Available online at sjuoz.uoz.edu.krd

journals.uoz.edu.krd

Science Journal of University of Zakho
Vol. 13, No.1 pp83–88January-March,2025

p-ISSN: 2663-628X
e-ISSN: 2663-6298

GLOBAL STABILITY AND HOPF BIFURCATION OF A DELAYED PREDATOR-PREY MODEL INCORPORATING ALLEE EFFECT AND FEAR EFFECT

Didar Akram Abdulqadr[1] and Arkan Nawzad Mustafa[1*]

[1]Department of Mathematics, College of Education, University of Sulaimani, Sulaymaniyah 46001, Iraq.
*Corresponding Author. arkan.mustafa@univsul.edu.iq

Received: 17 Oct 2024 / Accepted:9 Jan., 2025/ Published: 22 Jan., 2025. https://doi.org/10.25271/sjuoz.2025.13.1.1406

ABSTRACT:

This paper aims to discover the impact of the fear of predators in prey, Allee effect for predator reproduction and time delay corresponding to the gestation period on the dynamics of a predator- prey model. Existence, non-negativity, and boundedness of the model solutions are guaranteed. The criteria for asymptotically stability of all the biologically feasible steady state points are determined. It is also determined a critical value for time delay, where the model under goes Hopf -bifurcation near coexistence steady state point. Finally, with the help of the MATLAB program, to confirm the analytical results and discover the impact of fear, the Allee effect, and time delay, the model was solved numerically.it is observed that fear affect negatively on both prey and predator species and the time delay may system may induce a transition of the dynamics of system from the a stability situation to the state where the populations oscillate periodically or vice versa.

KEYWORD: Fear, Time delay, Allee effect, Stability analysis, Hopf bifurcation.

1. INTRODUCTION

Predator-prey interactions are an important feature of ecological communities, and many researchers have used mathematics to study the dynamic interactions between predators and prey. Studying the factors that affect the dynamics of predator-prey interactions through mathematical models has become an important area of research in ecology and theoretical biology.

Xiaoying et al. (2016) considered a predator–prey model with incorporating effect of fear on prey reproduction. In their paper, they consider two prey predator models. The first one incorporates the bilinear functional response while the second one incorporates Holling type II functional response, they observed that fear has no impact on the stability of first model, but for the second one, they showed that the fear can make the system become stable. Based on their model, Pal et al. (2019) discussed the stability and some bifurcation types of a prey predator model with effect of fear and harvesting. Huisen et al. (2019) showed that fear effect can stabilize a predator–prey model with prey refuge. Pingping et al. (2021) showed that fear can change the chaotic state of a food chain model to a stable state. Yipping et al. (2022) considered and studied the impact reduction of prey growth rate due to the anti-predator behavior on a predator-prey model when an epidemic disease is spread among the prey population. For more results about the fear effect, see (Jimil et al.,2023; Xiaoqin et al., 2020; Menxin et al., 2022; Soumitr et al.,2023; Yaseen et al.,2024).

The period of the time between the prey predation and predator response to the predation is called ecological time lag. Hague (2011) investigated effect of delay in a LotkaVolterra type predator-prey model with a transmissible disease in the predator species. Jliu (2021) studied the dynamics of a predator-prey model with the effect of both fear and time Delay. Dehingia (2022) investigated a tumor-macrophages interaction model with a discrete-time delay in the growth of pro-tumor M2 macrophages. For more results on time delay, see (Naji et al. 2020; Lavanya et al., 2022; Rihan et al., 2020; Dehingia et al., 2023; Das,2024; Dehingia et al., 2024).

The concept of fitness is central to the study of Allee effects. In particular, a demographic Allee effect refers to a positive correlation between the size or density of a population and the average fitness of the individuals in it**.** In other words, the greater the size or density of the population, the greater the average fitness. Alternatively, the lower the size or density of the population, the lower the average fitness (Alan, 2015).

Soura (2018) studied an ecological model with multiple Allee effects induced by fear factors. Yining et al. (1996) proposed a delay diffusive predator–prey model with a strong Allee effect in the prey and a fear effect on predator, they showed that the parameters of fear species. Alan (2015) considered the following prey predator modeling with Beddington-DeAngelis:

$$\frac{dN}{dt} = rN\left(1-\frac{N}{K}\right)-\frac{aNP}{1+bN+qP}$$
$$\frac{dP}{dt} = \frac{caNP}{1+bN+qP}\left(\frac{P}{h+P}\right)-dP-mP^2$$

(1)

Where, r is rate of intrinsic growth of prey. k is the carrying capacity, h is the intensity of Allee effect; a is the consumption rate of prey by predator; b is the effect of capture rate; c is the conversion and m is rate of predator aggression.

The aim of this paper is to discover the impact of the delay time between the prey predation and predator response to the predation and the predator fear on prey reproduction on the dynamic of trajectories of system (1). Therefore, by the aforementioned works, we modified system (1) by incorporating

---

* Corresponding author

it with the effect of both fear and time lag. The modified system (1) can be written as follows:

$$\frac{dN}{dt}=\frac{bN}{1+fP}-d_1N-cN^2-\frac{aNP}{1+rN+qP}$$
$$\frac{dP}{dt}=\frac{eaN(t-\tau)P}{1+rN(t-\tau)+qP}\left(\frac{P}{h+P}\right)-d_2P-mP^2$$

(2)

Where, $N(0)>0$, $P(0)>0$ and parameters are positive, their description are given in Table 1.

Writing this paper arranged as follow: in the next section, some property of the solution of system (2) are proved. Locally as well as globally, asymptotically stability conditions as well as of all feasible equilibrium points are determined, in section three. In section four, Hopf- bifurcations, near all steady state points, are discussed and the critical value for time delay, where the model undergoes Hopf -bifurcation near coexistence equilibrium points is founded. In section five, the model is solved numerically using modified Euler method. Finally, in section six, a brief conclusion on the whole work is given.

Table 1. Parameter description of system (2)

parameters	Description
b	Prey Birth rate in absence of fear of predators
f	Level of fear due to prey response to anti-predators.
d_1,d_2	Mortality rate of prey and predator, respectively
c,m	Intraspecific competition rates of prey and predator, respectively in
a	Rate of predation predator.
e	Conversion efficiency from biomass of prey to biomass of predator
r	capture rate
q	Rate of reciprocal interaction among predators

2. SOME PROPERTIES OF THE SOLUTIONS OF SYSTEM (2)

The function in the right-hand side system (2) is continuous and has partial derivatives on the spaceR^2. Therefore, system (2) satisfies the Lipschitzian condition. Therefore, it has a unique solution. Further, the time derivative of N is zero when $N=0$ and the time derivative of P is zero when$P=0$. Therefore, if the solution of system (2) initiates at a non-negative point, then the components N and P of the solution points of system (2) cannot cross $N-axis$ and $P-axis$ of the solution points. Hence components $N$and P are always non negative.

From system (2), it gets

$$\left(b-d_1-\frac{1}{q}\right)N\left(1-\frac{cqN}{bq-d_1q-1}\right)\le\frac{dN}{dt}\le(b-d_1)N\left(1-\frac{cN}{b-d_1}\right)$$

$$\frac{dP}{dt}\le eaN(t-\tau)P-mP^2$$

Therefore, the following Theorem can be derived.

Theorem 1. Any solution of system (2)initiate positively, satisfies the following:

1. If $b<d_1$, then $\lim_{t\to\infty}N(t)=\lim_{t\to\infty}P(t)=0$.
2. If $b>d_1$, then $\lim_{t\to\infty}Sup\,X(t)\le\frac{b-d}{c}$ and $\lim_{t\to\infty}Sup\,P(t)\le\frac{ea(b-d)}{mc}$.
3. If $b>d_1+\frac{1}{q}$, then $\lim_{t\to\infty}Inf\,X(t)\ge\frac{bq-d_1q-1}{qc}>0$.

Note. The first and the second part of the above theorem, tell us that all solution of system (2) are bounded, while the third part makes clear that under condition $b>d_1+\frac{1}{q}$, the prey species persist continuously.

i.

3. STEADY STATES AND THEIR STABILITY ANALYSIS

System (2) has the most three steady states. They are the total extinction steady state$S_0(0,0)$, which always exists, the Predator-free steady state $S_1\left(\frac{b-d_1}{c},0\right)$, which exists, if $b>d_1$and coexistence steady state $S_2(N^*,P^*)$, where $N^*=\frac{(d_2+m\,P^*)(h+P^*)(1+q\,P^*)}{ea\,P^*-r(d_2+m\,P^*)(h+P^*)}$ and P^* is a positive root of $G(P)$ where,

$$G(P)=\frac{b}{1+fP}-\left(d_1+cF(P)\right)-\frac{aP}{1+rF(P)+qP}$$ with $$F(P)=\frac{(d_2+mP)(h+P)(1+qP)}{eaP-r(d_2+mP)(h+P)}$$

From Theorem 1, we have $\lim_{t\to\infty}SupP(t)\le\frac{ea|b-d|}{mc}$. So, we search P^* in $\left[0,\frac{ea|b-d|}{mc}\right]$. Mean value theorem guaranteed that $G(P)$ has a positive root P^* if, $G(0)<0$ and $G\left(\frac{ea|b-d|}{mc}\right)>0$ or $G(0)<0$ and $G\left(\frac{ea|b-d|}{mc}\right)>0$. Further, if $ea\,P^*>r(d_2+m\,P^*)(h+P^*)$, the coexistence steady state exists.

To investigate the locally asymptotical stability (LAS) and globally asymptotical stability (GAS) for each steady state, firstly, let linearize system (2) around a point (N,P). Using the perturbed variables $U(t)=N(t)-N$ and$V(t)=P(t)-P$, system (2) can be linearized as follows:

$$\begin{pmatrix}\frac{dU(t)}{dt}\\ \frac{dV(t)}{dt}\end{pmatrix}=J_1(N,P)\begin{pmatrix}U(t)\\ V(t)\end{pmatrix}+J_2(N,P)\begin{pmatrix}U(t-\tau)\\ V(t-\tau)\end{pmatrix}$$

Where, $J_2(N,P)=\begin{pmatrix}0 & 0\\ \frac{ea(1+qP)P}{(1+rN+qP)^2} & 0\end{pmatrix}$

and $J_1(N,P)=$

$$\begin{pmatrix}\frac{b}{1+fP}-d_1-2cN-\frac{a(1+qP)P}{(1+bN+qP)^2} & -\frac{bfN}{(1+fP)^2}-\frac{a(1+rN)N}{(1+rN+qP)^2}\\ 0 & \frac{ca(1+rN)NP}{(1+rN+qP)^2(h+P)}+\frac{eahNP}{(1+rN+qP)(h+P)^2}-d_2-2mP\end{pmatrix}$$

The total extinction steady state $S_0(0,0)$

The eigenvalues of $J_1(0,0)+e^{-\lambda\tau}J_2(0,0)$, are $b-d_1$ and $-d_2$

So, $S_0(0,0)$ is LAS if and only if, $b<d_1$

Further, from theorem 1, it is proved that for any initial value of $N(t)$ and

$$\lim_{t\to\infty} N(t) = \lim_{t\to\infty} P(t) = 0, \text{ if } \ b < d_1.$$

Therefore, $S_0(0,0)$ is GAS if and only if, $b < d_1$

ii. The predator-free steady state $S_1\left(\frac{b-d_1}{c}, 0\right)$

The eigenvalues of $J_1\left(\frac{b-d_1}{c}, 0\right) + e^{-\lambda\tau} J_2\left(\frac{b-d_1}{c}, 0\right)$, are $d_1 - b$ and $-d_2$

So, $S_1\left(\frac{b-d_1}{c}, 0\right)$ is LAS if and only if, $b > d_1$.

Further, the GAS for $S_1\left(\frac{b-d_1}{c}, 0\right)$ is given in the following theorem

Theorem 2. If $S_1 = \left(\frac{b-d_1}{c}, 0\right)$ is exist, then it is GAS, if

$$f^2 b^2 < 4cm \qquad (4)$$

$$a(b - d_1) < d_2 c \qquad (5)$$

Proof. Consider the function

$$L_1(N,P) = N - \frac{b-d_1}{c} - \frac{b-d_1}{c} ln\left[\frac{cN}{b-d_1}\right] + P + \int_{t-\tau}^{t} \frac{eaN(s)P}{1 + rN(s) + qP}\left(\frac{P}{h+P}\right) ds$$

It is clear that $L_1(N,P)$ is positive and $L_1(N,P) = 0$,if and only if $N = \frac{b-d_1}{c}$ and $P = 0$. Further,

$$\frac{dL_1(N,P)}{dt} = \left(N - \frac{b-d_1}{c}\right)\left[\frac{-bfP}{1+fP} - c\left(N - \frac{b-d_1}{c}\right) - \frac{aP}{1+rN+qP}\right] + \frac{eaN(t-\tau)P}{1+rN(t-\tau)+qP}\left(\frac{P}{h+P}\right) - d_2 P - mP^2 + \frac{eaN(t)P}{1+rN(t)+qP}\left(\frac{P}{h+P}\right) - \frac{eaN(t-\tau)P}{1+rN(t-\tau)+qP}\left(\frac{P}{h+P}\right)$$

Accordingly,

$$\frac{dL_1(N,P)}{dt} < -c\left(N - \frac{b-d_1}{c}\right)^2 - \frac{bfP}{1+fP}\left(N - \frac{b-d_1}{c}\right) - mP^2 + \frac{a(e-1)NP}{1+rN+qP} + \left[\frac{a(b-d_1)}{c(1+rN(t)+qP)} - d_2\right] P$$

Conditions (4) and (5) guarantee that $\frac{dL_1(N,P)}{dt}$ is negative, this completes the proof.

iii. The coexistence steady state $S_2(N^*, P^*)$

The linearized system around $S_2(N^*, P^*)$ can be written as

$$\frac{dU}{dt} = N_1 U - N_2 V$$
$$\frac{dV}{dt} = P_1 U(t-\tau) + P_2 V \qquad (6)$$

Where, $U(t) = N(t) - N^*$, $V(t) = P(t) - P^*$, $N_1 = \frac{b}{1+fP^*} - d_1 - 2cN^* - \frac{a(1+qP^*)P^*}{(1+rN^*+qP^*)^2}$, $N_2 = \frac{bfN^*}{(1+fP^*)^2} + \frac{a(1+rN^*)N^*}{(1+rN^*+qP^*)^2}$, $P_1 = \frac{ea(1+qP^*)P^*}{(1+rN^*+qP^*)^2}$ and $P_2 = mP^* - (d_2 + mP^*)\left[\frac{h}{h+P^*} - \frac{qP^*}{1+rN^*+qP^*}\right]$

Theorem 3. If $S_2(N^*, P^*)$ is exist, then it is LAS if,

$$P_2 < -\mathrm{e}P_1 \qquad (7)$$

$$d_1 + 2cN^* > \frac{b}{1+fP^*} - \frac{1}{2}\left[\frac{bfN^*}{(1+fP^*)^2} + \frac{a(1+rN^*)N^*}{(1+rN^*+qP^*)^2}\right]^2 P_2 \qquad (8)$$

Proof. Consider the function

$$L_2(U,V) = \frac{1}{2}U^2 + \frac{1}{2}V^2 + \frac{P_1}{e}\int_{t-\tau}^{t} U^2(s) ds$$

It is clear that $L_2(U,V)$ is positive and$L_2(0,0)$, if and only if $U = 0$ and $V = 0$

$$\frac{dL_2(U,V)}{dt} = -\left(d_1 + 2cN^* - \frac{b}{1+fP^*}\right)U^2(t) - N_2 U(t)V(t) - \frac{1}{2}P_2 V^2(t) - \frac{P_1}{e}U^2(t-\tau) - P_1 U(t-\tau)V(t) - \frac{1}{2}P_2 V^2(t)$$

Due to conditions (7), (8), it gets

$$\frac{dL}{dt} < -\left[\sqrt{d_1 + 2cN^* - \frac{b}{1+fP^*}} + \sqrt{-\frac{P_2}{2}}\right]^2 - \left[\sqrt{\frac{P_1}{e}} + \sqrt{\frac{P_2}{2}}\right]^2$$

This completes the proof.

Theorem 4. If $S_2(N^*, P^*)$ exists, then it is GAS if,

$$\left[\frac{ahqP^*P - ah^2 - arh^2N^*}{G^*G(N,P)} - \frac{f}{F(P)}\right]^2 < \frac{4m}{e}\left[c - \frac{arh^2P^* + arhP^*p}{G^*G(N,P)}\right] \qquad (9)$$

$$\frac{G^*}{aN^{*2}P^*}\left(\frac{aN^*P^*P}{G^*} + 2 - \frac{aN^*P^2}{G(N,P)} - \frac{aNP^{*2}}{G^*}\right) > \frac{G(N,P)P}{G^*N} \qquad (10)$$

Where, $G(N,P) = (1 + rN + qP)(h + p)$, $G^* = (1 + rN^* + qP^*)(h + P^*)$ and

$$F(P) = (1 + fP^*)(1 + fP)$$

Proof. Consider the function $L(N,P) = L_3(N,P) + L_4(N,P)$ where,

$$L_3(N,P) = N - N^* - N^* \ln\frac{N}{N^*} + \frac{1}{e}\left[P - P^* - P^* \ln\frac{P}{P^*}\right]$$

$$L_4(N,P) = a\int_{t-\tau}^{t}\left[\frac{N(S)P^2}{G(N(S),P)} - \frac{N^*P^{*2}}{G^*} - \frac{N^*P^{*2}}{G^*}\ln\frac{G^*N(S)P^2}{G(N(S),P)N^*P^{*2}}\right] ds$$

It is clear that $L_3(N,P)$ and $L_4(N,P)$ are positive and$L_3(N^*, P^*) = 0$, if and only if $N = N^*$ and$P = P^*$. Further,

$$\frac{dL_3(N,P)}{dt} = -\left[c - \frac{arh^2P^* + arhP^*p}{G^*G(N,P)}\right](N - N^*)^2 + \left[\frac{ahqP^*P - ah^2 - arh^2N^*}{G^*G(N,P)} - \frac{f}{F(P)}\right](N - N^*)(P - P^*)$$

$$-\frac{m}{e}(P-P^*)^2+\frac{aN^*P^2}{G(N,P)}+\frac{aNP^{*2}}{G^*}-\frac{aNP^2}{G(N,P)}+\frac{aN(t-\tau)P^2}{G(N(t-\tau),P)}-\frac{aN(t-\tau)P^*P}{G(N(t-\tau),P)}-\frac{aN^*P^*P}{G^*}$$

and

$$\frac{dL_4(N,P)}{dt}=\frac{aNP^2}{G(N,P)}-\frac{aN(t-\tau)P^2}{G(N(t-\tau),P)}-\ln\frac{G^*NP^2}{G(N,P)N^*P^{*2}}+\ln\frac{G^*N(t-\tau)P^2}{G(N(t-\tau),P)N^*P^{*2}}$$

So,

$$\begin{aligned}\frac{dL(N,P)}{dt}&=-\left[c-\frac{arh^2P^*+arhP^*p}{G^*G(N,P)}\right](N-N^*)^2+\left[\frac{ahqP^*P-ah^2-arh^2N^*}{G^*G(N,P)}-\frac{f}{F(P)}\right](N-N^*)(P-P^*)\\&-\frac{m}{e}(P-P^*)^2+\frac{aN^*P^2}{G(N,P)}+\frac{aNP^{*2}}{G^*}-\frac{aN(t-\tau)P^*P}{G(N(t-\tau),P)}-\frac{aN^*P^*P}{G^*}-\frac{aN^*P^{*2}}{G^*}\ln\frac{G^*NP^2}{G(N,P)N^*P^{*2}}\\&+\frac{aN^*P^{*2}}{G^*}\ln\frac{G^*N(t-\tau)P^2}{G(N(t-\tau),P)N^*P^{*2}}\\&=-\left[c-\frac{arh^2P^*+arhP^*p}{G^*G(N,P)}\right](N-N^*)^2+\left[\frac{ahqP^*P-ah^2-arh^2N^*}{G^*G(N,P)}-\frac{f}{F(P)}\right](N-N^*)(P-P^*)\\&-\frac{m}{e}(P-P^*)^2\ -\frac{aN^*P^{*2}}{G^*}\left[\frac{N(t-\tau)G^*P}{N^*P^*G(N(t-\tau),P)}-1-\ln\frac{N(t-\tau)G^*P}{N^*P^*G(N(t-\tau),P)}\right]\\&-\frac{aN^*P^{*2}}{G^*}\left[-\frac{G^*}{aN^*P^{*2}}\left(\frac{aN^*P^2}{G(N,P)}+\frac{aNP^{*2}}{G^*}-\frac{aN^*P^*P}{G^*}-2\right)-1-\ln\frac{G(N,P)N^*P}{G^*NP^*}\right]\end{aligned}$$

Conditions (9) and (10) grantee that $\frac{dL(N,P)}{dt}$ is negative**.** This completes the proof.

4. HOPF-BIFURCATION

The necessary condition for undergoing Hopf bifurcation near a steady state point (N,P) of system (2) is that, the eigenvalues of $J_1(N,P)$ $+e^{-\lambda\tau}J_2(N,P)$ are two complex conjugate. Since $b-d_1$ and $-d_2$ are the eigenvalues of $J_1(0,0)+e^{-\lambda\tau}J_2(0,0)$ and d_1-b and $-d_2$ are the eigenvalues of $J_1\left(\frac{b-d_1}{c},0\right)+e^{-\lambda\tau}J_2\left(\frac{b-d_1}{c},0\right)$ so**,** there is no possibility to have a Hopf-bifurcation near $S_0(0,0)$ and$S_1\left(\frac{b-d_1}{c},0\right)$.

The conditions that guarantee the occurring of Hopf-bifurcation near the coexistence steady state $S_2=(N^*,P^*)$ are established in the following theorem.

Theorem 5. If $S_2(N^*,P^*)$ is exists and the following conditions hold:

$N_1<0$ and $P_2<0$ (11)

$N_1^2P_2^2<N_2^2P_1^2$ (12)

$(N_1+P_2)y_0<N_2P_1$ (13)

then, at $\tau=\bar{\tau}$,System (2) undergoes a Hopf-bifurcation near $S_2=(N^*,P^*)$, where $\bar{\tau}$ and y_0 are given in the proof.

Proof. The eigenvalues of $J_1(N^*,P^*)$ $+e^{-\lambda\tau}J_2(N^*,P^*)$ satisfy the equation

$$\lambda^2-(N_1+P_2)\lambda+N_1P_2+N_2P_1e^{-\lambda\tau}=0$$

Clearly, the roots of the above equation are neither zero nor positive. Therefore, the eigenvalues are negative or complex. Note that when$\tau=0$, condition 9, guarantees that all eigenvalues have negative real part. Suppose $\tau\neq 0$, $\lambda(\tau)=x(\tau)+iy(\tau)$ is the root of the equation 30, and $\bar{\tau}$ is least positive number such that $x(\bar{\tau})=0$,

Then

$$x^2(\tau)-y^2(\tau)-(N_1+P_2)x(\tau)+N_1P_2=N_2P_1e^{-x(\tau)\tau}\cos(\tau y(\tau)) \qquad (14)$$

$$2x(\tau)y(\tau)-(N_1+P_2)y(\tau)=N_2P_1e^{-x(\tau)\tau}\sin(\tau y(\tau)) \qquad (15)$$

Putting $\tau=\bar{\tau}$ in the above two equations, then adding and squaring them, the following equation get

$$y^4(\bar{\tau})+(N_1^2+P_2^2)y^2(\bar{\tau})+N_1^2P_2^2-N_2^2P_1^2=0 \qquad (16)$$

It is obvious that under condition (12), Eq. 1, always has one and only positive root, sayy_0.

From Eq. (15), it gets

$$(N_1+P_2)y_0=-N_2P_1\sin(\bar{\tau}y_0) \qquad (17)$$

Due to condition 13, Eq. 17 has much positive solution, let $\tau=\bar{\tau}$ be least positive satisfy Eq. 15.Further, suppose $\left[\frac{\partial x}{\partial\tau}\right]_{\tau=\bar{\tau}}=0$, then from Eq. 14 and Eq. 15, it gets

$y_0^2=-\frac{1}{2}(N_1^2+P_2^2)$, which is impossible because $y_0^2>0$, therefore $\left[\frac{\partial x}{\partial\tau}\right]_{\tau=\bar{\tau}}\neq 0$,

The proof is completed.

5. NUMERICAL COMPUTATION

In this section, some numerical simulations were conducted by using the method of modification Euler rule, with the help of MATLAB Program. The aim of numerical simulation is to confirm the analytical finding observed in the previous sections and discover the impact of fear, Allee effect, and time delay on the dynamics of components of system (2). First, lets choose the parameter values as follows:

$$b=2.5;\ d_1=\ d_2=0.01;c=0.1;a=0.09$$
$$e=0.8;m=0.01;h=0.5;f=1;q=0.1;\ \tau=7$$

Fig.1 shows that trajectory of system (2) approaches coexistence free steady state point, and since the parameter values given by (18), they satisfy the global stability condition in Theorem3. So Fig.1 confirms analytical result regarding to stability condition ofS_2.

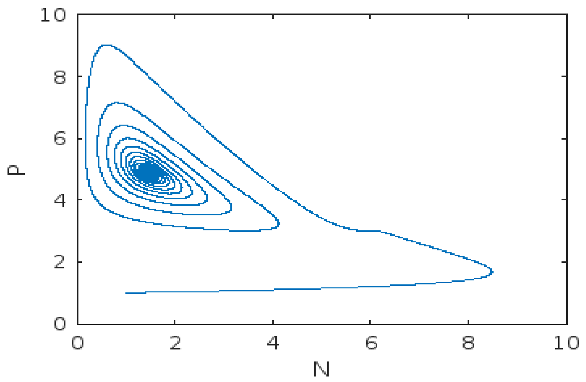

Figure 1: the phase portrait show that trajectory of system 2 approaches coexistence steady state point, when and other parameter values are as given in (18).

To show the impact of time lag, fear and Allee effect, and time lag on the dynamics of system 2. Lets solve system 2 with varying τ, and fixed others as given in (18). See Fig.2, Fig.2and Fig.3.

For the parameter values in (18), the bifurcation value of time delay in Theorem, is $\bar{\tau} \approx 7.8$, therefore, we solve system (2) when the time delay varying from7 to 9 and fixed other parameter values given in (18), the value of τ in range[7,9], see Fig.2.

In Fg.2, it has been shown that**,** dynamics of the system may induce a transition from the stability situation to the state where the populations oscillate periodically when the time delay value increases.

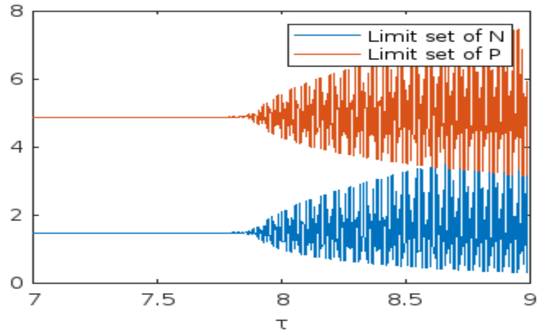

Figure 2: Illustration of bifurcation diagram for system 2, when $\boldsymbol{\tau}$ varies from 7 to 9 and other parameters are fixed as in (18).

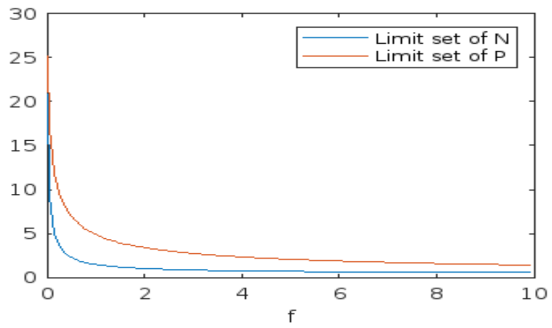

Figure 3: Illustration of bifurcation diagram for system (2), when $\boldsymbol{f}$ varies from 0 to10 and other parameters are fixed as in (18).

In Fig. 3, it has been discovered that, when f increases, the stability of coexistence steady limit value of both prey and predator decreases, which means fear directly affects prey dynamics as well as indirectly effects predator dynamics.

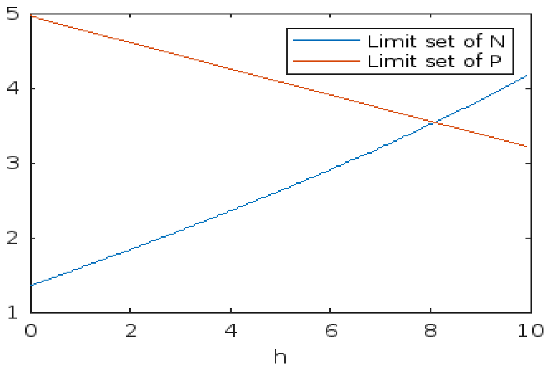

Figure 4 : Illustration of bifurcation diagrams for system 2, when $\boldsymbol{h}$ varies from 0 to 10 and other parameters are fixed as in (18).

In Fig.4, it has been observed that, when h increases, the limit value of prey density increases too while the limit value of predator density decreases.

In general, Fig.1 confirms the analytical results regarding to stability for the coexistence steady state, Fig.2 discovers that the time delay may induce a transition of the dynamics of system from the a stability situation to the state where the populations oscillate periodically or vice versa. Fear affects negatively on both prey and predator species, Fig.3 shows that the fear affects negatively on both prey and predator species and Fig.4 demonstrates that Allee effect for predator reproduction has positive impact on the prey density while it has negative impact on the predators.

CONCLUSION

In this paper, a predator- prey model has been proposed. For derivation purposes of the proposed model, it has been taken into account the time lag corresponding to the gestation period and the effect that the fear of predators has on prey and Allee effect for predator reproduction. Firstly, it is proved that the model solution is bounded and the prey species persist continuously under the condition $b > d_1 + \frac{1}{q}$. It is explored that the possible biological feasible steady states of system (2) are the total extinction steady state, the predator-free steady state, and coexistence steady state. It is proved that the total extinction steady state is LAS and GAS if and only if, $b < d_1$ and the Predator-free steady state is LAS if and only if $b > d_1$ the Local stability of both the total extinction steady and the Predator-free steady state are independent of fear levels. Alle effect and time lags on LAS for but big value of fear may destabilize Predator-free steady state for dome initial values of species because the Predator-free steady state is GAS if, $f^2 b^2 < 4cm$ and $a(b - d_1) < d_2 c$. According to coexistence steady state, the analytical and numerical result show the time delay may induce a transition of the dynamics of system from the a stability situation to the state where the populations oscillate periodically or vice versa, fear affect negatively on both prey and predator species and Allee effect for predator reproduction has positive impact on the prey density, while it has negative impact on the predators.

REFERENCES

Xiaoying, W., Liana Z., Xinfu Z.(2016).Modeling the fear effect in predator-prey interactions, Math. Bio, 73 (2), 1–38. https://doi.org/10.1007/s00285-016-0989-1

Pal S., Pal N, Samanta S.(2019) Chattopadhyay J, Effect of hunting cooperation and fear in a predator-prey model, Ecol.Complex39. https://doi.org/10.1016/j.ecocom.2019.100770

Huisen Z., Yangli C., Shengmao F., Weiming W.(2019) Impact of the fear effect in a prey-predator model Incorporating aprey refuge,Appl.Math.Comput,356,328–337. http://dx.doi.org/10.1016/j.amc.2019.03.034

Pingping G., Meng F., Ingfu Z. (2021)Dynamics of a three-species food chain model with fear effect, Commun. Nonlinear Sci. Number. Simulat 99. https://doi.org/10.1016/j.cnsns.2021.105809

Yiping T., Yongli C., Ruoxia Y., Maolin H., Weiming. W.(2022) Complex dynamics in an eco-Epidemiological model with the cost of anti-predator behaviors.NonlinearDyn,107,3127–3141. https://link.springer.com/article/10.1007%2Fs11071-021-07133-4

Jimil AR., Naji RK. (2023).Modeling and Analyzing the Influence of Fear on the Harvested Modified Leslie Gower Model. Baghdad Sci. J, 20(5),1701-1712. https://dx.doi.org/10.21123/bsj.2023.ID

Xiaoqin W., Yipng T, Yongli C, Weimng W.(2020). Impact of the fear effect on the stability and Bifurcation of a Leslie–Gower predator-prey model, Int.J.Bifur.Chaos30.doi:10.1142/S0218127420502107.

Mengxin He., zhong Li.(2022) stability of fear effect predator-prey model mutual interference or group defense, J. Bio.Dyn, 16,480-498. https://doi.org/10.1080/17513758.2022.2091800

Soumitra P., Pankaj .KT., Arvind KM., Hao W.(2023). Fear effect in a three-species food chain model with generalist predator, Math. Bio and Engineering, 21,1-33. https://doi.org/10.3934/mbe.2024001.

Yaseen, R.M., Helal, M.M., Dehingia, K., Mohsen, A.A.(2024). Effect of the Fear Factor and Prey Refuge in an Asymmetric Predator–Prey Model. Braz J Phys 54, 214. http://dx.doi.org/10.1007/s13538-024-01594-9.

Haque M., Sarwardi S., Preston S. and Venturino E.(2011) Effect of delay in a LotkaVolterra type predator-prey model with a transmissible disease in the predator species", Mathematical Biosciences, (1).234,47-57. https://doi.org/10.1016/j.mbs.2011.06.009

liu J., lv P., Li,, B., Zhang T.(2021). Dynamics of a Predator-Prey Model with Fear Effect and Time Delay,

Complexity,16pages.https://doi.org/10.1155/2021/9184193.

Dehingia, K., Hosseini, K., Salahshour, S., Baleanu, D .(2022) A Detailed Study on a Tumor Model with Delayed Growth of Pro-Tumor Macrophages. *Int. J. Appl. Comput. Math* **8**, 245 (2022). https://doi.org/10.1007/s40819-022-01433-y

Naji RK., Majeed SJ.(2020). the dynamical analysis of a delayed prey-predator model with arefuge-stage structure prey population, Iranian.J.Math.Sci and inf, 1(2020), 135-159. https://ijmsi.ir/article-1-1116-en.pdf

Lavanya R, Vinth S, Sathiyanathan, K, Zeric NT, Hammachukiatkul P, Vadivel R(2022) Dynamical behavior of a delayed Holling type-II predator prey model with predator Cannibalism, J. Mathematics,15 pages. https://doi.org/10.1155/2022/4071375.

Rihan FA, Alsakaji HJ, Rajivganthi C.(2020) Stability and Hopf Bifurcation of Three-Species Prey-Predator System with Time Delays and Allee Effect ,Complexity,(2020), 15pages. https://doi.org/10.1155/2020/7306412.

Alan J. Terry.(2015). Predator–prey models with component Allee effect for predator reproduction. J. Math. Biol,71,1325–1352**.** https://doi.org/10.1007/s00285-015-0856-5

Soura KS.(2018) Population dynamics with multiple allee effects induced by fear factors-A Mathematical study on prey-predator interactions, Appl.Math.Mode,64, 1–14. https://doi.org/10.1016/j.apm.2018.07.021

Yining X., Zhao J., Yang R.(1996).Stability Analysis and Hopf Bifurcation of a Delayed Diffusive Predator–Prey Model with a Strong Allee Effect on the Prey and the Effect of Fear on the Predator, , *Mathematics* 11,(9). https://doi.org/10.3390/math11091996.

Dehingia ,K., Das ,P., Upadhyay ,R., Misra ,A., Rihan, F. , Kamyar Hosseini .(2023)Modeling and analysis of delayed tumour–immune system with hunting T-cells, Mathematics,203,669-684. http://dx.doi.org/10.1016/j.matcom.2022.07.009.

Das, A., Dehingia, K ., Hinçal, E .,Özköse, F., Hosseini, K. (2024)A study on the dynamics of a breast cancer model with discrete-time delay, physica scripta, 99(3). http://dx.doi.org/10.1088/1402-4896/ad2753.

Dehingia, K ., Das, A ., Hinçal, E., Hosseini, K.(2024) The Effect of Time Delay on the Dynamics of a Plankton-Nutrient System with Refuge, Brazilian Journal of Physics, 55 (1). **doi:10.1007/s13538-024-01670-0**