ON GENERALIZED REGULAR LOCAL RING

Zubayda **M. I**braheem and **N**aeema **A**li **Sh**ereef
Dept. Of Mathematics ,College of Computer and mathematical Sciences, University of Mosul, Iraq.
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Abstract:
A ring R is called a generalized Von Neumann regular local ring (GVNL-ring) if for any a∈R, either a or (1-a) is π-regular element. In this paper, we give some characterization and properties of generalized regular local rings. And we studied the relation between generalized regular local rings, Von Neumann regular rings, Von Neumann regular local rings (VNL-rings) and exchange rings.

Key words: local, π-regular , exchange rings .

1- Introduction:

Throughout this paper, R will be an associative ring with identity. For a∈R, r(a), ℓ (a) denote the right (left) annihilator of a. We write Y(R), $\mathcal{J}$(R) for the right singular ideal and the Jacobson radical of R, respectively. An ideal ***I*** of a ring R is said to be essential if and only if ***I*** has a non-zero intersection with every non-zero ideal of an R.A ring R is reduced if R contains no non-zero nilpotent element. A ring R is said to be Von Neumann regular (or just regular) if and only if for each a in R ,there exists bin R such that a=aba [7]. In[3] Contessa first introduced and characterized a VNL-ring, and gave many properties, a ring R is called Von Neumann regular local rings (VNL-rings), if for any a∈R, either a or (1-a) is Von Neumann regular element. A ring R is called local ring, if it has exactly one maximal ideal [1]. A ring R is called π-regular ring if and only if for each a in R, there exists b in R and a positive integer n such that $a^n=a^n b a^n$ [4]. clearly that every π-regular ring is GVNL-ring. A ring R is said to be strongly commuting regular if for each x,y in R thereexista $\in R$ $such\ that$ $(xy)=(yx)^2a(yx)^2$[6]. A ring R is called Exchange ring if for any a∈R, there exists an idempotent element e ∈R such that e∈aR and (1-e) ∈(1-a)R [5].

2- GVNL-Rings

This section is devoted to give the definition of generalized Von Neumann regular local ring(GVNL-rings) with some of its characterization and basic properties .

Definition 2.1: [2]

A ring R is called a generalized Von Neumann regular local ring(GVNL-ring) if a or (1-a) is π-regular for every a∈R .

Examples:

1-) Let (Z_2 ,+, .) be a ring and let G=$\{g: g^{2=1}\}$is cyclic group, then
Z_2G=$\{0, 1, g, 1+g\}$ is not regular, but π-regular ring.

2-) Let R be the set of all matrix in Z_2 which defined as:

R=$\left\{\begin{bmatrix} a & b \\ 0 & d \end{bmatrix} : a, b, d \in Z_2\right\}$ It is easy to find the elements of R

R=$\left\{A_0, A_1, A_2, A_3, A_4, A_5, A_6, A_7\right\}$

$A_0 = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, A_1 = \begin{bmatrix} 1 & 0 \\ 0 & 0 \end{bmatrix}, A_2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, A_3 = \begin{bmatrix} 0 & 0 \\ 0 & 1 \end{bmatrix}$

$A_{4=} \begin{bmatrix} 0 & 1 \\ 0 & 1 \end{bmatrix}, A_5 = \begin{bmatrix} 1 & 1 \\ 0 & 1 \end{bmatrix}, A_6 = \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}, A_7 = \begin{bmatrix} 1 & 1 \\ 0 & 0 \end{bmatrix}$

Thus R is not regular but R is π-regular.

3-) Let : R=$Z_4 \oplus Z_4$, then it is easy to check that ,
$(\bar{3}, \bar{2})$ and$(\bar{1}, 1) - (\bar{3}, \bar{2})$ are not regular. So R is not a VNL-ring, but R is π-regular. Thus R is a GVNL-ring.

Proposition: 2.2:

A ring R is GVNL-ring iff a^n R is generated by an idempotent element for every a∈R and some positive integer n.

Proof:

Suppose that R is a GVNL-ring. Then either a or (1-a) is π-regular. Let a∈R. Then there exists an element b∈R and appositive integer n such that $a^n=a^n b\, a^n$, let $e=a^n b$. Clearly e is an idempotent element in R. We shall show that $a^n R=eR$. Let c $\in a^nR$, then, $c=a^nr$ for some r∈R. But, $a^n=a^n b\, a^n$, so

$C=a^n b\, a^n r = e\,.\, a^n r \in eR$, $a^n R \in R$. Therefore $a^nR \subseteq eR$ …(1)

On the other hand if d∈eR, then d=e s for some s∈R, but, $e=a^n b$, so, $d=a^n bs \in a^nR$

Therefore $eR\subseteq a^nR$ …(2). Thus $a^n=eR$

Now, if (1-a) is π-regular, then there exists z∈R and $n\in Z^+$ such that $(1-a)^n=(1-a)^n z\,(1-a)^n$,

let $e=(1-a)^n z$, we shall show that $(1-a)^nR=eR$.

Let $w\in(1-a)^n R$, then $w=(1-a)^n s$, for some s ∈R. But $(1-a)^n=(1-a)^n z(1-a)^n$. so ,$w=(1-a)^n z(1-a)^n s=(1-a)^n z(1-a)^n s=e(1-a)^n s \in eR$, therefore $(1-a)^n R \subseteq eR$.

Now if d∈eR, then d=es for some s∈R. But $e=(1-a)^n z$, so, $d=(1-a)^n zs \in a^n R$

Thus $eR\subseteq(1-a)^n R$. Therefor $(1-a)^n R=eR$.

Conversely, let $a\in R$, e is an idempotent in R such that $a^n R=eR$ for some positive integer n. Then $e= a^n b$ for some b∈R and $a^n=e\,c$ for some c∈R, so $e\, a^n=a^n b\, a^n=$, so $e\, a^n=a^n b\, a^n= e\, e\, c=e\, c=a^n$. Hence $a^n = ea^n = a^n b\, a^n$. Therefore a is π- regular.

Now, for $(1-a)\in R$, let e be an idempotent element in R such that

$(1-a)^n R=eR$, for some positive integer n .Then $e= (1-a)^n b$ for some b∈R

and $(1-a)^n=e\,c$,for some c∈ R . So, $e(1-a)^n =(1-a)^n b\,(1-a)^n$, and $e(1-a)^n=e\,e\,c$, hence $e(1-a)^n = e\,c=(1-a)^n$, then $(1-a)^n=e(1-a)^n=(1-a)^n b\,(1-a)^n$

Thus(1- a) is π-regular and therefore R is GVNL- ring. #

Proposition 2.3:

Let R be a commutative ring and let P be a primary ideal of R. Then P is a maximal ideal if R/P is a GVNL-ring .

***Proof*:**

Let $a\in R$, Then $a+P\in R/P$. Since R/P is GVNL-ring, then either (a+p) or ((1-a)+p) is π-regular element in R/P . Let (a+P) be π-regular in R/P. Then let (b+P)∈ R/P and $n\in Z^+$ such that $a^n+P=(a+P)^n=(a+P)^n(b+P)(a+P)^n$

$=(a^n+P)(b+P)(a^n+P)=a^n b\,a^n+P=a^n b\,a^n+P$

That is $(a^n-a^n b\,a^n)\in P$, then $a^n(1-b\,a^n)\in P$. Assume that $a^n\notin p$.

Since P is primary, then $(1-b\,a^n)^m\in P$, $m\in Z^+$ such that ,

$(1-b\,a^n)^m=1-[\sum_{k=1}^{m} C_k^m(-1)^{k-1} b^k a^{n(k-1)}]\, a^n \in p$ where $C_k^m=\frac{m!}{k!(m-k)!}$.

Now, let $z=\sum_{k-1}^{m} C_k^m(-1)^{k-1}b^k a^{n(k-1)}$.

Then $(1-z\,a^n)\in P$, Thus $1+p=(z+P)(a^n+P)=(z\,a^{n-1}+P)(a+P)$.

Therefore (a+p) has inverse in R/P. Now, if ((1-a)+P) is π-regular element in R/P, then there exists $n\in Z^+$ and $(\ell+P)\in R/P$ such that

$((1-a)^n+P)=((1-a)+P)^n=((1-a)+P)^n(\ell+P)((1-a)+P)^n=((1-a)^n+P)(\ell+P)((1-a)^n+P))$

$=(1-a)^n\,\ell\,(1-a)^n+P$

Then $(1-a)^n-(1-a)^n\,\ell(1-a)^n\in P$, that is $(1-a)^n(1-\ell(1-a)^n)\in P$

Let $(1-a)^n\notin P$, since Pis primary ideal, Then $(1-\ell(1-a)^n)^m\in P$, $m\in Z^+$ such that

$[1-\ell(1-a)^n]^m=1-[\sum_{k=1}^{m} C_k^m(-1)^{k-1}\ell^k\ (1-a)^{n(k-1)}]\ (1-a)^n \in p$

$=1-[\sum_{k=1}^{m} C_{k=1}^{m}\ )(-1)^{(k-1)}\left(\sum_{q=1}^{n(k-1)} c_q^{n(k-1)}(-1)^{(q-1)}a^{n(k-1)(q-1)}\right) a]\ (1-a)^n\in p$

$=1-[\sum_{k=1}^{m}\sum_{q=1}^{n(k-1)} C_k^m\, C_q^{n(k-1)}\,(-1)^{k+q-2}\, a^{n(k-1)(q-1)+1}]\,(1-a)^n \in p$

Where $C_k^m=\frac{m!}{k!(m-k)!}$

Now, let $w=\sum_{k=1}^{m}\sum_{q=1}^{n(k-1)} C_k^m\, C_q^{n(k-1)}\,(-1)^{k+q-2}\,a^{n(k-1)(q-1)+1}$

Then $1-w(1-a)^n \in P$ which implies that $1+p=(w+p)(1-a)^n+p = (w(1-a)^{n-1}+P)(a+P)$.

Hence ((1-a)+P) has inverse in R/P, thus R/P is a division ring. Therefore P is a maximal ideal in R . #

Theorem 2.4: Le t $\boldsymbol{I}$ be a regular ideal of a ring R. Then R is GVNL-ring if and only if R/$\boldsymbol{I}$ is GVNL-ring .

***Proof*:**

Let R be GVNL-ring. Then either a or (1-a) is π- regular element in R, for all $a\in$R.

Now, if a is π-regular, then there exists $b\in R$ and $n\in Z^+$ such that $a^n=a^n b a^n$

Hence $(a+\boldsymbol{I})^n= a^n+\boldsymbol{I} = a^n b a^n +\boldsymbol{I}$

$=(a^n+\boldsymbol{I}) (b+\boldsymbol{I}) (a^n +\boldsymbol{I})$

$= (a+\boldsymbol{I})^n (b+\boldsymbol{I})(a+\boldsymbol{I})^n$

Thus $(a+\boldsymbol{I})$ is π-regular element in R/$\boldsymbol{I}$

Now, if $(1-a)$ is π-regular in R, then there exists $d\in$ R and $n\in Z^+$ such that

$(1-a)^n=(1-a)^n d (1-a)^n$

Hence $((1-a)+\boldsymbol{I})^n=((1-a)+\boldsymbol{I})^n (d+\boldsymbol{I}) ((1-a)+\boldsymbol{I})^n$

$=((1-a)^n+\boldsymbol{I}) (d+\boldsymbol{I}) ((1-a)^n+ \boldsymbol{I})$

$=(1-a)^n d (1-a)^n+\boldsymbol{I}$

$= (1-a)^n+\boldsymbol{I}$

Therefore $((1-a+\boldsymbol{I})$ is π-regular in R/$\boldsymbol{I}$

Conversely, let R/$\boldsymbol{I}$ be GVNL and $a\in$R. Then either $(a+\boldsymbol{I})$ or $((1-a)+\boldsymbol{I})$ is π-regular element in R/$\boldsymbol{I}$. Then there exists $(b+\boldsymbol{I})\in$ R/$\boldsymbol{I}$ and a positive integer n such that

$(a^n+ \boldsymbol{I}) = (a+\boldsymbol{I})^n (b+\boldsymbol{I})(a+ \boldsymbol{I})^n = (a^n+ \boldsymbol{I})(b+\boldsymbol{I})(a^n+ \boldsymbol{I}) = a^n b a^n+\boldsymbol{I}$

Hence $a^n+\boldsymbol{I} = a^n b a^n+\boldsymbol{I}$, so $a^n - a^n b a^n\in \boldsymbol{I}$

Since $\boldsymbol{I}$ is regular, there exists $c\in\boldsymbol{I}$ such that

$a^n - a^n b a^n = (^n - a^n b a^n) c(a^n - a^n b a^n)$

$a^n = a^n b a^{n+}a^n c a^n - a^n b a^n c a^n - a^n c a^n b a^{n+}a^n b a^n c a^n ba^n$

$= a^n (b+c - b a^n c\text{-}c a^n b+b a^n c a^n b)a^n$

$=a^n w a^n$, where $w=(b+c\text{-}b a^n c\text{-}c a^n b+b a^n c a^n b)$

Thus a is π-regular in R.

Now, if (1-a)+$\boldsymbol{I}$ is π-regular, then there exists q+$\boldsymbol{I}$ and $n\in Z^+$ such that,

$(1\text{-}a)^n +\boldsymbol{I} = ((1\text{-}a)^n +\boldsymbol{I})(q+\boldsymbol{I})((1\text{-}a)^n +\boldsymbol{I})$

So $(1\text{-}a)^n\text{-}(1\text{-}a)^n q(1\text{-}a)^n \in\boldsymbol{I}$, since $\boldsymbol{I}$ is regular then

$(1-a)^n-((1-a)^n q (1-a)^n) =[(1-a)^n-(1-a)^n q (1-a)^n] w [(1-a)^n-(1-a)^n q(1-a)^n]$

$=[(1-a)^n w (1-a)^n - (1-a)^n w (1-a)^n q (1-a)^n]$

$-[(1-a)^n q (1-a)^n w (1-a)^n]+[(1-a)^n q (1-a)^n w (1-a)^n q (1-a)^n]$

$= (1-a)^n [w\text{-}w (1-a)^n q\text{-}q (1-a)^n w +q(1-a)^n w (1-a)^n q] (1-a)^n$

$(1-a)^n=(1-a)^n q(1-a)^n+(1-a)^n[w\text{-}w (1-a)^n q\text{-}q (1-a)^n w+ q(1-a)^n w (1-a)^n q] (1-a)^n$

$= (1-a)^n[q\text{-}w\text{-}w(1-a)^n q\text{-}q(1-a)^n w +q(1-a)^n w(1-a)^n q] (1-a)^n$

$= (1-a)^n Z (1-a)^n$

Thus $(1-a)$ is π- regular in R. Therefore R is a GVNL –ring. #

***Proposition*2.5:**

Let R be a GVNL-ring. Then $\mathcal{J}$(R) is nil ideal.

***Proof*:**

Let $0\neq a\in\mathcal{J}$(R) Since R is GVNL-ring, then either a or (1-a) is a π- regular element of R, if a is π-regular, then there exists an element $b\in R$ and $n\in Z^+$ such that $a^n= a^n b a^n$, then $a^n - a^n b a^n= 0$. Hence $(1- a^n b) a^n=0$ Since $a\in\mathcal{J}$(R), therefore $a^n \in\mathcal{J}$(R), and $a^n b \in\mathcal{J}$(R). Thus $(1- a^n b)$ is invertible, so there exists $u\in$ R such that $u (1- a^n b)=1$, it follows that $u (a^n - a^n b a^n) =a^n$, thus a is nilpotent element. Now, if (1-a) is π-regular, since a $\in\mathcal{J}$(R), then (1-a) is invertible, and then $(1\text{-}a)^n$ invert able, thus (1-a) is nilpotent element and Therefore $\mathcal{J}$(R) is a nilideal.

***Corollary*: 2.6:**

Let R be a reduced GVNL – ring Then $\mathcal{J}(R)=0$

Proof

Suppose that $\mathcal{J}(R) \neq 0$, then there exists a $\in\mathcal{J}$(R) and by (prop .2.5) a is a nilpotent element in R. But R is reduced, then a=0.therefore $\mathcal{J}(R) =0$. #

***Corollary 2.7*:**

Let R be a reduced GVNL- ring, Then R is VNL- ring .

Proof:

R is exchange ring [2,Theorem 2.2].

Now, R is reduced exchange ring for any a∈R .

.If a is π- regular in R, then $a^n = a^n x\, a^n$ for some positive integer n and $x \in$R. Clearly e = $a^n x$ where e is an idempotent element in R , so $((1\text{-}e)\, a)^n = (1\text{-}e)\, a^n = (1\text{-}e)\, e\, a^n = 0$, and hence (1-e)a=0 Therefore a= e a= $a(a^{n-1} x)\, a$, is regular.

Now, if (1-a) is π-regular, then $(1\text{-}a)^n = (1\text{-}a)^n z (1\text{-}a)^n$ for some positive integer n and z∈ R. Clearly e = $(1\text{-}a)^n$ z, so $(1\text{-}e)(1\text{-}a)^n = (1\text{-}e)\, e\, (1\text{-}a)^n = 0$.

Hence (1-e) (1- a) = o .

Thus $(1\text{-}a) = e(1\text{-}a) = (1\text{-}a)^n z(1\text{-}a) = (1\text{-}a)(1\text{-}a)^{n-1} z\, (1\text{-}a) = (1\text{-}a)w(1\text{-}a)$

That is (1- a) is regular. Therefore R is a VNL-ring. #

Proposition 2.8:

Let R be a GVNL-ring with $r(a) \subseteq r(1\text{-}a)^n$ for a∈R . Then Y(R) is a nilideal.

Proof:

Let a$\notin \{0, 1\}$ be an element in Y(R). Then Y(R) is an essential right ideal of R. Now, since $r(a) \subseteq r(a^n)$, then $r(a^n)$ is also an essential right ideal of R.

Since R is a GVNL-ring, then either a or (1-a) is π-regular element in R. If a is π-regular, then there exists b∈R and n∈Z^{+}, such that $a^n = a^n\, b\, a^n$.

Now, consider $r(a^n) \cap b\, a^n R$, and let $x \in r(a^n) \cap b a^n R$, Then $a^n x = 0$ and $x = b a^n r$ for some r∈R. So, $a^n b a^n r = 0$, which implies $a^n r = 0$, yielding x=0.

Therefore $r(a^n) \cap b a^n R = 0$, since $r(a^n)$ is a non-zero essential right ideal of R, then $b a^n = 0$, and hence $a^n = 0$.

Now, if (1-a) is π-regular, then there exists c∈R and n∈Z^+ such that $(1\text{-}a)^n = (1\text{-}a)^n\, c\, (1\text{-}a)^n$.

Now, since $r(a) \subseteq r(a^n) \subseteq r(1\text{-}a)^n$, then $r((1\text{-}a)^n)$ is also an essential ideal of R. Consider $r((1\text{-}a)^n) \cap c(1\text{-}a)^n R$. Let $y \in r((1\text{-}a)^n) \cap c((1\text{-}a)^n) R$, then $(1\text{-}a)^n y = 0$, and $y = c(1\text{-}a)^n r$ for some r∈R. So, $(1\text{-}a)^n c\, (1\text{-}a)^n r = 0$, Which implies $(1\text{-}a)^n r = 0$, yielding y=0. Therefore $r((1\text{-}a)^n) \cap c(1\text{-}a)^n R = 0$.

Since $r((1\text{-}a)^n)$ is a non-zero essential right ideal of R, then $c(1\text{-}a)^n = 0$.

And hence $(1\text{-}a)^n = 0$, and then 1=a, a contradiction .

Thus $a^n = 0$ and therefore Y(R) is nilideal. #

Theorem 2.9:

Let R be a ring with $(r(a^{n+1})) \subseteq r(a^n)$ and $r(a(1\text{-}a)^n) \subseteq r((1\text{-}a)^n)$ for a∈R, if R/r(a) is GVNL-ring. Then R is GVNL-ring

Proof:

Assume that R/r(a) be a GVNL-ring, then either a+r(a) ∈R/r(a) or

(1-a)+r(a) is π-regular element. Now, if a+r(a) is π-regular element for a∈R then there exists b+r(a) ∈R/r(a) and n∈Z^+, such that

$a^n + r(a) = (a + r(a))^n$

$= (a + r(a))^n (b + r(a))(a + r(a))^n$

$= (a^n + r(a))(b + r(a))(a^n + r(a))$

$= a^n b\, a^n + r(a)$

Then $a^n - a^n b\, a^n \in r(a)$, that is

$a\, (a^n - a^n b\, a^n) = 0$, hence $a^{n+1}(1\text{-}b\, a^n) = 0$.Then $(1\text{-}b\, a^n) \in r(a^{n+1}) \subseteq r(a^n)$, which implies that $a^n(1\text{-}b\, a^n) = 0$, then we get $a^n = a^n b\, a^n$

Thus a is π-regular element in R .

Now, if (1-a)+r(a) ∈R/ r(a) is π-regular for a∈R, then, there exists c+r(a) ∈R/ r(a) and n∈Z^+ such that $(1\text{-}a)^n + r(a) = ((1\text{-}a) + r(a))^n$

$=((1-a)+r(a))^n (c+r(a)) ((1-a)+r(a))^n = ((1-a)^n + r(a))(c+r(a))((1-a)^n + r(a))$
$=(1-a)^n c (1-a)^n + r(a)$, then $(1-a)^n - (1-a)^n c (1-a)^n \in r(a)$, that is
$a((1-a)^n - (1-a)^n c (1-a)^n)=0$, so $a(1-a)^n (1-c(1-a)^n)=0$ then
$(1-c(1-a)^n) \in r(a(1-a)^n) \subseteq r((1-a)^n$, hence $(1-a)^n (1-c(1-a)^n)=0$
Thus $(1-a)^n - (1-a)^n c (1-a)^n=0$. Hence $(1-a)$ is π- regular element in R
Therefore R is GVNL –ring. #

***Theorem 2.10*:**

If R is a reduced ring and every maximal ideal of R is a right annihilator.
Then R is GVNL-ring.

***Proof*:**

Let a∈R, we shall prove that $a^n R+r(a)=R$, if not there exists a right maximal ideal M containing $a^n R+r(a^n)$. If M=r(b) for some 0≠b∈R, we have
$b\in \ell(a^n R+r(a^n)) \subseteq \ell(a^n)=r(a^n)$, which implies that b∈M=r(b), then $b^2=0$,
and b=0 a contradiction. Therefore $a^n R+r(a^n)=R$
In particular $a^n c+d=1$, c∈R, $d\in r(a^n)$, then $a^n c a^n = a^n$ which proves
a is π-regular element. Now, if (1-a)∈ R, we shall prove that.
$((1-a)^n + r((1-a)^n)=R$, if not there exists a maximal right ideal M containing
$(1-a)^n R+r(1-a)^n$, if M=r(c) for some 0≠c∈R, we have
$c\in\ell(1-a)^n R+r(1-a)^n \subseteq \ell(1-a)^n = r(1-a)^n$. Which implies c∈M=r(r), then $c^2=0$.
And hence c=0 a ctraduction. Therefore $(1-a)^n R+r(1-a)^n=R$
Inparticular $(1-a)^n$ $x+y$ =1 Where x ∈R and y $\in r((1-a)^n)$ Then, $(1-a)^n x (1-a)^n + 0 = (1-a)^n$.
Thus $(1-a)$ is π-regular element in R. Therefore R is GVNL-ring. #

Now, we have the following result to obtain the relation between GVNL-ring and exchange ring .

***Proposition 2.11*:**

If R is GVNL-ring, then eRe is also GVNL-ring for every idempotent element e in R .

Proof:

For a∈eRe, a or (1-a) is π-regular in R. If a is π-regular, then there exists b in R and $n\in Z^+$ such that $a^n = a^n b a^n$, so, $a^n = (a^n e) b (e a^n) = a^n (ebe) a^n$. Thus a is π-regular in eRe.
If $(1-a)$ is π-regular, then there exists c∈R and $n\in Z^+$ such that $(1-a)^n=(1-a)^n c(1-a)^n$.
Now ,$(e-a)^n = (e-ea)^n = (e(1-a))^n = e(1-a)^n e$.
Hence e-a is π-regular in eRe.
Therefore, eRe is a GVNL-ring . #
Now, we give the following proposition which is due to YING Zhi- ling in [8]

***Proposition 2.12*:**

If R a GVNL-ring, then for every idempotent element e in R, either eRe or (1-e) R(1-e) is a π-regular ring .

***Proof*:**

Let R be a GVNL-ring and e ∈R be an idempotent element in R. Then

$$R \cong \begin{pmatrix} eRe & eR(1-e) \\ (1-e)Re & (1-e)R(1-e) \end{pmatrix}$$

If x ∈eRe and y ∈(1-e)R(1-e) are two non-π-regular elements. Then both

$$a=\begin{pmatrix} x & 0 \\ 0 & 1-y \end{pmatrix} \text{ and } 1-a=\begin{pmatrix} 1-x & 0 \\ 0 & y \end{pmatrix}$$

are also non-π-regular, a contradiction. #
From proposition 2.12 , clearly every GVNL-ring is an exchange ring .

The following corollary is given in [8]

***Corollary 2.13*:**
For an abelian ring R, R is GVNL-ring if R is an exchange ring

***Proof*:**
For $a \in$R, let $a_1 = a$ and $a_2 = 1-a$. Then
a_1 R+a_2R=R, then by [5,proposition 1-11] ,there exists an orthogonal idempotent
e_1 and e_2 such that $e_1 \in e_2$ R and $e_1 + e_2 = 1$.
Now, $e_1 = a_1 b_1$ and $e_2 = a_2 b_2$ for $b_1, b_2 \in$R, so $e_1 a_1 = (e_1 a_1)(e_1 b_1)(e_1 a_1)$
and $e_2 a_2 = (e_2 a_2)(e_2 b_2)(e_2 a_2)$ is regular.
Since R is abelian, $e_1 a_1$ and $e_2 a_2$ is also strongly regular.
Thus we can suppose that
$e_1 a_1 b_1 = e_1 b_1 a_1$ and $e_2 a_2 b_2 = e_2 b_2 a_2$, implying that, $e_1 a_1 = (e_1 a_1)^2 (e_1 b_1)$ and $e_2 a_2 = (e_2 a_2)^2 (e_2 b_2)$. By (proposition2.12) either e_1 R=e_1 R e_1 and e_2 R =e_2 R e_2 or (1-e_1)R(1-e_1)and(1-e_2)R(1-e_2)areπ-regularife_1R$e_1$and e_2 Re_2 are π-regular, then R being a finite direct product of e_1 Re_1 and e_2 Re_2 is π-regular. If (1-e_j)R(1-e_j) is π-regular for j=1, 2, then there exists m$>$0 and $C_j \in$R for j=1, 2 such that ($((1-e_j) a_j)^m C_j ((1-e_j) a_j)^m = ((1-e_j)a_j)^m = (1-e_j) a_j^m$),
$e_j a_j = (e_j a_j)^2 (e_j b_j)$, we have
$[e_j a_j + (1-e_j) a_j]^m [(e_j b_j)^m + (1-e_j) c_j] [e_j a_j + (1-e_j) a_j]^m$
$= [e_j a_j + (1- e_j) a_j]^m = a_j^m$
That is a_1, a_2 is also π-regular. Therefore R is GVNL-ring. #
Now, to give the relation between GVNL-ring and strongly commuting regular rings.

***Theorem 2.14*: [6]**
Suppose that R is a strongly commuting regular ring. Then R is π-regular .

Proof:
By strongly commuting regularity of R, for each $x \in$R, there exists an element c in R such that $x^2 = x^4 c x^4$. With suppose $\bar{c} = x^2 c x^2$. Then we have
$x^2 = x^2 \bar{c} x^2$ .Which implies that R is π–regular . #
We observe the following corollary .

***Corollary 2.15*:**
I f R is a strongly commuting regular. Then R is GVNL-ring .

3- The GVNL – Ring without zero Divisor element .

In this section, we give some results about GVNL-ring without zero –divisors and some relation with other rings like division ring, local ring, π-regular ring, simple ring, VNL-ring .

Proposition 3.1:
Let R be a GVNL-ring without zero divisors. Then every element a$\notin \{0, 1\}$ in R is invert able .

Proof:
Let Rbe a GVNL-ring and $0 \neq a \in R. Then\ either\ a\ or\ (1-a) is\ \pi - regular\ element\ in\ R$. If a is π-regular element in R, then there exists b$\in$R and n$\in Z^+$ Such that
$a^n = a^n b a^n$, and then we have $a^n - a^n b a^n = 0$.
That is, $a^n - a^n b a^n = a^n (1- b a^n) = a (a^{n-1} (1-b a^n)) = 0$. Since R is without zero divisors then ,
$a^{n-1} (1-b a^n) = 0$, thus $a(a^{n-2} (1-b a^n)) = 0$ and hence $\ldots a(1-b a^n) = 0$, then $1-b a^n = 0$ Thus $1 = b a^n$, which implies that $1 = (b a^{n-1}) a$, that a has left inverse.
Now, since $1 = (b a^{n-1}) a$, then, $a = a (b a^{n-1}) a$ and thus $(1- ab a^{n-1}) a = 0$.
Hence $(1-ab a^{n-1}) \in \ell (a) = 0$, which implies that $1 = a (b a^{n-1})$,
Thus a has right inverse and therefore a is invertable element in R .
Now, if (1-a) is π-regular element in R, then there exists c$\in$R and n$\in Z^+$ such that ,
$(1-a)^n - (1-a)^n c (1-a)^n = 0$,that is $(1-a)^n [1-c(1-a)^n] = (1-a)^n (1-a)^{n-1} [1-c(1-a)^n] = 0$.

Since R is without zero divisor and $a \notin \{0, 1\}$.
Thus (1-a) ((1-a)$^{n-2}$(1- c(1-a)^{n})=0, and hence … (1-a) (1-c(1-a)^{n})=0.
Then (1-c (1-a)^{n})=0, which implies that, 1=c(1-a)^{n}
That is 1=(c (1-a)$^{n-1}$) (1-a). Hence (1-a) has left inverse .
Now, since 1=(c (1-a)$^{n-1}$) (1-a), then,
(1-a)=(1-a) (c(1-a)$^{n-1}$) (1-a), and then (1-(1-a) c (1-a)$^{n-1}$) (1-a)=0 thus,
(1- (1-a)c (1-a)$^{n-1}$) ∈ ℓ (1-a) =0 which implies that 1=(1-a) (c (1-a)$^{n-1}$) .
And hence(1-a)has right inverse.Therefore (1-a)is invertible element in R. #

Corollary 3.2:

Let R be a GVNL-ring without zero-divisors. Then R is a division ring .

***Corollary 3.3*:**

Let R be a GVNL-ring without zero divisors. Then R is;

1- Local ring. 2- VN-regular ring and reduced ring.
2- VN-regular ring and reduced ring .
3- π-regular ring .

Finally we give the following result .

Proposition 3.4:

If ***I*** is a proper ideal of GVNL-ring R. Then every element of ***I*** is zero divisors in R. Especially every GVNL-ring has non-zero divisors is simple.

Proof:

Let 0≠a ∈ ***I***, since R is GVNL-ring, then either a or (1-a) is π-regular element in R. If a is π-regular element, then there exists an element b∈ R and a positive integer n such that $a^{n}=a^{n} b\, a^{n}$, if a is not zero divisors then a^{n} (1-b a^{n})=0 gives (1-b a^{n})=0, and hence 1∈***I***, a contradiction.

Hence, a is zero divisors . Now, if (1-a) is π-regular element, then there exists c∈R and a positive integer n such that (1- a)^{n} =(1- a)^{n} c (1- a)^{n} . If (1- a) is not zero divisor, then (1- a)^{n} (1- c (1- a)^{n}) =0 gives, 1-c(1-a)^{n} =0 thus

1=c (1- a)^{n} ∈***I*** a contradiction.

If R has nonzero divisors, there is not a proper ideal ***I*** of R. Hence R is simple. #

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پوختە:

دبێژنە خەلەکا (R) کو خەلەکەکا ناڤخوییا ڕێکخستی و گشتگیرە، بو هەر ئێك ژ (a ∈ R) یان (a) یان (1-a) هەیە کو توخمەکێ ڕێکخستیە ژ شێوازێ -π، دڤێ ڤەکولینێ دا مە هندەك تایبەندی و ساخلەتێن خەلەکا ناڤخویی یێن ڕێکخستی و ناڤخویی گشتگیر داینە، هەروەسا مە پەیوەندی دناڤبەرا خەلەکێن ناڤخو یێن گشتگیر و خەلەکێن ناڤخویی خاندیە، خەلەکێن ڕێکخستی ل پەی تێگەهشتنا فۆن نیومان، خەلەکێن فۆن نیومان یێن ڕێکخستیێن ناڤخویی (خەلەك ژ شێوازێ -VNL) و خەلەکێن هەڤ دژ.

المستخلص:

يقال للحلقة R بأنها حلقة محلية منتظمة معممة، إذا كان لكل a∈R إما a أو (1-a) هو عنصر منتظم من النمط – π. في هذا البحث أعطينا بعض مميزات و خواص الحلقة المحلية المنتظمة المعممة كذلك درسنا العلاقة بين الحلقات المحلية المنتظمة المعممة و الحلقات المحلية، الحلقات المنتظمة حسب مفهوم فون نيومان، حلقات فون نيومان المنتظمة المحلية(حلقات من النمط –VNL) والحلقات المقايضة.