cup Q)\subseteq Ext_{\varrho}(P)\cup Ext_{\varrho}(Q)$.
f. $Ext_{\varrho}(P\cap Q)\supseteq Ext_{\varrho}(P)\cap Ext_{\varrho}(Q)$.
g. $Ext_{\varrho}\left(Ext_{\varrho}(P)\right)=Int_{\varrho}\left(Cl_{\varrho}(P)\right)$.
h. $Ext_{\varrho}(P)=Ext_{\varrho}(Y-Ext_{\varrho}(P))$.
i. $Int_{\varrho}(P)\subseteq Ext_{\varrho}(Ext_{\varrho}(P))$.

Proof. The poof is obvious.
The following example shows that the reverse inequality of (e), (f) and (i) is not true in general.

Example 3.5.3.Consider the space $Y=\{p,q,r,s\}$ and $=\{\emptyset,Y,\{p,q\}\}$ $\delta P_SO(Y)=\{\emptyset,Y,\{r\},\{s\},\{r,s\}\}$. An operation $\varrho:\delta P_SO(Y)\to\mathcal{P}(Y)$ is defined as follows, for every $H\in\delta P_SO(Y)$

$$H^{\varrho}=\begin{cases}ClH, & if\ r\in H\\ H, & if\ r\notin H\end{cases}$$

Implies $\varrho\ O(Y)=\{\emptyset,Y,\{s\},\{r,s\}\}$.
For a subset $P=\{s\}$ and $Q=\{p,q,r\}$. Hence, $Ext_{\varrho}(\{s\}\cup\{p,q,r\})=Int_{\varrho}(Y-Y)=\emptyset\not\supseteq$
$Int_{\varrho}(Y-\{s\})\cup Int_{\varrho}(Y-\{p,q,r\})=\emptyset\cup\{s\}=\{s\}$.
Hence the reverse inequality of (e) is not true.
For a subset $P=\{s\}$ and $Q=\{p,q,r\}$. Hence, $Ext_{\varrho}(\{s\}\cap\{p,q,r\})=Int_{\varrho}(Y-\emptyset)=Y\not\subseteq$
$Int_{\varrho}(Y-\{s\})\cap Int_{\varrho}(Y-\{p,q,r\})=\emptyset\cap\{s\}=\emptyset$.
For a subset $P=\{s\}$.
Hence,$Ext_{\varrho}(Ext_{\varrho}(\{s\})=Ext_{\varrho}(Int_{\varrho}(Y-\{s\})=Ext_{\varrho}(\emptyset)=Y$.
Thus, $Int_{\varrho}(\{s\})=\{s\}\not\supseteq Y$.
Hence the reverse inequality of (i) is not true.

Corollary 3.5.4. $Ext_{\varrho}(P\cap Q)\supseteq Ext_{\varrho}(P)\cup Ext_{\varrho}(Q)$.
Proof. $Ext_{\varrho}(P\cap Q)=Y-(Int_{\varrho}(P\cap Q))=Int_{\varrho}\left((Y-P)\cup(Y-Q)\right)\supseteq Int_{\varrho}\left((Y-P)\right)\cup Int_{\varrho}\left((Y-Q)\right)=Ext_{\varrho}(P)\cup Ext_{\varrho}(Q)$.

Proposition 3.5.5. $Ext_{\varrho}(P\cup Q)\subseteq Ext_{\varrho}(P)\cap Ext_{\varrho}(Q)$, the equality holds if ϱ is a $regular_{\varrho}$-operation.
Proof. $Ext_{\varrho}(P\cup Q)\subseteq Int_{\varrho}\left(Y-(P\cup Q)\right)=Int_{\varrho}\left((Y-P)\cap(Y-Q)\right)\subseteq Int_{\varrho}(Y-P)\cap Int_{\varrho}(Y-Q)=Ext_{\varrho}(P)\cap Ext_{\varrho}(Q)$, by Proposition 3.4.5 as ϱ is $regular_{\varrho}$-operation the equality holds.

3.6. ϱ-BOUNDARY

Definition 3.6.1. Let P be a subset of a topological space(Y,τ) and ϱ be an operation on $O(Y)$, the
ϱ- boundary of P is defined as the family of points which do not belong to ϱ-interior or ϱ-exterior of P. It is denoted by $Bd_{\varrho}(P)$.

Remark 3.6.2. Let P be a subset of a topological space(Y,τ) and ϱ be an operation on $\delta P_SO(Y)$ then,
$Y-Ext_{\varrho}(P)=Int_{\varrho}(P)\cup Bd_{\varrho}(P)$.
Proof. Follows from Definition 3.6.1.

Proposition 3.6.3. Let ϱ be an operation on $\delta P_SO(Y)$. The following conditions are equivalent for a ϱ -closed subset:

a. $Y-Bd_{\varrho}(P)=Int_{\varrho}(P)\cup Ext_{\varrho}(P)$.
b. $Cl_{\varrho}(P)=Int_{\varrho}(P)\cup Bd_{\varrho}(P)$.
c. $Bd_{\varrho}(P)=Cl_{\varrho}(P)-Int_{\varrho}(P)=Cl_{\varrho}(P)\cap Cl_{\varrho}(Y-P)$.

Proof. $(a)\Rightarrow(b)$ Remark 3.6.2, $Y-Ext_{\varrho}(P)=Int_{\varrho}(P)\cup Bd_{\varrho}(P)$ gives $Y-(Int_{\varrho}(Y-P))=Int_{\varrho}(P)\cup Bd_{\varrho}(P)\Rightarrow Cl_{\varrho}(P)=Int_{\varrho}(P)\cup Bd_{\varrho}(P)$, by Proposition 3.4.6(d)
$(b)\Rightarrow(c)$ We known from the definition 3.6.1 that,
$Bd_{\varrho}(P)=\left(Y-Int_{\varrho}(P)\right)\cap\left(Y-Ext_{\varrho}(P)\right)$
$Bd_{\varrho}(P)=Y-\left((Int_{\varrho}(P)\cup Ext_{\varrho}(P)\right)$
$Y-Bd_{\varrho}(P)=Int_{\varrho}(P)\cup Ext_{\varrho}(P)=Int_{\varrho}(P)\cup Int_{\varrho}(Y-P)$
$=Y-(Cl_{\varrho}(Y-P))\cup Y-Cl_{\varrho}(P)$
[by Proposition 3.4.6 (c)]
Taking complement,
$Bd_{\varrho}(P)=Cl_{\varrho}(Y-P)\cap Cl_{\varrho}(P)$ -------------- (*)
$=\left(Y-Int_{\varrho}(P)\right)\cap Cl_{\varrho}(P)$ [by Proposition 3.4.6(b)]
$Bd_{\varrho}(P)=Cl_{\varrho}(P)-Int_{\varrho}(P)$.
$(c)\Rightarrow(a)$ We have, $Bd_{\varrho}(P)=\varrho\ Cl(P)-Int_{\varrho}(P)$
Consider
$Int_{\varrho}(P)\cup Int_{\varrho}(Y-P)$
$=\left(Y-\left(Y-Int_{\varrho}(P)\right)\right)$
$\cup\left(Y-\left(Y-Int_{\varrho}(Y-P)\right)\right)$
$=\left(Y-\left(Y-Int_{\varrho}(P)\right)\cap\left(Y-Int_{\varrho}(Y-P)\right)\right)$
$=Y-\left(Cl_{\varrho}(Y-P)\cap Cl_{\varrho}(P)\right)$[by Proposition 3.4.6 (b)] $=Y-Bd_{\varrho}(P)$ from (*).

Corollary 3.6.4. $Bd_{\varrho}(P)=Bd_{\varrho}(Y-P)$ for part of a subset P

of Y.
Proof. From (*) in Proposition 3.6.3, we get
$Bd_\varrho(P) = Cl_\varrho(Y-P) \cap Cl_\varrho(P)$.
Replacing P by $Y-P$, we get
$Bd_\varrho(Y-P) = Cl_\varrho(P) \cap Cl_\varrho(Y-P) = Bd_\varrho(P)$.

The following proposition characterises ϱ-open and ϱ-closed sets

Proposition 3.6.5. Let P be a subset of a topological space (Y,τ) and ϱ be an operation on $\delta P_S O(Y)$. Then, the following properties hold:

a. P is ϱ-open iff $P \cap Bd_\varrho(P) = \emptyset$.
b. P is ϱ-closed iff $Bd_\varrho(P) \subseteq P$.

Proof. (a) Let P be a ϱ-open set, then $Y-P$ is ϱ-closed (i.e) $Y-P = Cl_\varrho(Y-P)$ for,
$P \cap Bd_\varrho(P) = P \cap Cl_\varrho(X-P) \cap Cl_\varrho(P) = P \cap (Y-P) \cap Cl_\varrho(P) = \emptyset \cap Cl_\varrho(P) = \emptyset$.
On the other hand, Suppose $P \cap Bd_\varrho(P) = \emptyset$, $P \cap Cl_\varrho(Y-P) \cap Cl_\varrho(P) = \emptyset$ implies,
$P \cap Cl_\varrho(Y-P) = \emptyset \Rightarrow Cl_\varrho(Y-P) - (Y-P) = \emptyset$, thus $Y-P$ is ϱ-closed implies P is ϱ-open.
(b) Let P be a ϱ-closed set, thus $P = Cl_\varrho(P)$. Now, $Bd_\varrho(P) = Cl_\varrho(Y-P) \cap Cl_\varrho(P) \subseteq Cl_\varrho(P)$.
Hence, $Bd_\varrho(P) \subseteq P$.
On the other hand, Let $Bd_\varrho(P) \subseteq P$, then
$(Y-P) \cap Bd_\varrho(P) = \emptyset$. From Corollary 3.6.4, we have $Bd_\varrho(P) = Bd_\varrho(Y-P)$ thus $(Y-P) \cap Bd_\varrho(Y-P) = \emptyset$, by (a) $Y-P$ is ϱ-open. Hence P is ϱ-closed.

Proposition 3.6.6. Let P be a subset of a topological space (Y,τ) and ϱ be an operation on $\delta P_S O(Y)$. Then the following equality holds:

a. $Bd_\varrho(P) \cap Int_\varrho(P) = \emptyset$.
b. $Int_\varrho(P) = P - Bd_\varrho(P)$.

Proof. (a) $Bd_\varrho(P) \cap Int_\varrho(P) = Cl_\varrho(Y-P) \cap Cl_\varrho(P) \cap Int_\varrho(P)$ from Proposition 3.6.3(c)
$= \left(Y - Int_\varrho(P)\right) \cap Cl_\varrho(P) \cap Int_\varrho(P) = \emptyset$ from Proposition3.4.6(b)
(b) Consider $P - Bd_\varrho(P) = P - (Cl_\varrho(Y-P) \cap Cl_\varrho(P))$
$= P \cap \left(Y - Cl_\varrho(Y-P) \cap Cl_\varrho(P)\right)$
$= P \cap \left(Y - Cl_\varrho(Y-P)\right) \cup \left(Y - Cl_\varrho(P)\right)$
$= P \cap Int_\varrho(P) \cup \left(Y - Cl_\varrho(P)\right)$
$= \left(P \cap Int_\varrho(P)\right) \cup \left(P \cap \left(Y - Cl_\varrho(P)\right)\right)$
$= Int_\varrho(P) \cup \emptyset = Int_\varrho(P)$.

Proposition 3.6.7. Let ϱ be an operation on $\delta P_S O(Y)$Then for any two subsets P and Q of a topological space, if ϱ is ϱ-regular then the following equality holds:

a. $Bd_\varrho(P \cup Q) \subseteq \left[Bd_\varrho(P) \cap Cl_\varrho(Y-Q)\right] \cup \left[Bd_\varrho(Q) \cap Cl_\varrho(Y-P)\right]$
b. $Bd_\varrho(P \cap Q) \subseteq \left[Bd_\varrho(P) \cap Cl_\varrho(Q)\right] \cup \left[Bd_\varrho(Q) \cap Cl_\varrho(P)\right]$

Proof. (a) Consider, $Bd_\varrho(P \cup Q) = \left(Y - Cl_\varrho(P \cup Q)\right) \cap Cl_\varrho(P \cup Q) = Cl_\varrho\left((Y-P) \cap (Y-Q)\right) \cap Cl_\varrho(P \cup Q) \subseteq \left(Cl_\varrho(P) \cup Cl_\varrho(Q)\right) \cap \left(Cl_\varrho(Y-P) \cap Cl_\varrho(Y-Q)\right)$
$\subseteq \left(Cl_\varrho(P) \cap Cl_\varrho(Y-A) \cap Cl_\varrho(Y-Q)\right)$
$\cup \left(Cl_\varrho(Q) \cap Cl_\varrho(Y-P) \cap Cl_\varrho(Y-Q)\right)$
$\subseteq \left(Bd_\varrho(P) \cap Cl_\varrho(Y-Q)\right) \cup Bd_\varrho(Q) \cap Cl_\varrho(Y-P)$.
(b)Consider, $Bd_\varrho(P \cap Q) = \left(Y - Cl_\varrho\left((P \cap Q)\right)\right) \cap Cl_\varrho(P \cap Q)$
$= Cl_\varrho\left((Y-P) \cup (Y-Q)\right) \cap Cl_\varrho(P \cap Q)$
$\subseteq \left(Cl_\varrho(P) \cap Cl_\varrho(Q)\right) \cap \left(Cl_\varrho(Y-P) \cup Cl_\varrho(Y-Q)\right) \subseteq (Cl_\varrho(P) \cap Cl_\varrho(Y-P) \cap Cl_\varrho(Q)) \cup \left(Cl_\varrho(Q) \cap Cl_\varrho(P) \cap Cl_\varrho(Y-Q)\right)$
$\subseteq \left(Bd_\varrho(P) \cap Cl_\varrho(Q)\right) \cup \left(Bd_\varrho(Q) \cap Cl_\varrho(Y-P)\right)$.

3.7. ϱ-FRONTIER

Definition 3.7.1. Let P be a subset of a topological space (Y,τ) and ϱ be an operation on $\delta PsO(Y)$. The ϱ-frontier of P is defined as $Cl_\varrho(P) - Int_\varrho(P)$,
That is $Fr_\varrho(P) = Cl_\varrho(P) - Int_\varrho(P)$.

Proposition 3.7.2. For a subset P of a topological space (Y,τ) and ϱ be an operation on $\delta PsO(Y)$, the following is true:

a. $Cl_\varrho(P) = Int_\varrho(P) \cup Fr_\varrho(P)$.
b. $Int_\varrho(P) \cap Fr_\varrho(P) = \emptyset$.
c. $Fr_\varrho(P) = Cl_\varrho(P) \cap Cl_\varrho(Y-P)$.
d. $Fr_\varrho(P) = Fr_\varrho(Y-P)$.
e. $Fr_\varrho(P)$ is ϱ-cloed .
f. $\delta Ps\, Fr(Fr_\varrho(P)) \subseteq Fr_\varrho(P)$.

Proof.(a)$Int_\varrho(P) \cup Fr_\varrho(P) = Int_\varrho(P) \cup \left(Cl_\varrho(P) - Int_\varrho(P)\right)$
$= Int_\varrho(P) \cup \left(Cl_\varrho(P) \cap \left(Y - Int_\varrho(P)\right)\right)$
$= (Int_\varrho(P) \cup Cl_\varrho(P)) \cap (Int_\varrho(P) \cup \left(Y - Int_\varrho(P)\right) = Cl_\varrho(P)$.
(b)$Int_\varrho(P) \cap Fr_\varrho(P) = Int_\varrho(P) \cap (Cl_\varrho(P) - Int_\varrho(P)) = Int_\varrho(P) \cap \left(Cl_\varrho(P) \cap Int_\varrho(Y-P)\right) = \emptyset$.
(c) From theorem 3.6.3 (c) $Cl_\varrho(P) \cap Cl_\varrho(Y-P) = Cl_\varrho(P) - Int_\varrho(P) = Fr_\varrho(P)$, by definition 3.7.1.
(d)$Fr_\varrho(Y-P) = Cl_\varrho(Y-P) - Int_\varrho(Y-P) = \left[Y - Int_\varrho(P)\right] - \left[Y - Cl_\varrho(P)\right]$
$= [Y - Int_\varrho(P)] \cap Cl_\varrho(P) = Cl_\varrho(P) \cap [Y - Int_\varrho(P)]$
$= Cl_\varrho(P) - Int_\varrho(P) = Fr_\varrho(P)$.
(e) $Fr_\varrho(P) = Cl_\varrho(P) \cap Cl_\varrho(Y-P)$, which is an intersection of ϱ-closed sets and thus it is a ϱ-closed set.
(f) $\delta Ps\, Fr\left(Fr_\varrho(P)\right) = \delta Ps\, Cl\left(Fr_\varrho(P)\right) \cap \delta Ps\, Cl\left(Y - Fr_\varrho(P)\right) \subseteq \delta Ps\, Cl\left(Fr_\varrho(P)\right) \subseteq \varrho\, Cl\left(Fr_\varrho(P)\right) = Fr_\varrho(P)$ by (e), therefore $\delta Ps\, Fr(Fr_\varrho(P)) \subseteq Fr_\varrho(P)$.

Proposition 3.7.3.

a. $P \subseteq Q$ and $Int_\varrho(Q) = \emptyset$ *then* $Fr_\varrho(P) \subseteq Fr_\varrho(Q)$.
b. Let P be a subset of Y, then $Y = Int_\varrho(P) \cup Ext_\varrho(P) \cup Fr_\varrho(P)$.

Proof. (a) Let $P \subseteq Q$ and $Int_\varrho(P) = \emptyset$.
Let $p \in Fr_\varrho(P) = Cl_\varrho(P) - Int_\varrho(P)$ gives, $p \in Cl_\varrho(P) \subseteq Cl_\varrho(Q) = Cl_\varrho(Q) - Int_\varrho(Q) = Fr_\varrho(Q)$ as $Int_\varrho(Q) = \emptyset$, finally $p \in Fr_\varrho(Q)$. So, $Fr_\varrho(P) \subseteq Fr_\varrho(Q)$.
(b) $Int_\varrho(P) \cup Ext_\varrho(P) \cup Fr_\varrho(P) =$
$Int_\varrho(P) \cup Int_\varrho(Y-P) \cup (Cl_\varrho(P) - Int_\varrho(P)) = Int_\varrho(P) \cup (Y - Cl_\varrho(P)) \cup Cl_\varrho(P) = Y$, by Proposition3.7.3 (a).
Hence, $Y = Int_\varrho(P) \cup Ext_\varrho(P) \cup Fr_\varrho(P)$.

3.8. ϱ-SATURATED

Definition 3.8.1. Let P be a subset of a topological space (Y,τ) and ϱ be an operation on $\delta P_S O(Y)$, P is called ϱ-saturated if $\varrho Cl(\{p\}) \subseteq P$, for every point p in P.
The family of ϱ-saturated sets in (Y,τ) is defined as $\varrho\, Sat(Y)$.

Remark 3.8.2. In (Y, τ) if P is saturated set $Y - P$ is also a saturated set. But if $A \in \varrho\ Sat(Y)$ then $(Y - P)$ need not to be a $\varrho\ Sat(Y)$.

Proposition 3.8.3. Every ϱ-closed set is a ϱ-saturated set.
Proof. Let P be a ϱ-closed set and let $x \in P$,then $\{p\} \subseteq P \Rightarrow \varrho\ Cl(\{p\})\} \subseteq \varrho\ Cl(P) = P$, as P is a ϱ- closed set $\Rightarrow \varrho\ Cl(\{p\})\} \subseteq P \Rightarrow$ P is a ϱ-saturated set.
The converse is not true. That is if P is ϱ-saturated set then P is need not to be ϱ-closed set.

Example 3.8.4. Consider the space $Y = \{p, q, r, s\}$,
$\tau = \{ \emptyset, Y, \{p\}\}$ and
$\delta P_S O(Y) = \{ \emptyset, Y, \{q\}, \{r\}, \{s\}, \{q, r\}, \{q, s\}, \{r, s\}, \{q, r, s\}\}$. An operation $\varrho: \delta P_S O(Y) \rightarrow \mathcal{P}(Y)$ is defined as follows, for every $H \in \delta P_S O(Y)$

$$H^{\varrho} = \begin{cases} Cl\ H, & \text{if } q \in H \\ H, & \text{if } q \notin H \end{cases}$$

Implies, $\varrho\ O(Y) = \{ \emptyset, Y, \{r\}, \{s\}, \{q, r, s\}\}$ and
$\varrho\ C(Y) = \{ \emptyset, Y, \{p\}, \{p, q, r\}, \{p, q, s\}\}$.
$\varrho\ Sat(Y) = \{ \emptyset, Y, \{p\}, \{p, q\}, \{p, q, r\}, \{p, q, s\}\}$.
Here $\{p, q\} \in \varrho\ Sat(Y)$ but $\{p, q\} \notin \varrho\ C(Y)$.

In conclusion, In this paper, some new classes of sets via ϱ-open sets on an operation ϱ on δP_S-open sets in topological spaces such as closure, $Closure_{\varrho}$, limit points, derived sets, neighbourhood, interior, exterior, boundary and frontier have been defined. Moreover, some important properties based on these sets have been investigated. In addition, a new class of set via an operation ϱ on δP_S-open sets called saturated set have been presented. Finally, some relationships among these classes of sets have been obtained and some examples have been given.

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