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COEFFICIENT ESTIMATES OF A NEW SPECIAL SUBCATEGORY OF BI-UNIVALENT FUNCTIONS

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ABSTRACT:

In this article, based on the concepts of subordination and q-derivative operator, we introduce and study a new subcategory $\mathcal{H}_\Sigma(q,\beta;\varsigma)$ of analytic and bi-univalent functions in the open unit disk D. Upper bounds for the second and third coefficients of the functions belonging to the subcategory $\mathcal{H}_\Sigma(q,\beta;\varsigma)$ are found and several particular outcomes of the main finding are also presented.

KEYWORDS: Analytic Function, Bi-univalent Function, q-derivative, Subordination.

1. INTRODUCTION

Suppose that $\mathbb{C}$ is the set of all complex numbers in the plane and $D = \{\xi \in \mathbb{C} : |\xi| < 1\}$ represents the open unit disk in $\mathbb{C}$, we signify $\mathcal{H}$ the class of all functions that are analytic in D and normalized by $f(0) = f'(0) - 1 = 0$. Then every function f in $\mathcal{H}$ has power series representation:

$$f(\xi) = \xi + \sum_{k=2}^{\infty} a_k \xi^k = \xi + a_2\xi^2 + a_3\xi^3 + \cdots. \qquad (1.1)$$

An analytic function f is called univalent in D if it is injective in D. The Koebe function $K(\xi) = \frac{\xi}{(1-\xi)^2}$ is an example of univalent functions (see (Duren, 1983)). For more than a century, the theory of univalent functions is a very active field of study. A significant portion of its history is connected to the well-known Bieberbach conjecture that $|a_k| \le k$ for $k \ge 2$. This well-known conjecture from 1916 rose to prominence as one of mathematics' most well-known problems.

Now, we denote the class of all functions in $\mathcal{H}$ that they are univalent in D by S (Srivastava & Owa, 1992; Ma & Minda, 1992). Let the functions f and g be analytic in D, then we say that the function f is subordinate to the function g and we write $f(\xi) \prec g(\xi)$, if there is an analytic function ψ in D such that $f(\xi) = g(\psi(\xi))$ where $|\psi(\xi)| < 1$, $\xi \in D$, and $\psi(0) = 0$ (ψ is a Schwarz function). Specifically, if g is univalent in D then the following equivalence relationship is valid

$$f(\xi) \prec g(\xi) \Leftrightarrow f(0) = g(0) \; and \; f(D) \subset g(D).$$

Using the subordination concept, Ma and Minda (1992) introduced the subcategories of starlike and convex functions. Here, we assume a function ς has a positive real part in D, $\varsigma(D)$ is symmetric about the real axis, $\varsigma(0) = 1$, $\varsigma'(0) = m_1 > 0$ and using the power series representation

$$\varsigma(\xi) = 1 + m_1\xi + m_2\xi^2 + m_3\xi^3 + \ldots; \; (\xi \in D). \qquad (1.2)$$

The Ma-and-Minda subcategories of functions are introduced as follows:

$$S^*(\varsigma) = \left\{ f \in \mathcal{H} \; ; \; \frac{\xi f'(\xi)}{f(\xi)} \prec \varsigma(\xi); \quad \xi \in D \right\},$$

and

$$\mathcal{K}(\varsigma) = \left\{ f \in \mathcal{H} \; ; \; 1 + \frac{z f''(\xi)}{f'(\xi)} \prec \varsigma(\xi); \quad \xi \in D \right\}.$$

Here, we need to recall the Koebe one-quarter theorem that stated by Duren (2001):
The range of every function of the class S contains the disk $\{\tau : \; |\tau| < 1/4\}$.

We note that the Koebe one-quarter theorem sensures that the image of D under any function $f \in S$ contains a disk with the center at the origin and the radius $\frac{1}{4}$. Thus, every univalent function $f \in S$ has an inverse $f^{-1}: f(D) \to D$, such that

$$f^{-1}(f(\xi)) = \xi; \;\; (\xi \in D)$$

and

$$f(f^{-1}(\tau)) = \tau; \quad \left(|\tau| < r_0(f) \; such \; that \; r_0(f) \ge \frac{1}{4} \right).$$

where the radius $r_0(f)$ depends on the function f. Note that the inverse function f^{-1} is defined by

$$f^{-1}(\tau) = g(\tau) = \tau - a_2\tau^2 + (2a_2^2 - a_3)\tau^3 - (5a_2^3 - 5a_2a_3 + a_4)\tau^4 + \cdots. \qquad (1.3)$$

We say that the function $f \in \mathcal{H}$ is bi-univalent in D if f and f^{-1} are univalent functions in D and we denote by $\sum$ the class of all bi-univalent functions in D given by (1.1). Some examples of bi-univalent functions in $\sum$ are:

$$\frac{1}{2}\log\left(\frac{1+\xi}{1-\xi}\right), \; -\log(1-\xi), \; \frac{\xi}{1-\xi},$$

(see Alrefai & Ali, 2020). However, $\sum$ does not contain the renowned Koebe function. Additionally, several functions that belong to the class S, like $\xi - \frac{\xi^2}{2}$ and $\frac{\xi}{1-\xi^2}$ are not in the class $\sum$.

A long time ago, scientists were tried to estimate the coefficients of the power series of some classes of bi-univalent functions. For example, it has been proved by Lewin (1967) that $|a_2| < 1.51$ and then conjectured by Brannan and Clunie (1980) that $|a_2| \le \sqrt{2}$. Furthermore, Netanyahu (1969) proved that $max_{f\in\Sigma} |a_2| = \frac{4}{3}$. All that we have mentioned above is related to the geometric properties of analytic functions. In this last decade, some efforts have been made in this field, some of which are mentioned here.

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Subclasses of analytic functions have been investigated from different perspectives. For example, Alimohammadi *et al.* (2020) investigated the strong starlikeness properties as well as the close-to-convexity properties for a new subcategory $G(\alpha,\delta)$ which is a subclass of analytic functions. Also, we mention a work of Mohammed *et al.* (2022) who introduced a subcategory of normalized analytic functions that is defined using a differential inequality and they studied several geometric properties of it.

Lately, the q-calculus (quantum calculus) has become crucial in univalent functions theory, especially for estimating sharp inequality bounds for different subcategory of univalent and bi-univalent functions. The idea of this merger is inspired by Jackson's works. Jackson (1909, 1910) introduced and studied q-derivative operator D_q of a function $f(\xi)$ as follows:

Assume that $q \in (0,1)$

$$\mathfrak{D}_q f(\xi) = \frac{f(q\xi) - f(\xi)}{(q-1)\xi}, \quad (\xi \neq 0)$$

and $\mathfrak{D}_q f(0) = f'(0)$ if $f'(0)$ exists.

For any function $f \in \mathcal{H}$, the simple computation implies

$$\mathfrak{D}_q f(\xi) = 1 + \sum_{k=2}^{\infty} [k]_q \, a_k \, \xi^{k-1} \,; \ (\xi \in D),$$

where,

$$[k]_q = \frac{1-q^k}{1-q}.$$

We observe that

$$\mathfrak{D}_q g(\tau) = 1 + (-a_2)[2]_q \tau + (2a_2^2 - a_3)[3]_q \tau^2 - (5a_2^3 - 5a_2 a_3 + a_4)\,[4]_q \tau^3 + \cdots \quad (1.4)$$

The first one that used this idea in relation with univalent functions was Srivastava (1989). (see also Seoudy, 2014; Toklu, 2019).

However, the problem of estimating the coefficients for every Taylor-Maclaurin series coefficients $|a_k|$ $(k \geq 3;\ k \in \mathbb{N})$ is still an open problem.

In this paper, the authors presented a new subclass $\mathcal{H}_\Sigma(q,\beta;\varsigma)$ of the function class Σ based on q-derivative operator (Jackson-derivative operator) D_q for functions $f(\xi)$ in this new subclass and estimated the upper bounds for the coefficients $|a_2|$ and $|a_3|$, using the techniques previously used by Frasin and Aouf (2011) and Saravanan and Muthunagai (2019) (see also Mohammed, 2021; Abdullah, 2022).

Now, we introduce the category $\mathcal{H}_\Sigma(q,\beta;\varsigma)$ as follows:

Definition 1.1 A function $f \in \Sigma$, as defined by (1.1), belongs to the class $\mathcal{H}_\Sigma(q,\beta;\varsigma)$ if it satisfies the following two conditions:

$$1 + \xi\left(\mathfrak{D}_q f(\xi)\right)' + \beta\xi^2\left(\mathfrak{D}_q f(\xi)\right)'' \prec \varsigma(\xi); \quad \xi \in D,$$

and

$$1 + \tau\left(\mathfrak{D}_q g(\tau)\right)' + \beta\tau^2\left(\mathfrak{D}_q g(\tau)\right)'' \prec \varsigma(\tau); \quad \tau \in D,$$

where $0 < q < 1$, $\beta \geq 0$, $g = f^{-1}$ and ς is the function given by (1.2). The main result can be demonstrated using the following lemma.

Lemma 1.2. (Duren, 2001) If $p \in \mathcal{P}$, where $\mathcal{P}$ represents the class of all functions that are analytic in D, with

$$p(\xi) = 1 + p_1\xi + p_2\xi^2 + \cdots, \ (\xi \in D,\ Re(p(\xi)) > 0) \quad (1.5)$$

then $|p_k| \leq 2$ for every $k \geq 1$.

2. MAIN RESULTS

This section presents some interesting estimate coefficients for the functions in the mentioned subcategory of Σ. Now let us to explain our main result.

Theorem 2.1. Let f be a function in the class $\mathcal{H}_\Sigma(q,\beta,\varsigma)$, then

$$|a_2| \leq \sqrt{\frac{m_1(m_1 + |m_2|)}{2m_1(1+\beta)[3]_q + \left([2]_q\right)^2}}\,,$$

also,

$$|a_3| \leq \frac{m_1(m_1 + |m_2|)}{2\,m_1(1+\beta)[3]_q + \left([2]_q\right)^2} + \frac{|m_2|}{2\,(1+\beta)[3]_q}.$$

Proof. Suppose that the function $f \in \mathcal{H}_\Sigma(q,\beta;\varsigma)$ and $g = f^{-1}$ in this case there are two Schwarz functions $u, v: D \longrightarrow D$, such that

$$1 + \xi\left(\mathfrak{D}_q f(\xi)\right)' + \beta z^2\left(\mathfrak{D}_q f(\xi)\right)'' = \varsigma\big(u(\xi)\big); \ \xi \in D, \quad (2.1)$$

and

$$1 + \tau\left(\mathfrak{D}_q g(\tau)\right)' + \beta\tau^2\left(\mathfrak{D}_q g(\tau)\right)'' = \varsigma\big(v(\tau)\big); \ \tau \in D. \quad (2.2)$$

Now, we define two auxiliary functions h_1 and h_2 by

$$h_1(\xi) = \frac{1+u(\xi)}{1-u(\xi)} = 1 + c_1\xi + c_2\xi^2 + \cdots$$

and

$$h_2(\tau) = \frac{1+v(\tau)}{1-v(\tau)} = 1 + d_1\tau + d_2\tau^2 + \cdots.$$

In other words, we have

$$u(\xi) = \frac{h_1(\xi)-1}{h_1(\xi)+1} = \frac{1}{2}\left(c_1\xi + \left(c_2 - \frac{c_1^2}{2}\right)\xi^2 + \cdots\right), \quad (2.3)$$

and

$$v(\tau) = \frac{h_2(\tau)-1}{h_2(\tau)+1} = \frac{1}{2}\left(d_1\tau + \left(d_2 - \frac{d_1^2}{2}\right)\tau^2 + \cdots\right). \quad (2.4)$$

Then the two functions h_1 and h_2 are analytic in D, $h_1(0) = 1 = h_2(0)$. Given that $u, v: D \longrightarrow D$, the real parts of h_1 and h_2 are nonnegative in D and by using lemma (1.2), $|c_k| \leq 2$ and $|d_k| \leq 2$ for every $k \geq 1$. Recall that

$$\mathfrak{D}_q f(\xi) = 1 + [2]_q\, a_2\xi + [3]_q\, a_3\xi^2 + [4]_q a_4\xi^3 + [5]_q a_5\xi^4 + \cdots,$$

so, we have

$$1 + \xi\left(\mathfrak{D}_q f(\xi)\right)' + \beta\xi^2\left(\mathfrak{D}_q f(\xi)\right)'' =$$

$$1+[2]_q\,a_2\xi+2(1+\beta)[3]_q a_3\xi^2+3(1+2\beta)[4]_q a_4\xi^3+4(1+3\beta)[5]_q a_5\xi^4+\cdots, \quad (2.5)$$

However,

$$\varsigma(u(\xi))=\varsigma\left(\frac{h_1(\xi)-1}{h_1(\xi)+1}\right)=1+\frac{1}{2}m_1c_1\xi+\left(\frac{1}{2}m_1\left(c_2-\frac{c_1^2}{2}\right)+\frac{1}{4}m_2c_1^2\right)\xi^2+\cdots. \quad (2.6)$$

The equations (2.1), (2.3), (2.5), and (2.6) can be equated to obtain:

$$1+[2]_q\,a_2\xi+2(1+\beta)[3]_q\,a_3\xi^2+3(1+2\beta)[4]_q a_4\xi^3+4(1+3\beta)[5]_q a_5\xi^4+\cdots$$
$$=1+\frac{1}{2}m_1c_1\xi+\left(\frac{1}{2}m_1\left(c_2-\frac{c_1^2}{2}\right)+\frac{1}{4}m_2c_1^2\right)\xi^2+\cdots.$$

By comparing the coefficients in the above equation, we get:

$$[2]_q\,a_2=\tfrac{1}{2}m_1c_1, \quad (2.7)$$

and

$$2(1+\beta)[3]_q\,a_3=\frac{1}{2}m_1\left(c_2-\frac{c_1^2}{2}\right)+\frac{1}{4}m_2c_1^2. \quad (2.8)$$

Once more, given

$\mathfrak{D}_q g(\tau)=1+(-a_2)[2]_q\tau+(2a_2^2-a_3)[3]_q\tau^2-(5a_2^3-5a_2a_3+a_4)\,[4]_q\tau^3+\cdots,$

we have

$$1+\tau\left(\mathfrak{D}_q g(\tau)\right)'+\beta\tau^2\left(\mathfrak{D}_q g(\tau)\right)''=$$
$$1+(-a_2)[2]_q\tau+2(1+\beta)(2a_2^2-a_3)[3]_q\tau^2-3(1+2\beta)(5a_2^3-5a_2a_3+a_4)[4]_q\tau^3+\cdots. \quad (2.9)$$

Once more, since

$$\varsigma(v(\tau))=\varsigma\left(\frac{h_2(\tau)-1}{h_2(\tau)+1}\right)=1+\frac{1}{2}m_1d_1\tau+\left(\frac{1}{2}m_1\left(d_2-\frac{d_1^2}{2}\right)+\frac{1}{4}m_2d_1^2\right)\tau^2+\cdots. \quad (2.10)$$

The equations (2.2), (2.4), (2.9) and (2.10) can be equated to obtain:

$$1+(-a_2)[2]_q\tau+2(1+\beta)(2a_2^2-a_3)[3]_q\tau^2-3(1+2\beta)(5a_2^3-5a_2a_3+a_4)[4]_q\tau^3+\cdots$$
$$=1+\tfrac{1}{2}m_1d_1\tau+\left(\tfrac{1}{2}m_1\left(d_2-\tfrac{d_1^2}{2}\right)+\tfrac{1}{4}m_2d_1^2\right)\tau^2+\cdots.$$

Hence, by comparing the coefficients in the above equation, we get:

$$[2]_q\,a_2=-\frac{1}{2}m_1d_1, \quad (2.11)$$

and, also

$$4(1+\beta)[3]_q a_2^2=2(1+\beta)[3]_q a_3+\frac{1}{2}m_1\left(d_2-\frac{d_1^2}{2}\right)+\frac{1}{4}m_2d_1^2. \quad (2.12)$$

By comparing the equations (2.7) and (2.11) we have

$$c_1=-d_1, \quad (2.13)$$

also

$$\left([2]_q\right)^2a_2^2=\frac{1}{4}\,m_1^2d_1^2. \quad (2.14)$$

Now, apply the equations (2.8) and (2.13) in (2.12) to obtain

$$4(1+\beta)[3]_q a_2^2+\frac{1}{2}m_1d_1^2=\frac{1}{2}\left(m_1(c_2+d_2)+m_2d_1^2\right).$$

Given that $m_1>0$, (2.14) yields

$$4(1+\beta)[3]_q a_2^2+\frac{2\left([2]_q\right)^2}{m_1}a_2^2=\frac{1}{2}m_1(c_2+d_2)+\frac{1}{2}m_2d_1^2,$$

and we obtain

$$\frac{4m_1(1+\beta)[3]_q a_2^2+2\left([2]_q\right)^2a_2^2}{m_1}=\frac{1}{2}\left(m_1c_2+m_2d_2+m_2d_1^2\right).$$

Alternatively, obtain that

$$a_2^2=\frac{m_1^2(c_2+d_2)+m_1m_2d_1^2}{8m_1(1+\beta)[3]_q+4\left([2]_q\right)^2} \quad (2.15)$$

It is easy to conclude that

$$|a_2|^2\le\frac{m_1^2|c_2+d_2|+m_1\,|m_2|\,d_1^2}{8m_1(1+\beta)[3]_q+4\left([2]_q\right)^2}.$$

Using Lemma (1.2), we have $|c_n|\le 2$ and $|d_n|\le 2$ and after some e computation this implies that

$$|a_2|^2\le\frac{4\,m_1^2+4\,m_1\,|m_2|}{4\left(2\,m_1(1+\beta)[3]_q+\left([2]_q\right)^2\right)}.$$

Finally, we get

$$|a_2|\le\sqrt{\frac{m_1(m_1+|m_2|)}{2\,m_1(1+\beta)[3]_q+\left([2]_q\right)^2}}.$$

The next step is to find an upper bound for $|a_3|$ and for this purpose we must subtract the equation (2.12) from the equation (2.8) to obtain

$$4(1+\beta)[3]_q\,a_3=4\,(1+\beta)[3]_q\,a_2^2+\frac{1}{2}m_1(c_2-d_2).$$

Alternatively, we conclude that

$$a_3=a_2^2+\frac{m_1(c_2-d_2)}{8\,(1+\beta)[3]_q}. \quad (2.16)$$

Substituting the equation (2.15) in (2.16), we obtain

$$a_3=\frac{m_1^2(c_2+d_2)+m_1m_2d_1^2}{8\,m_1(1+\beta)[3]_q+4\left([2]_q\right)^2}+\frac{m_2(c_2-d_2)}{8\,(1+\beta)[3]_q}.$$

It is easy to conclude that

$$|a_3|\le\frac{m_1^2\,|c_2+d_2|+m_1|m_2|\,d_1^2}{8\,m_1(1+\beta)[3]_q+4\left([2]_q\right)^2}+\frac{|m_2|\,|c_2-d_2|}{8\,(1+\beta)[3]_q}.$$

Again, by using Lemma (1.2) $|c_n| \le 2$ and $|d_n| \le 2$ and this implies that

$$|a_3| \le \frac{4\, m_1^2 + 4\, m_1 |m_2|}{8\, m_1(1+\beta)[3]_q + 4\left([2]_q\right)^2} + \frac{4\,|m_2|}{8\,(1+\beta)[3]_q},$$

Finally, we have

$$|a_3| \le \frac{m_1(m_1 + |m_2|)}{2\, m_1(1+\beta)[3]_q + \left([2]_q\right)^2} + \frac{|m_2|}{2\,(1+\beta)[3]_q}.$$

■

Now, we shall give upper bounds concerning the initial two coefficients of the function f^{-1}. Since $b_2 = -a_2$ (by (1.3)), the upper bound that is obtained for $|a_2|$ also holds for $|b_2|$. Additionally, in order to obtain the upper bound for $|b_3|$ we must perform some calculations based on the equation $b_3 = 2a_2^2 - a_3$ that we will explain it in the following corollary.

Corollary 2.2. If the function f is in the category $\mathcal{H}_\Sigma(q,\beta,\varsigma)$, then

$$|b_3| \le \frac{m_1(m_1 + |m_2|)}{2\, m_1(1+\beta)[3]_q + \left([2]_q\right)^2} + \frac{m_1}{2\,(1+\beta)[3]_q}.$$

Proof. Using the equation (2.15) and the equation (2.16) in the proof of the Theorem 2.1, we have

$$b_3 = \frac{m_1^2(c_2 + d_2) + m_1 m_2 d_1^2}{8\, m_1(1+\beta)[3]_q + 4\left([2]_q\right)^2} + \frac{m_1(d_2 - c_2)}{8\,(1+\beta)[3]_q}.$$

Then, by using Lemma (1.2) and triangle inequality, we conclude that

$$|b_3| \le \frac{m_1(m_1 + |m_2|)}{2\, m_1(1+\beta)[3]_q + \left([2]_q\right)^2} + \frac{m_1}{2\,(1+\beta)[3]_q}.$$

■

At this step, we want to highlight some interesting findings for some special cases of ς in Theorem 2.1. We consider the Taylor-Maclaurin expansion of

$$e^\xi = 1 + \xi + \frac{1}{2!}\xi^2 + \frac{1}{3!}\xi^3 + \cdots$$

and then we obtain a special case of Theorem 2.1 by assuming $\varsigma(\xi) = e^\xi$, $(\xi \in D)$. In this case $m_1 = 1$, $m_2 = \frac{1}{2}$ and we acquire the following result.

Corollary 2.3 If the function f is in the category $\mathcal{H}_\Sigma\left(q,\beta,e^\xi\right)$, then simple computations yield

$$|a_2| \le \sqrt{\frac{3}{4(1+\beta)[3]_q + \left([2]_q\right)^2}},$$

Also,

$$|a_3| \le \frac{3}{4(1+\beta)[3]_q + \left([2]_q\right)^2} + \frac{1}{4(1+\beta)[3]_q}.$$

■

Now, by taking

$$\varsigma(\xi) = \frac{1 + (1-2\delta)\xi}{1-\xi}$$

$$= 1 + 2(1-\delta)\xi + 2(1-\delta)\xi^2 + \cdots;\quad 0 \le \delta < 1,\ \ \xi \in D,$$

we have $m_1 = m_2 = 2(1-\delta)$ and we obtain the next result.

Corollary 2.4. If the function f is in the category $\mathcal{H}_\Sigma\left(q,\beta,\frac{1+(1-2\delta)\xi}{1-\xi}\right)$ such that $0 \le \delta < 1$ then simple computations yield

$$|a_2| \le \sqrt{\frac{8(1-\delta)^2}{4\,(1-\delta)(1+\beta)[3]_q + \left([2]_q\right)^2}},$$

and

$$|a_3| \le \frac{8(1-\delta)^2}{4\,(1-\delta)(1+\beta)[3]_q + \left([2]_q\right)^2} + \frac{(1-\delta)}{(1+\beta)[3]_q}.$$

■

Finally, if we take

$$\varsigma(\xi) = \left(\frac{1+\xi}{1-\xi}\right)^\alpha$$

$$= 1 + 2\alpha\xi + 2\alpha^2\xi^2 + \cdots;\ 0 < \alpha \le 1,\quad \xi \in D$$

we get $m_1 = 2\alpha$, $m_2 = 2\alpha^2$ and we acquire the following result.

Corollary 2.5. If the function f is in the category $\mathcal{H}_\Sigma\left(q,\beta,\left(\frac{1+\xi}{1-\xi}\right)^\alpha\right)$ such that $0 < \alpha \le 1$, then simple computations yield

$$|a_2| \le \sqrt{\frac{4\alpha^2(1+\alpha)}{4\alpha(1+\beta)[3]_q + \left([2]_q\right)^2}},$$

and

$$|a_3| \le \frac{4\alpha^2(1+\alpha)}{4\alpha(1+\beta)[3]_q + \left([2]_q\right)^2} + \frac{\alpha^2}{(1+\beta)[3]_q}.$$

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REFERENCES

Alimohammadi, D., Cho, N. E., Adegani, E. A., & Motamednezhad, A. (2020). Argument and coefficient estimates for certain analytic functions. *Mathematics*, *8*(1), 88. *https://doi.org/10.3390/math8010088*.

Alrefai, O., & Ali, M. (2020). General coefficient estimates for bi-univalent functions: a new approach. *Turkish Journal of Mathematics, 44(1), 240-251.* 10.3906/mat-1910-100.

Brannan, D. A., & Clunie, J. (1980). Aspects of contemporary complex analysis. (No Title).

Duren, P. L. (1983). Grundlehren der Mathematischen Wissenschaffen. Univalent Functions; Springer: New York, NY, USA; Berlin/Heidelberg, Germany, 259.

Duren, P. L. (2001). Univalent functions (Vol. 259). *Springer Science & Business Media.*

Frasin, B. A., & Aouf, M. K. (2011). New subclasses of bi-univalent functions. Applied Mathematics Letters, 24(9), 1569-1573. *https://doi.org/10.1016/j.aml.2011.03.048.*

Jackson, F. H. (1909). XI.—On q-functions and a certain difference operator. *Earth and Environmental Science Transactions of the Royal Society of Edinburgh*, 46(2), 253-281. *https://doi.org/10.1017/S0080456800002751.*

Jackson, F. H. (1910). On q-definite integrals. *Quart. J. Pure Appl. Math,* 41(15), 193-203.

Lewin, M. (1967). On a coefficient problem for bi-univalent functions. *Proceedings of the American mathematical society*, *18*(1), 63-68. *https://doi.org/10.1090/S0002-9939-1967-0206255-1.*

Ma, W. (1992). A unified treatment of some special classes of univalent functions. In Proceedings of the Conference on Complex Analysis, 1992. *International Press Inc.*

Hameed Mohammed, N. H. (2021). Coefficient Bounds for a New Class of Bi-Univalent Functions Associated with Subordination. *Mathematical Analysis and Convex Optimization,* 2(2), 73-82. *10.52547/maco.2.2.8.*

Mohammed, N. H. (2022). Bounds for the coefficients of two new subclasses of bi-univalent functions. *Science Journal of University of Zakho*, *10*(2), 66-69. *https://doi.org/10.25271/sjuoz.2022.10.2.922.*

Mohammed, N. H., Adegani, E. A., Bulboacă, T. E. O. D. O. R., & Cho, N. E. (2022). A family of holomorphic functions defined by differential inequality. *Math. Inequal. Appl*, *25*, 27-39. *http://dx.doi.org/10.7153/mia-2022-25-03.*

Netanyahu, E. (1969). The minimal distance of the image boundary from the origin and the second coefficient of a univalent function in $|z| < 1$. *Archive for rational mechanics and analysis*, 32(2), 100-112. *10.1007/BF00247676.*

Saravanan, G., & Muthunagai, K. (2019). Coefficient bounds for a new subclass of bi-univalent functions defined by q-fractional derivative operator. *Recent Developments in Mathematical Analysis and Computing*, *2095*(1), 030023. *10.1063/1.5097534.*

Seoudy, T. M., & Aouf, M. K. (2014). Convolution Properties for Certain Classes of Analytic Functions Defined by q-Derivative Operator. *In Abstract and Applied Analysis* (Vol. 2014, No. 1, p. 846719). Hindawi Publishing Corporation. *https://doi.org/10.1155/2014/846719.*

Srivastava, H. M. (1989). Univalent functions, fractional calculus, and associated generalized hypergeometric functions. *Univalent Functions, Fractional Calculus, and Their Applications; Srivastava, HM, Owa, S., Eds,* 329-354.

Srivastava, H. M., & Owa, S. (1992). Current topics in analytic function theory. World Scientific.

Toklu, E. (2019). A new subclass of bi-univalent functions defined by q-derivative. *TWMS Journal of Applied and Engineering Mathematics*, 9(1), 84-90.