ON THE ENERGY OF SOME COMPOSITE GRAPHS

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(Accepted for publication: May 20, 2015)

ABSTRACT:

Eigenvalues of a graph are the eigenvalues of its adjacency matrix. The energy of a graph is the sum of the absolute values of its eigenvalues, was studied by (Gutman 1978). This paper divided in to three parts, in part one spectra and nullity of graphs are defined (Brouwer and Haemers, 2012) and (Harary, 1969). In the second part graph products an their spectra is studied (Shibata and Kikuchi 2000) and (Balakrishnan and Ranganathan , 2012). In the last part, we proves the energy of some graph products including Cartesian, tensor, strong, skew and inverse skew which are applied of some graphs.

Keywords: Graph product, Spectra, Energy.

1. INTRODUCTION

Let G be a graph of p vertices with adjacency matrix A, then A is a real symmetric matrix and so the eigenvalues of A are real and hence can be ordered. The eigenvalues of A, $\lambda_1 \geq \lambda_2 \geq \cdots \geq \lambda_p$ are called the eigenvalues of G and form the spectrum of G. The energy E(G) of a graph G is defined as the sum of absolute values of its eigenvalues. That is $E(G)=\sum_{i=1}^{p}|\lambda_i|$.The study of properties of E(G) was initiated by (Gutman, 1978). All graphs considered in this paper are finite, simple and undirected.

In this part, we look at the properties of graphs from their eigenvalues. The set of eigenvalues of a graph G with its multiplicities is known as the spectrum of G and denoted by $S_p(G)$.

Definition 1.1: The **adjacency matrix** A(G) or $A=[a_{ij}]$ of a labeled graph G with vertex set V(G), $V(G)=\{v_1, v_2, \ldots, v_p\}$ is a p×p matrix in which $a_{ij}=1$ if v_i and v_j are adjacent, and 0 if they are not.

Adjacency matrices define graphs up to isomorphism. Moreover, the adjacency matrix of a graph G is a symmetric 0, 1 matrix having zero entries along the main diagonal, and in which the sum of the entries in any row or column is equal to the degree of the corresponding vertex. Because of this correspondence between graphs and matrices, any graph theoretic concept is reflected in the adjacency matrix.

Definition 1.2: The **characteristic polynomial** of the adjacency matrix A(G) of a graph G with p vertices is called the characteristic polynomial of G, denoted by φ(G; x) with the convention that the coefficient of the highest order term is positive:

$\varphi(G;x)=Det(xI_p-A(G))=(-1)^p Det(A(G)-xI_p)$.

Therefore, the characteristic polynomial of a graph G of order p is a polynomial of degree p:

$\varphi(G; x) =a_0x^p+a_1x^{p-1}+\ldots+a_{p-1}x+a_p$.

It has two practical forms, explicitly as a polynomial in the variable x, or as product of linear factors. Thus,

$$\varphi(G; x)=\sum_{i=0}^{p} a_i x^{p-i}=\prod_{i=1}^{p}(x-\lambda_i).$$

Definition 1.3: The **eigenvalues** of a graph G of order p are defined to be the eigenvalues of the adjacency matrix associated with the graph G. That is, if G has adjacency matrix A(G), then the eigenvalues of G are those p (not necessarily distinct) numbers λ which satisfy the determinant equation $Det(A(G)-\lambda I_p)=0$, viz each λ is a root of the polynomial equation $|A(G)-\lambda I_p| = 0$.

Equivalently, a number λ is an eigenvalue of G if there exists a non-zero p×1 vector X (called an eigenvector of λ) such that A(G)X = λX.

Since A(G) is a real symmetric (0, 1) matrix, its eigenvalues $\{\lambda_1, \lambda_2, \ldots, \lambda_p\}$ must be real by the next theorem and can be ordered as $\lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_p$. See (Balakrishnan, 2004)

Theorem 1.4: The eigenvalues of a real symmetric matrix are real. ■

Theorem 1.5: The sum of the eigenvalues of any simple graph is zero. ■

Definition 1.6: Let G be a graph of order p and H is a subgraph of G of order m, $m \le p$, then H is called a **partial cover** of G if every component of H is isomorphic with K_2 or a cycle graph. If m = p, then H is called a **spanning cover** of G.

Theorem 1.7: (Sachs' Theorem) The coefficients a_i's of φ(G; x) are given by

$a_i=\sum_{H}(-1)^{k(H)}\, 2^{c(H)}$,

Where the summation extends over all partial covers H on i vertices of G, and where k(H) and c(H) denote respectively, the number of components and of cycles in H.

Definition 1.8: The **spectrum** $S_p(G)$ of a graph G is defined as the eigenvalues of its adjacency matrix, that is, another matrix of two rows, the first row consists of the eigenvalues of the graph G and the second row consists of the multiplicities of the corresponding eigenvalues. That is if the distinct eigenvalues of G are $\lambda_1, \lambda_2, \ldots, \lambda_k$ and their multiplicities are $m_1, m_2, \ldots, m_k$, respectively, then we write

$$S_p(G)=\begin{pmatrix} \lambda_1 & \lambda_2 & \ldots & \lambda_k \\ m_1 & m_2 & \ldots & m_k \end{pmatrix}$$

Or just as $\lambda_1^{m_1}, \lambda_2^{m_2}, \ldots, \lambda_k^{m_k}$.

If G is a disconnected graph with components $G_1, G_2, \ldots, G_k$, then the spectrum of G is the "union" of the eigenvalues of the components of G in some manner because of the fact that

$\varphi(G; x)=\prod_{i=1}^{k} \varphi(G_i; x)$.

Spectra of graphs can be obtained using the fact that the coefficients of the characteristic polynomial are integers. It follows that the sum of k-th powers of eigenvalues are integers too. Since the coefficient of the highest power term x^p of the characteristic polynomial φ(G; x) is equal to 1, hence any eigenvalue of G which is rational must be an integer, and for any square matrix with real entries, the sum of its eigenvalues is equal to its trace.

To specify the spectrum of a graph G with order p, the coefficients a_i's of the characteristic polynomial $\varphi(G; x)=\sum_{i=0}^{p} a_i\, x^{p-i}$ are of important use. Thus, we seek the coefficients a_i's first. Certainly $a_0 =1$, and using Theorem 1.7, we can easily verify that $a_1=0$, $-a_2 = q$ and $-a_3$ is twice the number of triangles in G. And to find the remaining coefficients apply Theorem 1.7.

Definition 1.9: A graph G is said to be a **singular** graph provided that its adjacency matrix A(G) is a singular matrix. The algebraic multiplicity of the number zero in the spectrum of the graph G is called its **nullity** (**degree of singularity**), and is denoted by η(G).

Lemma 1.10:

i) The eigenvalues of the cycle graph C_p are of the form $2\cos\frac{2\pi i}{p}$, $i = 0, \ldots, p-1$, and

$$\eta(C_p)=\begin{cases} 2, & \text{if } p \equiv 0 \pmod 4, \\ 0, & \text{otherwise.} \end{cases}$$

ii) The eigenvalues of the path graph P_p are of the form $2\cos\frac{\pi i}{p+1}$, $i = 1, \ldots, p$, and

$$\eta(P_p)=\begin{cases} 1, & \text{if p is odd,} \\ 0, & \text{if p is even.} \end{cases}$$

iii) The spectrum of the complete graph K_p, $S_p(K_p)=\begin{pmatrix} p-1 & -1 \\ 1 & p-1 \end{pmatrix}$,and $\eta(K_p)=\begin{cases} 1, \text{if } p = 1, \\ 0, \text{if } p > 1. \end{cases}$

iv) The spectrum of the complete bipartite graph $K_{p1,p2}$, $S_p(K_{p1,p2}) = \begin{pmatrix} \sqrt{p_1p_2} & 0 & -\sqrt{p_1p_2} \\ 1 & p_1+p_2-2 & 1 \end{pmatrix}$, and $\eta(K_{p1,p2})=p_1+p_2-2$, for all p_1, p_2.

2. Graph Products

In this part, we study some graph products and determine the spectra of some of them.

Let $G_1 =(V_1, E_1)$ and $G_2=(V_2, E_2)$ be vertex disjoint non-trivial graphs.

Definition 2.1: The **Cartesian product** $G_1\times G_2$ of the two graphs G_1 and G_2 is the graph with vertex set $V(G_1\times G_2)=V_1\times V_2$ and two vertices (u_1, v_1) and (u_2, v_2) are adjacent in $G_1\times G_2$ if, and only if, $[u_1 \equiv u_2$ and $v_1v_2\in E(G_2)]$ or $[u_1u_2\in E(G_1)$ and $v_1 \equiv v_2]$.

It is clear that: $p(G_1\times G_2)=p(G_1)p(G_2)$ and $q(G_1\times G_2)=p(G_1)q(G_2)+q(G_1)\, p(G_2)$

Definition 2.2: The **tensor product** $G_1\otimes G_2$ of the two graphs G_1 and G_2 is the graph with vertex set $V(G_1\otimes G_2)=V_1\times V_2$ and two vertices (u_1, v_1) and (u_2, v_2) are adjacent in $G_1\otimes G_2$ if, and only if, $[u_1 u_2 \in E(G_1)$ and $v_1v_2\in E(G_2)]$.

It is clear that: $p(G_1\otimes G_2)=p(G_1)p(G_2)$ and $q(G_1\otimes G_2)=2q(G_1)\, q(G_2)$.

Definition 2.3: The **strong product** $G_1\boxtimes G_2$ of the two graphs G_1 and G_2 is the graph with vertex set $V(G_1\boxtimes G_2)=V_1\times V_2$ and two vertices (u_1, v_1) and (u_2, v_2) are adjacent in $G_1\boxtimes G_2$ if, and only if, $[u_1 \equiv u_2$ and $v_1v_2\in E(G_2)]$ or $[u_1u_2\in E(G_1)$ and $v_1 \equiv v_2]$ or $[u_1 u_2 \in E(G_1)$ and $v_1v_2\in E(G_2)]$.

It is clear that: $p(G_1\boxtimes G_2)=p(G_1)p(G_2)$ and $q(G_1\boxtimes G_2)=p(G_1)q(G_2)+q(G_1)\, p(G_2)+2q(G_1)\, q(G_2)$.

Definition 2.4: The **skew product** $G_1\blacklozenge G_2$ of the two graphs G_1 and G_2 is the graph with vertex set $V(G_1\blacklozenge G_2) = V_1\times V_2$ and where (u_1, v_1) and (u_2, v_2) are adjacent in $G_1\blacklozenge G_2$ if, and only if, $[u_1 \equiv u_2$ and $v_1v_2\in E(G_2)]$ or $[u_1u_2\in E(G_1)$ and $v_1v_2\in E(G_2)]$.

It is clear that: $p(G_1\blacklozenge G_2) = p(G_1)p(G_2)$ and $q(G_1\blacklozenge G_2) = p(G_1)q(G_2)+2q(G_1)q(G_2)$.

Definition 2.5: Let $G_1 = (V_1, E_1)$ and $G_2 = (V_1, E_1)$ be vertex disjoint non-trivial graphs. The **inverse**

skew product $G_1\lozenge G_2$ of the two graphs G_1 and G_2 is the graph with vertex set $V(G_1\lozenge G_2) = V_1\times V_2$ and where (u_1, v_1) and (u_2, v_2) are adjacent in $G_1\lozenge G_2$ if, and only if, $[u_1u_2 \in E(G_1)$and $v_1 \equiv v_2]$ or $[u_1u_2\in E(G_1)$ and $v_1v_2\in E(G_2)]$.

It is clear that: $p(G_1\lozenge G_2) = p(G_1)p(G_2)$ and $q(G) = q(G_1)p(G_2)+2q(G_1)q(G_2)$.

Lemma 2.6: (Shibata and Kikuchi, 2000) Let G_1 and G_2 be two graphs with orders p_1 and p_2, respectively. Then

i) $A(G_1\blacklozenge G2) = (I_{p1} *A_2)+(A_1*A_2)$.

ii) $A(G_1\lozenge G_2) = (A_1*I_{p2})+(A_1*A_2)$. ■

Corollary 2.7: Let $S_p(G_1)=\{ \lambda_1, ..., \lambda_{p1}\}$ and $S_p(G_2)=\{\mu_1,..., \mu_{p2}\}$, and let A_1 and A_2 be the adjacency matrices of G_1 and G_2; respectively. Then

(i) $A(G_1\times G_2)=(I_{p1} *A_2)+(A_1*I_{p2})$ and $S_p(G_1\times G_2)= \{\lambda_i+\mu_j: 1\le i \le p_1;\ 1\le j \le p_2\}$.

(ii) $A(G_1\otimes G_2)=(A_1*A_2)$ and $S_p(G_1\otimes G_2)= \{\lambda_i\mu_j : 1\le i \le p_1;\ 1\le j \le p_2\}$.

(iii) $A(G_1 \boxtimes G_2) =(A_1*A_2)+(I_{p1}*A_2)+(A_1*I_{p2})$ and $S_p(G_1 \boxtimes G_2)= \{ \lambda_i\mu_j + \lambda_i+\mu_j : 1\le i \le p_1;\ 1\le j \le p_2\}$.■

3. Energy of a Product Graphs

In this part, we discuss another application of eigenvalues of graphs. The energy E(G) of a graph G is defined as follows:

Definition 3.1: Let G be a graph on p vertices, and let its ordinary spectrum (i. e., the spectrum of its adjacency matrix) consist of the numbers $\lambda_1, \lambda_2, ..., \lambda_p$. Then, The energy E(G) of a graph G is defined as follows:

$E(G) = \sum_{i=1}^{p} |\lambda_i|$.

In the next proposition, the energy of some special graphs is studied by (Balakrishnan, 2004) and (Li, Shi, and Gutman, 2012).

Proposition 3.2: The energy of some special graphs is defined as follows:

1) $E(K_p)= 2p-2$.

2) $E(K_{p1,p2})=2\sqrt{p_1p_2}$.

3) $E(P_p)=\begin{cases} \frac{2}{\sin(\frac{\pi}{2(p+1)})}-2\text{, if } p=0(\text{mod } 2),\\ \frac{2\cos(\frac{\pi}{2(p+1)})}{\sin(\frac{\pi}{2(p+1)})}-2\text{, if } p=1(\text{mod } 2).\end{cases}$

4) $E(C_p)=\begin{cases} \frac{4\cos(\frac{\pi}{p})}{\sin(\frac{\pi}{p})}\text{, if } p=0(\text{mod } 4),\\ \frac{4}{\sin(\frac{\pi}{p})}\text{, if } p=2(\text{ mod } 4),\\ \frac{2}{\sin(\frac{\pi}{2p})}\text{, if } p=1(\text{ mod } 2).\end{cases}$

In the following, we determine the spectra of skew and inverse skew products.

Let A_1 and A_2 be the $p_1\times p_1$ and $p_2\times p_2$ adjacency matrices of G_1 and G_2 have eigenvalues $\lambda_i, 1\leq i\leq p_1$ and μ_j, $1\leq j\leq p_2$, respectively.

Theorem 3.3: The p_1p_2 eigenvalues of the skew product $G_1\blacklozenge G_2$ are $\mu_j+\lambda_i\mu_j$, $\forall$ i, j; $1\leq i\geq p_1;1\leq j\geq p_2$.

Moreover, if $X_1, ..., X_{p1}$ are the eigenvectors of A_1 corresponding to $\lambda_1, ..., \lambda_{p1}$, and $Y_1, ...,Y_{p2}$ are the eigenvectors of A_2 corresponding to$\mu_1,...,\mu_{p2}$, then X_i*Y_j are the eigenvectors of $G_1\blacklozenge G_2$ corresponding to $\mu_j+\lambda_i\mu_j$, $1\leq i\leq p_1$; $1\leq j\leq p_2$.

Proof: By Lemma 2.6, we have:

$A(G_1\blacklozenge G_2)=[(I_{p1}*A_2)+(A_1*A_2)]$

Assume that X(Y) is an eigenvector of $G_1(G_2)$ corresponding to the eigenvalue $\lambda(\mu)$. Then

$$\begin{aligned}A(G_1\blacklozenge G_2)(X*Y)&=[(I_{p1}*A_2)+(A_1*A_2)](X*Y)\\ &=(X*A_2Y)+(A_1X*A_2Y)\\ &=(X*\mu Y)+(\lambda X*\mu Y)\\ &=(\mu+\lambda\mu)(X*Y).\ \blacksquare\end{aligned}$$

Theorem 3.4: The p_1p_2 eigenvalues of the inverse skew product $G_1\lozenge G_2$ are $\lambda_i+\lambda_i\mu_j$, for all $1\leq i\leq p_1$; $1\leq j\leq p_2$.

Moreover, if $X_1, ..., X_{p1}$ are the eigenvectors of A_1 corresponding to $\lambda_1, ..., \lambda_{p1}$, and $Y_1, ...,Y_{p2}$ are the eigenvectors of A_2 corresponding to $\mu_1, ..., \mu_{p2}$, then X_i*Y_j are the eigenvectors of $G_1\lozenge G_2$ corresponding to $\lambda_i+\lambda_i\mu_j$, $1\leq i\leq p_1$; $1\leq j\leq p_2$.

Proof: By Lemma 2.6, we have:

$A(G_1\lozenge G_2)=[(A_1*I_{p2})+(A_1*A_2)]$

Also, assume that X(Y) is an eigen vector of $G_1(G_2)$ corresponding to the eigen value $\lambda(\mu)$. Then

$$\begin{aligned}A(G_1\lozenge G_2)(X*Y)&=[(A_1*I_{p2})+(A_1*A_2)](X*Y)\\ &=(A_1X*Y)+(A_1X*A_2Y)\\ &=(\lambda X*Y)+(\lambda X*\mu Y)\\ &=(\lambda+\lambda\mu)(X*Y).\ \blacksquare\end{aligned}$$

Thus, as a result for the spectra of the above two products we have:

Corollary 3.5: Let $S_p(G_1)=\{\lambda_1, ..., \lambda_{p1}\}$ and $S_p(G_2)=\{\mu_1, ..., \mu_{p2}\}$. Then

(i) $S_p(G_1\blacklozenge G_2)=\{\mu_j+\lambda_i\mu_j : 1\leq i\leq p_1; 1\leq j\leq p_2\}$.

(ii) $S_p(G_1\lozenge G_2)=\{\lambda_i+\lambda_i\mu_j: 1\leq i\leq p_1; 1\leq j\leq p_2\}$.■

Proposition 3.6: The energy of the Cartesian product of Kp_1 and Kp_2 where $p_1, p_2>1$, is given by $E(Kp_1\times Kp_2)=4(p_1p_2-p_1-p_2+1)$.

Proof: By Corollary 2.7, we have:

$S_p(Kp_1 \times Kp_2)$

$$= \begin{pmatrix} p_1-1 & -1 \\ 1 & p_1-1 \end{pmatrix} + \begin{pmatrix} p_2-1 & -1 \\ 1 & p_2-1 \end{pmatrix} = \begin{pmatrix} p_1+p_2-2 & p_1-2 & p_2-2 & -2 \\ 1 & p_2-1 & p_1-1 & p_1p_2-p_1-p_2+1 \end{pmatrix}$$

Thus, by Definition 3.1, we get:

$E(Kp_1 \times Kp_2) = p_1+p_2-2+p_1p_2-p_1-2p_2+2+p_1p_2-p_2-2p_1+2+2p_1p_2-2p_1-2p_2+2$
$= 4(p_1p_2-p_1-p_2+1)$.

Therefore, $Kp_1 \times Kp_2$ is singular, if and only if $p_1=p_2=1$, this gives the case $K_1 \times K_1 = K_1$. ■

Proposition 3.7: The energy of the Cartesian product of K_p and $K_{p1,p2}$ is

$E(K_p \times K_{p1,p2}) = 2(p-1)(p_1+p_2+\sqrt{p_1p_2}-1)$.

Proof: By Lemma 1.10 and Corollary 2.7 it follows that the spectrum of $K_p \times K_{p1,p2}$ is

$$\begin{pmatrix} p-1 & -1 \\ 1 & p-1 \end{pmatrix} + \begin{pmatrix} \sqrt{p_1p_2} & 0 & -\sqrt{p_1p_2} \\ 1 & p_1+p_2-2 & 1 \end{pmatrix} =$$

$$\begin{pmatrix} p+\sqrt{p_1p_2}-1 & p-1 & p-\sqrt{p_1p_2}-1 & \sqrt{p_1p_2}-1 & -1 & -(\sqrt{p_1p_2}+1) \\ 1 & p_1+p_2-2 & 1 & p-1 & (p-1)(p_1+p_2-2) & p-1 \end{pmatrix}$$

Thus, by Definition 3.1, we have:

$E(K_p \times K_{p1,p2}) = (p+\sqrt{p_1p_2}-1) + (pp_1+pp_2-2p-p_1-p_2+2) + (p-\sqrt{p_1p_2}-1) + (\sqrt{p_1p_2}\,p-\sqrt{p_1p_2}-p+1) + (pp_1+pp_2-2p-p_1-p_2+2) + (\sqrt{p_1p_2}\,p-\sqrt{p_1p_2}+p-1) = 2pp_1+2pp_2+2\sqrt{p_1p_2}\,p-2p-2p_1-2p_2+2-2\sqrt{p_1p_2}$

$= 2p(p_1+p_2+\sqrt{p_1p_2}-1) - 2(p_1+p_2+\sqrt{p_1p_2}-1) = 2(p-1)(p_1+p_2+\sqrt{p_1p_2}-1)$. ■

Moreover, if $p \neq \sqrt{p_1p_2}+1 \leq 0$, so

i) If $p_1=p_2=1$, then $K_p \times K_{1,1}$ is (p -1) singular.

ii) If either p_1 or p_2 is not 1 while $p=\sqrt{p_1p_2}+1$, then $K_p \times K_{p1,p2}$ is 1- singular.

Lemma 3.8: The energy of the Cartesian product of Kp_1 and $K_{p2,p2}$ is

$E(Kp_1 \times K_{p2,p2}) = (p_1-1)(6p_2-2)$.

Proof: Put $p_1=p_2$ in Proposition 3.7, we get the result. ■

Proposition 3.9: The energy of the Cartesian product of $K_{p1,p2}$ and $K_{p1,p2}$ is

$E(K_{p1,p2} \times K_{p1,p2}) = 4\sqrt{p_1p_2}(p_1+p_2-1)$.

Proof: By Corollary 2.7, we have

$$\begin{pmatrix} \sqrt{p_1p_2} & 0 & -\sqrt{p_1p_2} \\ 1 & p_1+p_2-2 & 1 \end{pmatrix} + \begin{pmatrix} \sqrt{p_1p_2} & 0 & -\sqrt{p_1p_2} \\ 1 & p_1+p_2-2 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 2\sqrt{p_1p_2} & \sqrt{p_1p_2} & 0 & \sqrt{p_1p_2} & 0 & -\sqrt{p_1p_2} & 0 & -\sqrt{p_1p_2} & -2\sqrt{p_1p_2} \\ 1 & p_1+p_2-2 & 1 & p_1+p_2-2 & 2p_1+2p_2-4 & p_1+p_2-2 & 1 & p_1+p_2-2 & 1 \end{pmatrix}$$

$$= \begin{pmatrix} 2\sqrt{p_1p_2} & \sqrt{p_1p_2} & 0 & -\sqrt{p_1p_2} & -2\sqrt{p_1p_2} \\ 1 & 2p_1+2p_2-4 & 2p_1+2p_2-2 & 2p_1+2p_2-4 & 1 \end{pmatrix}$$

Thus, by Definition 3.1, we have:

$= 2\sqrt{p_1p_2} + 2p_1\sqrt{p_1p_2} + 2p_2\sqrt{p_1p_2} - 4\sqrt{p_1p_2} + 2p_1\sqrt{p_1p_2} + 2p_2\sqrt{p_1p_2} - 4\sqrt{p_1p_2} + 2\sqrt{p_1p_2}$

$E(K_{p1,p2} \times K_{p1,p2}) = 4p_1\sqrt{p_1p_2} + 4p_2\sqrt{p_1p_2} - 4\sqrt{p_1p_2} = 4\sqrt{p_1p_2}\,(p_1+p_2-1)$. ■

Moreover, the nullity of $K_{p1,p2} \times K_{p1,p2}$ is $2p_1+2p_2-2$.

Lemma 3.10: The energy of the Cartesian product of $K_{p1,p1}$ and $K_{p2,p2}$ is
$E(K_{p1,p1}\times K_{p2,p2}) = 2(4p_1p_2 - p_1 - p_2)$.

Proof: By Corollary 2.7, we have:

$$\begin{pmatrix} p_1 & 0 & -p_1 \\ 1 & 2p_1-2 & 1 \end{pmatrix} + \begin{pmatrix} p_2 & 0 & -p_2 \\ 1 & 2p_2-2 & 1 \end{pmatrix} =$$

$$\begin{pmatrix} p_1+p_2 & p_1 & p_1-p_2 & p_2 & 0 & -p_2 & p_2-p_1 & -p_1 & -(p_1+p_2) \\ 1 & 2p_2-2 & 1 & 2p_1-2 & (2p_1-2)(2p_2-2) & 2p_1-2 & 1 & 2p_2-2 & 1 \end{pmatrix}$$

Thus, by Definition 3.1, we get:
$E(K_{p1,p1}\times K_{p2,p2}) = 8p_1p_2 - 2p_1 - 2p_2 = 2(4p_1p_2 - p_1 - p_2)$. ■

Moreover, the nullity of $K_{p1,p1}\times K_{p2,p2}$ is $(2p_1\text{-}2)(2p_2\text{-}2)$, provided that neither p_1 nor p_2 is 1. If $p_1=p_2=1$, then the nullity of $K_{1,1}\times K_{1,1}$ is 2.

Proposition 3.11: Let G_1 and G_2 be two graphs on p_1 and p_2 vertices, respectively. Then $E(G_1\times G_2) \le p_2 E(G_1)+p_1E(G_2)$.

Proof: Let $\lambda_1, \lambda_2, \ldots, \lambda_{p1}$ and $\mu_1, \mu_2, \ldots, \mu_{p2}$ be the eigenvalues of G_1 and G_2; respectively. Then

$$E(G_1\times G_2)= \sum_{i=1}^{p_1}\sum_{j=1}^{p_2}[|\lambda_i+\mu_j|] \le \sum_{i=1}^{p_1}\sum_{j=1}^{p_2}|\lambda_i|+|\mu_j| \le \sum_{i=1}^{p_1}\sum_{j=1}^{p_2}|\lambda_i| + \sum_{i=1}^{p_1}\sum_{j=1}^{p_2}|\mu_j|$$
$$\le p_2 E(G_1)+p_1E(G_2). \ ■$$

Proposition 3.12: The energy of the tensor product $Kp_1\otimes Kp_2$ is given by:
$E(Kp_1\otimes Kp_2) = 4(p_1\text{-}1)(p_2\text{-}1)$.

Proof: By Lemma 1.10 and Corollary 2.7, we have:

$$\begin{pmatrix} p_1-1 & -1 \\ 1 & p_1-1 \end{pmatrix}\begin{pmatrix} p_2-1 & -1 \\ 1 & p_2-1 \end{pmatrix}$$
$$= \begin{pmatrix} (p_1-1)(p_2-1) & -p_1+1 & -p_2+1 & 1 \\ 1 & p_2-1 & p_1-1 & (p_1-1)(p_2-1) \end{pmatrix}$$

Thus, by Definition 3.1, we get:
$E(Kp_1\times Kp_2) = p_1p_2\text{-}p_1\text{-}p_2\text{+}1\text{+}p_1p_2\text{-}p_2\text{-}p_1\text{+}1\text{+}p_1p_2\text{-}p_2\text{-}p_1\text{+}1\text{+}p_1p_2\text{-}p_1\text{-}p_2\text{+}1$
$= 4(p_1p_2\text{-}p_1\text{-}p_2\text{+}1) = 4(p_1\text{-}1)(p_2\text{-}1)$. ■

Proposition 3.13: The energy of the tensor product $K_{p1,p2}\otimes K_{p1,p2}$ is given by:
$E(K_{p1,p2}\otimes K_{p1,p2}) = 4p_1p_2$.

Proof: By Corollary 2.7, we have:

$$\begin{pmatrix} \sqrt{p_1p_2} & 0 & -\sqrt{p_1p_2} \\ 1 & p_1+p_2-2 & 1 \end{pmatrix}\begin{pmatrix} \sqrt{p_1p_2} & 0 & -\sqrt{p_1p_2} \\ 1 & p_1+p_2-2 & 1 \end{pmatrix} =$$

$$\begin{pmatrix} p_1p_2 & 0 & -p_1p_2 & 0 & 0 & 0 & -p_1p_2 & 0 & p_1p_2 \\ 1 & p_1+p_2-2 & 1 & p_1+p_2-2 & (p_1+p_2-2)^2 & p_1+p_2-2 & 1 & p_1+p_2-2 & 1 \end{pmatrix}$$

$$Sp((K_{p1,p2}\otimes K_{p1,p2}) = \begin{pmatrix} p_1p_2 & 0 & -p_1p_2 \\ 2 & 4(p_1+p_2-2)+(p_1+p_2-2)^2 & 2 \end{pmatrix}.$$

Then, by Definition 3.1, we get:
$E(K_{p1,p2}\otimes K_{p1,p2}) = 4p_1p_2$.
And the nullity of $K_{p1,p2}\otimes K_{p1,p2}$ is $4(p_1+p_2-2)+(p_1+p_2-2)^2$. ■

Proposition 4.14: (Balakrishnan 2004) Let G_1 and G_2 be two graphs on p_1 and p_2 vertices, respectively. Then $E(G_1\otimes G_2) = E(G_1)E(G_2)$.

Proof: By Definition 3.1, we have:

$$E(G_1\otimes G_2) = \sum_{i=1}^{p_1}\sum_{j=1}^{p_2}|\lambda_i\mu_j| = \sum_{i=1}^{p_1}\sum_{j=1}^{p_2}|\lambda_i||\mu_j| = \sum_{j=1}^{p_2}|\lambda_i| \ \sum_{j=1}^{p_2}|\mu_j| = E(G_1)E(G_2). \ ■$$

Proposition 3.15: The energy of the strong product of two graphs is the sum of energy of their Cartesian and tensor products. ■

Proposition 3.16: The energy of the skew product of two graphs G_1 and G_2 is related as:

$E(G_1 \blacklozenge G_2) \leq p_1 E(G_2) + E(G_1) E(G_2)$.

Proof: Let the spectra of G_1 and G_2 be $\{\lambda_1, ..., \lambda_{p1}\}$ and $\{\mu_1, ..., \mu_{p2}\}$; respectively.

Then $E(G_1 \blacklozenge G_2) = \sum_{i=1}^{p_1} \sum_{j=1}^{p_2} [|\mu_j + \lambda_i \mu_j|] \leq \sum_{i=1}^{p_1} \sum_{j=1}^{p_2} [|\mu_j| + |\lambda_i \mu_j|] \leq \sum_{i=1}^{p_1} \sum_{j=1}^{p_2} |\mu_j| + \sum_{i=1}^{p_1} \sum_{j=1}^{p_2} |\lambda_i \mu_j| \leq p_1 E(G_2) + E(G_1) E(G_2)$. ■

Example 3.17: Let $G_1=K_2$ and $G_2=P_3$, then the skew product of K_2 and P_3 is given by:

$S_p(K_2)=\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$ and $S_p(P_3)=\begin{pmatrix} \sqrt{2} & 0 & -\sqrt{2} \\ 1 & 1 & 1 \end{pmatrix}$

For j=1 and i=1, 2, we have: $\mu_1 + \lambda_1\mu_1 = 2\sqrt{2}, \mu_1 + \lambda_2\mu_1 = 0$.

For j=2 and i=1, 2, we have: $\mu_2 + \lambda_1\mu_2 = 0, \mu_2 + \lambda_2\mu_2 = 0$.

For j=3 and i=1, 2, we have: $\mu_3 + \lambda_1\mu_3 = -2\sqrt{2}, \mu_3 + \lambda_2\mu_3 = 0$.

Then, $E(K_2 \blacklozenge P_3)=4\sqrt{2}$.

And by Proposition 3.16, we have $E(K_2 \blacklozenge P_3) \leq 2 * 2\sqrt{2} + 2 * 2\sqrt{2} = 8\sqrt{2}$.

Moreover, equality does not hold for any pair of non-empty simple graphs.

Proposition 3.18: The energy of the inverse skew product is given by

$E(G_1 \lozenge G_2) \leq p_2 E(G_1) + E(G_1) E(G_2)$.

Proof: Let the spectra of G_1 and G_2 be $\{\lambda_1, ..., \lambda_{p1}\}$ and $\{\mu_1, ..., \mu_{p2}\}$, respectively.

Then $E(G_1 \lozenge G_2)=\sum_{i=1}^{p_1} \sum_{j=1}^{p_2} [|\lambda_i + \lambda_i \mu_j|] \leq \sum_{i=1}^{p_1} \sum_{j=1}^{p_2} [|\mu_j| + |\lambda_i \mu_j|] \leq \sum_{i=1}^{p_1} \sum_{j=1}^{p_2} |\lambda_i| + \sum_{i=1}^{p_1} \sum_{j=1}^{p_2} |\lambda_i \mu_j| \leq p_2 E(G_1) + E(G_1) E(G_2)$. ■

Example 3.19: Let $G_1=K_2$ and $G_2=P_3$, then the inverse skew product of K_2 and P_3 is given by

$S_p(K_2)=\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}$ and $S_p(P_3)=\begin{pmatrix} \sqrt{2} & 0 & -\sqrt{2} \\ 1 & 1 & 1 \end{pmatrix}$

For i=1 and j=1, 2, 3, we have: $\lambda_1 + \lambda_1\mu_1 = 1 + \sqrt{2}, \lambda_1 + \lambda_1\mu_2 = 1, \lambda_1 + \lambda_1\mu_3 = 1 - \sqrt{2}$.

For i=2 and j=1, 2, 3, we have: $\lambda_2 + \lambda_2\mu_1 = -1 - \sqrt{2}, \lambda_2 + \lambda_2\mu_2 = -1, \lambda_2 + \lambda_2\mu_3 = -1 + \sqrt{2}$.

Then, $E(K_2 \lozenge P_3) = 2+4\sqrt{2}$.

And by Proposition 3.18, we have $E(K_2 \lozenge P_3) \leq 3 * 2 + 2 * 2\sqrt{2} = 6 + 4\sqrt{2}$.

REFERENCES

- A.E. Brouwer and W.H. Haemers; (2012), *Spectra of Graphs*, Springer, NewYork.
- F. Harary, ;(1969), *Graph Theory*, Addison-Wesley, Reading, MA. *Fundamentals,* Vol. E83-A, No. 3, pp.459-464.
- I.Gutman; (1978), The Energy of a Graph, *Ber. Math. Statist. Sekt. For schungszenturm Graz.* 1031–22.
- R. Balakrishnan, and K. Ranganathan; (2012), *A Textbook of Graph Theory*, Springer, New York.
- R. Balakrishnan; (2004), The energy of a graph, *Linear Algebra and its Applications* 387, pp.287–295.
- X. Li, Y. Shi, and I. Gutman; (2012), *Graph Energy*, Springer. http://books.google.iq/books?id=90U4suPlWS4C&printsec=frontcover&hl=ar#v=onepage&q&f=false.
- Y. Shibata, and Y. Kikuchi,; (2000), Graph products based on the distance in graphs, *IEICE Trans.*

حول الطاقة لبعض البيانات المركبة

الخلاصة

القيم الذاتية للبيان هي القيم الذاتية لمصفوفة التجاور له. الطاقة للبيان هو مجموع القيم المطلقة للقيم الذاتية له و التي تم دراستها من قبل(Gutman ١٩٧٨). هذا البحث قُسم على ثلاثة اجزاء، في الجزء الاول نعرف ألاطياف و البطلان للبيانات (Brouwer and Haemers 2012)و(Harary1969). في الجزء الثاني تم دراسة جداء البيانات واطياف جداءاتها (Shibata and Kikuchi 2000) و (Balakrishnan and Ranganathan 2012) . في الجزء الاخير اُثبت أطياف الانحراف الضربي و المعكوس الانحراف الضربي والطاقة لبعض المضروبات بما في ذلك الجداء الديكارتي، والجداء تينسري، والجداء قوي و الجداء الانحراف والانحراف المعكوس.

ل دوّر هيّزا هەندەك گرافيّن ئاويّتە

كورتى:

بەهاى خويەتى يى گرافى ئەو بەهاى خويەتيە بوّ ريّزكراوا ويّ، هيّزا گرافى ئەو كوّمەلا بەهاى رووتيە يا بەهاييّن خويەتى، وا هاتيە خويندن ژ لايّ (Gutman١٩٧٨). ئەڤ ڤەكولينە هاتە دابەشكرن ل سەر سيّ پشكا، د پشكا ئيّكيّدا سيبەر و پلا ناباو يا گرافان هاتە دياركرن (Brouwer and Haemers 2012)و(Harary1969) . د پشكا دوويدا ليّكدانيّن هەندەك گرافا و سيبەرا وان هاتە خواندن (Shibata and Kikuchi 2000) و(Balakrishnan and Ranganathan 2012). د پشكا دوماهييدا سيبەرا ليّكدانا خوار (skew) و ليّكدانا خوار يا زڤرى (inverse skew) مە سەلماندن و هيّزا هەندەك گرافيّن ئاويّتە مە دياركر ل سەر هەندەك ليكدانيّن گرافا.