Available online at sjuoz.uoz.edu.krd
journals.uoz.edu.krd
Science Journal of University of Zakho
Vol. 13, No.1 pp52–57 January-March,2025
p-ISSN: 2663-628X
e-ISSN: 2663-6298

SMARANDACHE ANTI ZERO DIVISORS

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Received:27 Jul 2024 / Accepted:8 Nov., 2024 / Published: 12 Jan., 2025. https://doi.org/10.25271/sjuoz.2024.12.3.1342

ABSTRACT:

In this paper, we study and discuss the concept of Smarandache anti zero divisor (SAZD) element of the ring $\mathbb{Z}_n$ and the group ring $\mathbb{Z}_nG$, where G is a cyclic group of order m generated by g. Moreover, we introduce and discuss the concept of SAZD ideal of the ring $\mathbb{Z}_n$. Some results related to the given concepts are proved in detail. Accordingly, a Computer Algebra System (GAP) is used to verify the results of this study.

KEYWORDS: Zero Divisors, Unit Element, SAZD Element, SAZD Ideal.

1. INTRODUCTION

Smarandache concepts were first introduced by Smarandache (2000). These concepts have been widely studied by many authors (Padilla, 1998; Srinivas & Rao, 2009; Yongxing, 2005; Kandasamy, 2002a). Kandasamy has published many books and papers about Smarandache concepts by creating the Smarandache analogue for the various mathematical theoretical concepts. In 2001, Kandasamy and 2002, Kandasamy and Chetry introduced Smarandache zero divisor elements in semigroups, rings, and group rings. A nonzero element x in a ring R is called a Smarandache zero divisor if $xy = 0$, for some $0 \neq y \in R$, and there exist $a, b \in R \setminus \{0, x, y\}$ such that

1. $ax = 0$ or $xa = 0$,
2. $by = 0$ or $yb = 0$ and
3. $ab \neq 0$ or $ba \neq 0$.

In 2002, Kandasamy published a book entitled "Smarandache Semirings, Semifields, and Semivector Spaces" (Kandasamy, 2002b). She introduced many Smarandache elements in this book, such as Smarandache idempotents, Smarandache units, and SAZD elements. An element x in the semiring S (or any ring R) is said to be SAZD (Kandasamy, 2002b) if there exists an element y such that $xy \neq 0$, and $a, b \in S\setminus\{0, x, y\}$ such that

1. $ax \neq 0$ or $xa \neq 0$,
2. $by \neq 0$ or $.b \neq 0$, and
3. $ab = 0$ or $ba = 0$.

A semiring is a non-empty set S with two binary operations addition '+' and multiplication '·' satisfying the following conditions:

1. (S, +) is a commutative monoid.
2. (S, ·) is a semigroup.
3. $(a + b) \cdot c = a \cdot c + b \cdot c$ and $a \cdot (b + c) = a \cdot b + a \cdot c, \forall\ a, b, c \in S$.

In this study, the SAZD elements of the ring $\mathbb{Z}_n$ and of the group ring $\mathbb{Z}_nG$ are considered, where G is a cyclic group of order m. In addition, we introduce and discuss the concept of SAZD ideals of the ring $\mathbb{Z}_n$. A necessary and sufficient conditions are given that which element is a SAZD element, and which ideal is a SAZD ideal.

The structure of this study is as follows. In Section 2, the concept of SAZD elements is explained with its results. The idea of SAZD ideals with its results is shown in Section 3. Finally, the computational code is given in the Appendix.

SAZD Elements

In this section, we find the SAZD elements of the ring $\mathbb{Z}_n$ and in the group ring $\mathbb{Z}_nG$, where G is a cyclic group of order m generated by an element g. Throughout this paper all rings are finite commutative rings with identity 1. Also, all groups are commutative cyclic groups with identity 1. So the three conditions of SAZD elements are considered as $ax \neq 0$, $by \neq 0$ and $ab = 0$.

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Remark 2.1. The ring $\mathbb{Z}_p$, p is prime, which has no SAZD, since it has no zero divisor.

Proposition 2.2. In $\mathbb{Z}_n$, with the prime factorization of $n = {p_1}^{\alpha_1}{p_2}^{\alpha_2} \dots {p_r}^{\alpha_r}$, for $\alpha_i > 1$ for some i, where $i = 1, \dots, r$, every unit is a SAZD.

Proof: Suppose x is a unit, then there exist a unit $y \in \mathbb{Z}_n$, such that $xy \equiv 1(mod\ n)$.

Now, suppose $a = p_1$, and $b = {p_1}^{\alpha_1 - 1}{p_2}^{\alpha_2} \dots {p_r}^{\alpha_r}$, we see $a, b \in Z_n \backslash \{0, x, y\}$, then

$$xa \not\equiv 0(mod\ n),$$
$$yb \not\equiv 0(mod\ n) \text{ and}$$
$$ab \equiv 0(mod\ n).$$

Hence a unit x is a SAZD of $\mathbb{Z}_n$.

Proposition 2.3. If x is a zero divisor of the ring $\mathbb{Z}_{p^2}$, p is prime, then x is not a SAZD element.

Proof: Suppose x is a zero divisor. Then $x = kp$ for $1 \leq k \leq p - 1$, for each zero divisor $a \in \mathbb{Z}_{p^2} \backslash \{0, x, y\}$, we have $ax \equiv 0(mod\ p^2)$. Hence x is not a SAZD of $\mathbb{Z}_{p^2}$.

Proposition 2.4. In $\mathbb{Z}_{p^k}$, p is prime and $k > 2$, a zero divisor x is a SAZD if and only if $p^{k-1} \nmid x$.

Proof: Suppose that x is a zero divisor such that $p^{k-1} \nmid x$. Then $x = p^s l$, for $p \nmid l$ and $1 \leq s \leq \mathrm{k} - 2$. Then there exist a unit $y \in \mathbb{Z}_{p^k}$, such that $xy \not\equiv 0(mod\ p^k)$.

For each zero divisors $a, b \in \mathbb{Z}_{p^k} \backslash \{0, x, y\}$, such that $a = tp$, such that $t \neq l$ and $p \nmid t$ and $b = p^{k-1}$, then $a \neq x$ and $b \neq x$, we have

$$ax \not\equiv 0(mod\ p^k),$$
$$by \not\equiv 0(mod\ p^k) \text{ and}$$
$$ab \equiv 0(mod\ p^k).$$

Hence x is a SAZD of $\mathbb{Z}_{p^k}$.

Now suppose that x is a SAZD. If $p^{k-1} \mid x$, then for every zero divisor $a \in \mathbb{Z}_{p^k}$, we have $ax \equiv 0(mod\ p^k)$, which is a contradiction with assumption.

Hence a zero divisor x is a SAZD of $\mathbb{Z}_{p^k}$ if and only if $p^{k-1} \nmid x$.

Theorem 2.5 In $\mathbb{Z}_n$, with the prime factorization of $n = {p_1}^{\alpha_1}{p_2}^{\alpha_2} \dots {p_r}^{\alpha_r}$ where p_i are distinct odd primes for $i = 1, \dots, r$, every zero divisor is a SAZD element.

Proof: Suppose x is a zero divisor of the form $x = {p_1}^{\alpha_1}{p_2}^{\alpha_2} \dots {p_k}^{\alpha_k} l$, with $p_j \nmid l$ for $k + 1 \leq j \leq r$. Then there exist a unit $y \in Z_n$, such that $xy \not\equiv 0(mod\ n)$

Consider two zero divisors $a = p_1 t$ and $b = {p_1}^{\alpha_1 - 1}{p_2}^{\alpha_2} \dots {p_r}^{\alpha_r}$ for $p_s \nmid t$,

For $2 \leq s \leq r$ and $t \neq l$, then

$$xa \not\equiv 0(mod\ n),$$
$$by \not\equiv 0(mod\ n) \text{ an}$$
$$ab \equiv 0(mod\ n).$$

Hence every zero divisor of $\mathbb{Z}_n$ is a SAZD.

Corollary 2.6 In $\mathbb{Z}_{pq}$, p and q are distinct odd primes, every zero divisor is a SAZD.

Proof: Let x be a zero divisor of $\mathbb{Z}_{pq}$. We take $x = kp$, for $1 \leq k \leq q - 1$ and we take a unit $y \in \mathbb{Z}_{pq}$, we have $xy \not\equiv 0(mod\ pq)$.

Consider two zero divisors $a, b \in \mathbb{Z}_{pq} \backslash \{0, x, y\}$ such that $a = tp$, and $b = sq$,

for $t \neq k$, and $1 \leq t \leq q - 1$ and $1 \leq s \leq p - 1$ then

$$ax \not\equiv 0(mod\ pq),$$
$$by \not\equiv 0(mod\ pq), \text{ and}$$
$$ab \equiv 0(mod\ pq)$$

The proof is similar for $x = lq$, for $1 \leq k \leq p - 1$.

Hence every zero divisor of $\mathbb{Z}_{pq}$ is a SAZD.

Corollary 2.7 In $\mathbb{Z}_{pqr}$, where $p < q < r$ are distinct odd primes, every zero divisor is a SAZD.

Proof: Let x be a zero divisor, then we take $x = kp$, for $1 \leq k \leq qr - 1$. For a unit $y \in \mathbb{Z}_{pqr}$, we have $xy \not\equiv 0(mod\ pqr)$.

Consider two zero divisors $a, b \in \mathbb{Z}_{pqr} \backslash \{0, x, y\}$ such that $a = tp$, and $b = qr$, where $t \neq k$, and $1 \leq t \leq qr - 1$, we have

$$ax \not\equiv 0(mod\ pqr),$$
$$by \not\equiv 0(mod\ pqr), \text{ and}$$
$$ab \equiv 0(mod\ pqr),$$

The proof is similar for $x = lq$, where $1 \leq l \leq pr - 1$, and $x = mr$, where $1 \leq m \leq pq - 1$.

Hence every zero divisor of $\mathbb{Z}_{pqr}$ is a SAZD.

Proposition 2.8 Every zero divisor x of the ring $\mathbb{Z}_{2p}$, p is an odd prime, is a SAZD if and only if $p \nmid x$.

Proof: Suppose $p \nmid x$, then $x = 2l$, for $1 \leq l \leq p - 1$. Then there exist a unit $y \in \mathbb{Z}_n$, such that $xy \not\equiv 0(mod\ n)$

Consider two zero divisors $a, b \in Z_{2p} \backslash \{0, x\}$, such that $a = 2t$ and $b = p$, for $t \neq l$ and $1 \leq t \leq p - 1$. Now

$$xa \not\equiv 0(mod\ 2p),$$
$$by \not\equiv 0(mod\ 2p) \text{ and}$$
$$ab \equiv 0(mod\ 2p).$$

Hence x is a SAZD of $\mathbb{Z}_{2p}$.

Conversely: Suppose x is a SAZD

If $p \mid x$, then $x = p$ and for every zero divisor $a \in \mathbb{Z}_{2p}$, we have $ax \equiv 0(mod\ 2p)$, which is a contradiction with assumption, therefore $p \nmid x$.

Hence x is a SAZD of $\mathbb{Z}_{2p}$ if and only if $p \nmid x$.

In what follows, we study SAZD elements in some type of group rings. Note that in the following results we mean by G a cyclic group of order n generated by g, where n is any positive integer.

Proposition 2.9 Consider the group ring $\mathbb{Z}_n G$, where G is a cyclic group of order m except the case $m = n = 2$, The element $x = sg^k$ is SAZD, where $0 \neq s \in \mathbb{Z}_n$ and $1 \leq k \leq m$.

Proof: Suppose $y = g$. Then $xy = sg^{k+1} \neq 0$. We take two elements

$a, b \in \mathbb{Z}_n G \backslash \{0, x, y\}$ such that $a = 1 + (n-1)g$ and $b = 1 + g + \cdots + g^{m-1}$, then

$$a.x \neq 0$$
$$b.y \neq 0 \text{ and}$$
$$a.b = 0$$

Hence x is a SAZD of $\mathbb{Z}_n G$.

In the above proposition, if we take $n = 4, m = 3, s = 1$ and $k = 1$, then

$x = g^2, y = g, a = 1 + 3g$ and $b = 1 + g + g^2$.

Now $x.y = g^2.g = g^3 \neq 0$,

$a.x = (1+3g).g^2 \neq 0$,

$b.y = (1 + g + g^2).g \neq 0$, and

$a.b = (1+3g).(1 + g + g^2) = 4 + 4g + 4g^2 = 0$.

Hence x is a SAZD element.

Proposition 2.10 Consider the group ring $\mathbb{Z}_2 G$, where G is a cyclic group of order $m > 3$. If an element x has an odd number of summands, then it is a SAZD element.

Proof: Suppose that x has an odd number of summands, then either $x = g^{i_1} + g^{i_2} + \cdots + g^{i_k}$, where k is odd, or $x = 1 + g^{i_1} + g^{i_2} + \cdots + g^{i_k}$, where k is even and, in both cases $1 \leq i_j \leq m - 1$ and $i_h \neq i_s$ for $j, h, s \in \{1,2, \ldots, k\}$.

For each case of x we take $y = g$, $a = 1 + g$ and $b = 1 + g + \cdots + g^{m-1}$, then $a, b \in \mathbb{Z}_2 G \backslash \{0, x, y\}$

Case1. If $x = g^{i_1} + g^{i_2} + \cdots + g^{i_k}$, then

$x.y = 1 + g^{i_1+1} + g^{i_2+1} + \cdots + g^{i_k+1} \neq 0$,

$a.x = g^{i_1+1} + g^{i_2+1} + \cdots + g^{i_k+1} + g^{i_1} + g^{i_2} + \cdots + g^{i_k} \neq 0$,

$b.y = b \neq 0$ and

$a.b = 0$.

Case 2. If $x = 1 + g^{i_1} + g^{i_2} + \cdots + g^{i_k}$, then

$x.y = g + g^{i_1+1} + g^{i_2+1} + \cdots + g^{i_k+1} \neq 0$,

$a.x = 1 + g^{i_1} + g^{i_2} + \cdots + g^{i_k} + g + g^{i_1+1} + g^{i_2+1} + \cdots + g^{i_k+1} \neq 0$,

$b.y = b \neq 0$ and

$a.b = 0$.

Then x is a SAZD element of $\mathbb{Z}_2 G$.

Proposition 2.11 Every non zero element of the group ring $\mathbb{Z}_p G$, where p is an odd prime and G is a cyclic group of order 2 generated by g, is a SAZD element.

Proof: Let $0 \neq x = a_0 + b_0 g \in \mathbb{Z}_p G$.

Case I: if $a_0 + b_o \neq 0$. Let $y = 1, a = 1 + g$ and $b = 1 + (p-1)g$.

Where $a, b \in \mathbb{Z}_p G \backslash \{0, x, y\}$, then $x.y = a_0 + b_0 g \neq 0$,

$a.x = (a_0 + b_0) + (a_o + b_0)g \neq 0$, because $a_0 + b_0 \neq 0$,

$b.y = 1 + (p-1)g \neq 0$

$a.b = (1+g)(1 + (p-1)g) = 0$.

(In case if $x = 1 + g$, take $a = 2 + 2g$).

Case II: If $a_0 + b_0 = 0$. Let $y = 1, a = b_0 + a_0 g$ and $b = 1 + g$, Where $a, b \in \mathbb{Z}_p G \backslash \{0, x, y\}$, then

$x.y = a_0 + b_0 g \neq 0$,

$a.x = 2a_0 b_0 + (a_0^2 + b_0^2)g$

since $a_0 + b_0 = 0$, So $a_0^2 = b_0^2$ and since $\mathbb{Z}_p$ has no zero divisors, so $2a_0 b_0 \neq 0$ and $2a_0^2 \neq 0$, then

$a.x = 2a_0 b_0 + 2a_0^2 g \neq 0$,

$b.y = 1 + g \neq 0$ and

$a.b = (a_0 + b_0) + (a_0 + b_0)g = 0$.

Therefore, in both cases x is a SAZD element of $\mathbb{Z}_p G$.

Remark 2.12 Consider the group ring $\mathbb{Z}_{\boldsymbol{p}}\boldsymbol{G}$**,** where G is a cyclic group of order p generated by g, and consider the zero divisor $x = a_0 + a_1 g + a_2 g^2 + \cdots + a_{p-1} g^{p-1}$. Then

$x.(1 + g + g^2 + \cdots + g^{p-1}) = 0$ if and only if $a_0 + a_1 + a_2 + \cdots a_{p-1} = 0$.

Proof: Suppose that $a_0 + a_1 + a_2 + \cdots a_{p-1} = 0$.

Now, $x.(1 + g + g^2 + \cdots + g^{p-1}) = (a_0 + a_1 g + \cdots + a_{p-1} g^{p-1})(1 + g + \cdots + g^{p-1})$.

$$= a_0 + a_0 g + a_0 g^2 + \cdots + a_o g^{p-1} + a_1 g + a_1^2 g^2 + \cdots + a_1 g^{p-1} + \cdots + a_{p-1} g^{p-1} + a_{p-1} g + \cdots + a_{p-1} g^{2(p-1)}$$

$$= (a_0 + a_1 + \cdots + a_{p-1}) + (a_0 + a_1 + \cdots + a_{p-1})g + (a_0 + a_1 + \cdots + a_{p-1})g^2 + \cdots + (a_0 + a_1 + \cdots + a_{p-1})g^{p-1}$$

So $x.(1 + g + g^2 + \cdots + g^{p-1}) = (a_0 + a_1 + \cdots + a_{p-1})(1 + g + \cdots g^{p-1})$. Since

$a_0 + a_1 + \cdots + a_{p-1} = 0$, then $x.(1 + g + g^2 + \cdots + g^{p-1}) = 0$.

Conversely: Suppose that $x.(1 + g + g^2 + \cdots + g^{p-1}) = 0$. We have to show that

$a_0 + a_1 + \cdots + a_{p-1} = 0.$,

Now, $x.(1 + g + g^2 + \cdots + g^{p-1}) = (a_0 + a_1 + \cdots + a_{p-1})(1 + g + g^2 + \cdots g^{p-1}) = 0$, then

$a_0 + a_1 + \cdots + a_{p-1} = 0.$ (By: If $k + kg + \cdots kg^{p-1} = 0, then\ k = 0$).

Proposition 2.13 Consider the group ring $\mathbb{Z}_pG$, where G is a cyclic group of order p generated by g. Then a non-zero element $a_0 + a_1g + \cdots + a_{p-1}g^{p-1}$ is a zero divisor of $\mathbb{Z}_pG$ if and only if $a_0 + a_1 + \cdots + a_{p-1} = 0$. Where $a_i \in \mathbb{Z}_p$, for $i \in \{0,1,2, \ldots, p-1\}$.

Proof: Let $x = a_0 + a_1g + \cdots + a_{p-1}g^{p-1}$ and suppose that $a_0 + a_1 + \cdots + a_{p-1} = 0$. Let $y = 1 + g + g^2 + \cdots + g^{p-1}$.

Now $x.y = (a_0 + a_1g + \cdots + a_{p-1}g^{p-1})(1 + g + \cdots + g^{p-1})$

$$= a_0 + a_0g + a_0g^2 + \cdots + a_og^{p-1} + a_1g + a_1^2g^2 + \cdots + a_1g^{p-1} + \cdots + a_{p-1}g^{p-1} + a_{p-1}g + \cdots + a_{p-1}g^{2(p-1)}$$

$$= (a_0 + a_1 + \cdots + a_{p-1}) + (a_0 + a_1 + \cdots + a_{p-1})g + (a_0 + a_1 + \cdots + a_{p-1})g^2 + \cdots + (a_0 + a_1 + \cdots + a_{p-1})g^{p-1}$$

So $x.y = (a_0 + a_1 + \cdots + a_{p-1})(1 + g + \cdots g^{p-1})$, since $a_0 + a_1 + \cdots + a_{p-1} = 0$.

So $x.y = 0$, therefor x is a zero divisor of $\mathbb{Z}_pG$.

Conversely: Suppose that $x = a_0 + a_1g + \cdots + a_{p-1}g^{p-1}$ is a zero divisor of $\mathbb{Z}_pG$. We have to show that $a_0 + a_1 + \cdots + a_{p-1} = 0$

Since x is a zero divisor of $\mathbb{Z}_pG$, So by **Remark 2.12,** $x.(1 + g + \cdots + g^{p-1}) = 0$

$$(a_0 + a_1g + \cdots + a_{p-1}g^{p-1})(1 + g + \cdots + g^{p-1}) = 0$$

So $(a_0 + a_1 + \cdots + a_{p-1})(1 + g + \cdots + g^{p-1}) = 0$

Since $1 + g + \cdots + g^{p-1} \neq 0$. So $a_0 + a_1 + \cdots + a_{p-1} = 0$

Proposition 2.14 consider the group ring $\mathbb{Z}_pG$, where p is a prime and G is a cyclic group of order p generated by g. Then an element $x = a + ag + \cdots + ag^{p-1}$ is not a SAZD element.

Proof: From definition x is a SAZD element, if $\exists\, a, b \in \mathbb{Z}_pG\backslash\{0, x, y\}$ such that $a.x \neq 0, b.y \neq 0$ and $a.b = 0$.

Since a and b are zero divisors, so by **Proposition 2.13,** a and b must be of the form $a = a_0 + a_1g + \cdots + a_{p-1}g^{p-1}, b = b_0 + b_1g + .. + b_{p-1}g^{p-1}$ such that

$a_0 + a_1 + \cdots + a_{p-1} = 0$ and $b_0 + b_1 + \cdots + b_{p-1} = 0$.

But $a.x = (a_0 + a_1g + \cdots + a_{p-1}g^{p-1})(a + ag + \cdots + ag^{p-1})$

$$= (a_0 + a_1 + \cdots + a_{p-1})(a + ag + \cdots + ag^{p-1}),$$

Since $a_0 + a_1 + \cdots + a_{p-1} = 0$, So $a.x = 0$

Therefore x is not a SAZD element.

Remark 2.15 If x is a zero divisor of the group ring $\mathbb{Z}_{2^n}G$, where G is a cyclic group of order 2^m generated by g, then $x.2^{n-1}(1 + g + \cdots + g^{2^m-1}) = 0$.

Lemma 2.16 Consider the group ring $\mathbb{Z}_{2^n}G$, where G is a cyclic group of order 2^m. Then an element $x = a_0 + a_1g + \cdots + a_{2^{m-1}}g^{2^{m-1}}$ is a zero divisor of $\mathbb{Z}_{2^n}G$, if and only if $a_0 + a_1 + \cdots + a_{2^{m-1}} = 2^k$, where $k \geq 1$ and $a_i \in \mathbb{Z}_{2^n}$ for $i \in \{1,2, \ldots, 2^{m-1}\}$.

Proof: Let $x = a_0 + a_1g + \cdots + a_{2^{m-1}}g^{2^{m-1}}$.

Suppose that $a_0 + a_1 + \cdots + a_{2^{m-1}} = 2^k$, where $k \geq 1$. We have to show that x is a zero divisor. Let $y = 2^{n-1}(1 + g + g^2 + \cdots + g^{2^{m-1}})$

Now $x.y = (a_0 + a_1g + \cdots + a_{2^{m-1}}g^{2^{m-1}})(2^{n-1}(1 + g \ldots + g^{2^{m-1}})$

$$= 2^{n-1}(a_0 + a_0g + \cdots + a_0g^{2^{m-1}} + a_1g + a_1g^2 + \cdots + a_1 + \cdots + a_{2^{m-1}}g^{2^{m-1}} + a_{2^{m-1}} + a_{2^{m-1}}g + \cdots + a_{2^{m-1}}g^{2(2^{m-1})})$$

$$= 2^{n-1}\left(a_0(1 + g + \cdots + g^{2^{m-1}}) + a_1(1 + g + \cdots + g^{2^{m-1}}) + \cdots + a_{2^{m-1}}(1 + g + \cdots + g^{2^{m-1}})\right)$$

$$= 2^{n-1}(a_0 + a_1 + \cdots + a_{2^{m-1}})(1 + g + \cdots + g^{2^{m-1}})$$

Since $a_0 + a_1 + \cdots + a_{2^{m-1}} = 2^k$, where $k \geq 1$

So $x.y = 2^{n-1}.2^k(1 + g + \cdots + g^{2^{m-1}}) = 0$

Therefore x is a zero divisor of $\mathbb{Z}_{2^n}G$.

Conversely: Suppose that x is a zero divisor of $\mathbb{Z}_{2^n}G$.

Then by **Remark 2.15**, $x.(2^{n-1}(1 + g + g^2 + \cdots + g^{2^{m-1}}) = 0$

$\left(a_0 + a_1 g + a_2 g^2 + \cdots + a_{2^{m-1}} g^{2^{m-1}}\right) 2^{n-1} \left(1 + g + g^2 + \cdots + g^{2^{m-1}}\right) = 0.$

So $2^{n-1}\left(a_0 + a_1 + a_2 + \cdots + a_{2^{m-1}} + (a_0 + a_1 + a_2 + \cdots + a_{2^{m-1}})g + \cdots + (a_0 + a_1 + a_2 + \cdots + a_{2^{m-1}})g^{2^{m-1}}\right) = 0.$

Then $2^{n-1}(a_0 + a_1 + a_2 + \cdots + a_{2^{m-1}})\left(1 + g + \cdots + g^{2^{m-1}}\right) = 0$

Therefore $2^{n-1}(a_0 + a_1 + a_2 + \cdots + a_{2^{m-1}}) = 0 \equiv 2^n (mod\ 2^n)$.

Thus $a_0 + a_1 + a_2 + \cdots + a_{2^{m-1}} = 2^k$, where $k \geq 1$.

Proposition 2.17. Consider the group ring $\mathbb{Z}_{2^n}G$, where G is a cyclic group of order 2^m. Then the element $x = 2^{n-1} + 2^{n-1}g + 2^{n-1}g^2 + \cdots + 2^{n-1}g^{2^{m-1}}$ is not a SAZD element.

Proof: Suppose that x is a SAZD element. Then there exist $y \in \mathbb{Z}_{2^n}G$ and $a, b \in \mathbb{Z}_{2^n}G \backslash \{0, x, y\}$ such that $xy \neq 0$, $a.x \neq 0$ and $a.b = 0$.

Since a is a zero divisor of $\mathbb{Z}_{2^n}G$, then by **Lemma 2.16,**

$a = a_0 + a_1 g + \cdots + a_{2^{m-1}} g^{2^{m-1}}$ such that $a_0 + a_1 + a_2 + \cdots + a_{2^{m-1}} = 2^h$, where $h \geq 1$. Now:

$a.x = \left(a_0 + a_1 g + a_2 g^2 + \cdots + a_{2^{m-1}} g^{2^{m-1}}\right)\left(2^{n-1} + 2^{n-1}g + \cdots + 2^{n-1}g^{2^{m-1}}\right)$

$= 2^{n-1}a_0 + 2^{n-1}a_1\, g + \cdots + 2^{n-1}a_{2^{m-1}} g^{2^{m-1}} + 2^{n-1}a_0 g + 2^{n-1}a_1 g^2 + 2^{n-1}a_2 g^3 + \cdots + 2^{n-1}a_{2^{m-1}} + 2^{n-1}a_0 g^2 + 2^{n-1}a_1 g^3 + \cdots + 2^{n-1}a_{2^{m-1}} g + \cdots + 2^{n-1}a_0 g^{2^{m-1}} + 2^{n-1}a_1 + \cdots + 2^{n-1}a_{2^{m-1}} g^{2(2^{m-1})}.$

So $a.x = 2^{n-1}(a_0 + a_1 + \cdots + a_{2^{m-1}}) + 2^{n-1}g(a_0 + a_1 + \cdots + a_{2^{m-1}}) + \cdots + 2^{n-1}g^{2^{m-1}}(a_0 + a_1 + \cdots + a_{2^{m-1}}))$.

Since $a_0 + a_1 + \cdots + a_{2^{m-1}} = 2^h, h \geq 1$

So $a.x = 2^{n-1}.2^h\left(1 + g + g^2 + \cdots + g^{2^{m-1}}\right) = 0$, which contradicts with the assumption.

So x is not a SAZD element of the group ring $\mathbb{Z}_{2^n}G$.

Smarandache Anti Zero Divisor (SAZD) Ideals

In this section, we introduce the SAZD ideals. We determine those conditions which make a given ideal to be a SAZD ideal of the ring $\mathbb{Z}_n$. The ideals are symbolled in capital letters, and (0) represents the zero ideal.

Definition 3.1. An ideal I of the ring R is said to be a SAZD ideal, if we can find an ideal J such $IJ \neq (0)$ and there exist two proper ideals K, L of R which are different from I and J such that:

1. $IK \neq (0) or\ KI \neq (0)$.
2. $LJ \neq (0) or\ JL \neq (0)$.
3. $KL = (0) or\ LK = (0)$.

Note 3.2. Since the ring $\mathbb{Z}_n$ is commutative for any positive integer n, then the three conditions in **Definition.** are considered as $IK \neq (0)$, $LJ \neq (0)$ and $KL = (0)$.

Remark 3.3. Trivially the ideal (p^{k-1}) is not a SAZD ideal of the ring $\mathbb{Z}_{p^k}$.

Theorem 3.4. In $\mathbb{Z}_{p^k}$, p is prime and $k > 6$, every non zero ideal is a SAZD ideal except (p^{k-1}).

Proof: The proper ideals of $\mathbb{Z}_{p^k}$ are of the form (p^i), for $i = 1,2, \dots, k-1$.

Case1: If $I = (p)$, Then for $J = (p^2), K = (p^{k-3})$ and $L = (p^3)$ we see that all of them are distinct.

Now $IJ \neq (0), IK \neq (0), JL \neq (0)$ and $KL = (0)$.

Hence $I = (p)$ is a SAZD ideal of $\mathbb{Z}_{p^k}$.

Case2: Let $I = (p^{k-2})$. Then for $J = (p), K = (p^{k-3})$ and $L = (p^3)$ we see that all of them are distinct.

Now $IJ \neq (0), IK \neq (0), JL \neq (0)$ and $KL = (0)$.

Hence I is a SAZD ideal of Z_{p^k}.

Case3: Let $I = (p^s)$ for $1 < s < k-2$. Then for $J = (p), K = (p^2)$ and $L = (p^{k-2})$ we see that all of them are distinct.

Now $IJ \neq (0), IK \neq (0), JL \neq (0)$ and $KL = (0)$.

Hence I is a Smarandache anti zero divisor ideal of Z_{p^k}.

By **Remark 3.3**, we have (p^{k-1}) is not a SAZD ideal.

Therefore, every non zero ideal of Z_{p^k} anti zero divisor ideal except (p^{k-1}).

Remark 3.5. If $p < q$ are primes, then $\mathbb{Z}_{pq}$ has no SAZD ideal.

Proposition 3.6. In $\mathbb{Z}_{pqr}$, where $p < q < r$ are primes, every non zero ideal is a SAZD ideal.

Proof: Clearly proper ideals of $\mathbb{Z}_{pqr}$ are of the form $(p), (q), (r), (pq), (pr)$ and (qr).

We proof for the case $I = (P)$. Take $J = (q), k = (pq)$. and $L = (r)$. We see that ideals I, J, K and L are distinct and $IJ \neq (0), IK \neq (0), JL \neq (0)$ and $KL = (0)$.

The proof is similar for all other ideals.

Hence every non zero ideal of $\mathbb{Z}_{pqr}$ is a SAZD ideal .

Proposition 3.7. In $\mathbb{Z}_{p_1 p_2 \dots p_r}$, where $p_1 < p_2 < \cdots < p_r$ are primes, every proper ideal is a SAZD ideal.

Proof: Let $I = \left(p_1 p_2 \dots p_{j-3} p_j p_{j+3} \dots p_r\right)$ be a proper ideal of $Z_{p_1 p_2 \dots p_r}$. Then we take ideals $J =$

$(p_1p_2 \dots p_{j-3}p_{j-2}p_jp_{j+2}p_{j+3} \dots p_r)$, $L = (p_1p_2 \dots p_{j-3}p_{j-2}p_{j+2}p_{j+3} \dots p_r)$ and $K = (p_{j-1}p_jp_{j+1})$. We see that I, J, K and L are distinct ideals. Now $IJ \neq (0), IK \neq (0), JL \neq (0)$ and $KL = (0)$.

Hence I is a SAZD ideal of $\mathbb{Z}_{p_1p_2 \dots p_r}$.

Therefore, every non zero ideal of $\mathbb{Z}_{p_1p_2 \dots p_r}$ is a SAZD ideal.

Theorem 3.8. In $\mathbb{Z}_n$, with the prime factorization of $n = {p_1}^{\alpha_1}{p_2}^{\alpha_2} \dots {p_r}^{\alpha_r}$, for $\alpha_i \geq 1$ where $i = 1,2, \dots, r$, every proper ideal is a SAZD ideal.

Proof: Let $I = ({p_1}^{\alpha_1 - s_1}{p_2}^{\alpha_2 - s_2} \dots {p_{j-3}}^{\alpha_{j-3} - s_{j-3}}{p_j}^{\alpha_j}{p_{j+3}}^{\alpha_{j+3} - s_{j+3}} \dots {p_r}^{\alpha_r - s_r})$, for $s_i \geq 1$, and $i = 1, \dots, r$. Then we take

$J = ({p_{j-3}}^{s_{j-3}}), K = ({p_1}^{\alpha_1}{p_2}^{\alpha_2} \dots {p_{j-3}}^{\alpha_{j-3}}{p_{j+3}}^{s_{j+3}} \dots {p_r}^{\alpha_r}$ and $L = ({p_{j-2}}^{\alpha_{j-2}}{p_j}^{\alpha_j}{p_{j-2}}^{s_{j-2}})$, we see that I, J, K and L are distinct ideals. Now

$IJ \neq (0), IK \neq (0), JL \neq (0)$ and $KL = (0)$.

Hence every non zero ideal I of $\mathbb{Z}_n$ is a SAZD ideal.

Appendix

The well-known Computer Algebra System (GAP)-code was studied in The GAP Group (2016), we use this code in this study to check the results. If this code is applied, one can see that which element of the ring $\mathbb{Z}_n$ or which element of the group ring $\mathbb{Z}_nG$ is a SAZD element, and which element is not a SAZD element, for any positive integer n. For instance, if we put $m = 2$, and $k = (1,2,3,4)$ in the first row of the code, it means that which element of the group ring $\mathbb{Z}_2G$ is a SAZD element, where G is a cyclic group of order 4 generated by g. After running the code, we get the following results:

$\mathbb{Z}_2G = \{0, 1, g, g^2, g^3, 1+g, 1+g^2, 1+g^3, g+g^2, g+g^3, g^2+g^3, 1+g+g^2, 1+g+g^3, 1+g^2+g^3, g+g^2+g^3, 1+g+g^2+g^3\}$,

where

0 and $1+g+g^2+g^3$ are not SAZD elements, and the other elements are SAZD.

```
#SAZD
GR:=GroupRing(ZmodnZ(m),Group(k));
A:=[];
B:=[];
F:=[];
e:=Elements(GR);
  for i in [1..Size(e)] do
     for j in [1..Size(e)] do
      x:=e[i];
      y:=e[j];
     if x*y<>Zero(GR)then
      for k in [2..Size(e)] do
       for h in [2..Size(e)] do
        a:=e[k];
        b:=e[h];
        if a<>x and a<>y then
         if b<>x and b<>y then
          if (a*b=Zero(GR)and a*x<>Zero(GR)and
b*y<>Zero(GR)) then
          #Print(a,"*",b,"    ");
            Add(B,x);
            Add(A,["x=",x,"          y=",y,"         a=",a,"
b=",b]);
            Print(x,"  ",y,"  ",a,"  ",b,"     ");
          fi;
         fi;
        fi;
        if x in B then
         break;
        fi;
       od;
       if x in B then
        break;
       fi;
      od;
      if x in B then
       break;
      fi;
     fi;
    od;
if not x in B then
Add(F,x);
fi;
  od;
A:=AsSet(A);
B:=AsSet(B);
F:=AsSet(F);
Print(F,"are not SAZDs","\n");
Print(B,"are SAZD","\n");
Print(Size(F),"\n",Size(B),"\n",Size(GR));
```

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