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ON NIL-SYMMETRIC RINGS AND MODULES SKEWED BY RING ENDOMORPHISM

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ABSTRACT:

The symmetric property plays an important role in non-commutative ring theory and module theory. In this paper, we study the symmetric property with one element of the ring $\mathfrak{R}$ and two nilpotent elements of $\mathfrak{R}$ skewed by ring endomorphism б on rings, introducing the concept of a right б-$\mathcal{N}^{\mathcal{L}}$-symmetric ring and extend the concept of right б-$\mathcal{N}^{\mathcal{L}}$-symmetric rings to modules by introducing another concept called the right б-$\mathcal{N}^{\mathcal{L}}$-symmetric module which is a generalization of б-symmetric modules. According to this, we examine the characterization of a right б-$\mathcal{N}^{\mathcal{L}}$-symmetric ring and a right б-$\mathcal{N}^{\mathcal{L}}$-symmetric module and their related properties including ring and explore their connections to other classes of rings and modules. Furthermore, we investigate the concept of б-$\mathcal{N}^{\mathcal{L}}$-symmetric on some ring extensions and localizations like $\mathfrak{R}[\varkappa]$, $\mathfrak{R}[\varkappa, \varkappa^{-1}]$, Dorroh extension, Jordan extension and module localizations like $\Omega^{-1}\widehat{\mathcal{M}}^{\Omega^{-1}\mathfrak{R}}$.

KEYWORDS: Reduced-Ring, Symmetric Ring, Flat Module, б-Reduced Module, Polynomial Module.

1. INTRODUCTION

Every ring in this study has a unique identity, and every module that is investigated is a unital module. $\bar{\mathcal{Z}}, \bar{\mathcal{Z}}_n$ and $\mathcal{N}^{\mathcal{L}}(\mathfrak{R})$ denotes the ring of integers, integers modulo n and the set of nilpotent elements in $\mathfrak{R}$, respectively. Furthermore,$1_{\mathfrak{R}}$, б, $\widehat{\mathcal{M}}^{\mathfrak{R}}$ denote the identity endomorphism, an endomorphism of an arbitrary ring $\mathfrak{R}$ (For short, *endo*) and right $\mathfrak{R}$-module respectively. ℓ_m $(\mathfrak{R}) = \{m \in \widehat{\mathcal{M}} : m\mathfrak{R} = 0\}$ is the left annihilator of $\mathfrak{R}$ in $\widehat{\mathcal{M}}$.

A ring $\mathfrak{R}$ is reduced (For short red-ring), if it has no nonzero nilpotent elements. However, if $\breve{u}б(\breve{u}) = 0$ implies $\breve{u} = 0$ for $\breve{u} \in \mathfrak{R}$, then *endo* б of the ring $\mathfrak{R}$ is said to be rigid (For short, rg-ring endo) (Krempa, 1996). If there is a rg-ring *endo* б of ring $\mathfrak{R}$, then $\mathfrak{R}$ is said to be б-rigid ring (For short, б-rg-ring) (Suarez H., *et al*., 2024). Note that, б-rg-rings are red-rings by [(Hong *et al*., 2000), Proposition 5]. and any rg-ring *endo* of a ring is a monomorphism. Cohn introduced a ring $\mathfrak{R}$ as *reversible*, if whenever $\breve{u}\acute{\rho} = 0$, then $\acute{\rho}\breve{u} = 0$, for $\breve{u}, \acute{\rho} \in \mathfrak{R}$ (Cohn, 1999). Lembek referred to a ring $\mathfrak{R}$ as symmetric (For short, $\mathcal{S}$-ring), if whenever $\breve{u}\acute{\rho}\tilde{\omega} = 0$, then $\breve{u}\tilde{\omega}\acute{\rho} = 0$, for $\breve{u}, \acute{\rho}, \tilde{\omega} \in \mathfrak{R}$ (Lambek, 1971). According to [(Shin, 1973), Lemma 1.1], every red-ring is symmetric; however, the convers does not true in general [(Anderson & Camillo, 1999), Example 11.5]. Although, it is clear that $\mathcal{S}$-rings are reversible and commutative rings are symmetric, the convers of each of them does not true in general [(Anderson & Camillo, 1999), Example 1.5 and 11.5] and [(Marks, 2002), Example 5 and 7]. As an extension of $\mathcal{S}$-rings and a specific instance of $\mathcal{N}^{\mathcal{L}}$-semi-commutative rings, Chakraborty and Das presented the idea of $\mathcal{N}^{\mathcal{L}}$-symmetric rings in (Chakraborty & Das, 2014). A ring $\mathfrak{R}$ is right(R) (left(L)) $\mathcal{N}^{\mathcal{L}}$-symmetric (For short, R(L)-$\mathcal{N}^{\mathcal{L}}\mathcal{S}$-ring), if for $\breve{u} \in \mathfrak{R}$, and $\acute{\rho}, \tilde{\omega} \in \mathcal{N}^{\mathcal{L}}(\mathfrak{R})$ with $\breve{u}\acute{\rho}\tilde{\omega} = 0 (\tilde{\omega}\breve{u}\acute{\rho} = 0)$, then $\breve{u}\tilde{\omega}\acute{\rho} = 0$. A ring is $\mathcal{N}^{\mathcal{L}}\mathcal{S}$-ring if it is both L(R) $\mathcal{N}^{\mathcal{L}}\mathcal{S}$-ring.

The concept of an б-symmetric ring was first proposed by Kwak, T. K. in 2007, as an extension of $\mathcal{S}$-rings and a generalization of б-rg rings. In (Kwak, 2007) an *endo* б of a ring $\mathfrak{R}$ is called L(R)-б-symmetric ring(For short, б-$\mathcal{S}$-ring), if $\breve{u}\acute{\rho}\tilde{\omega} = 0$ imply $\breve{u}\tilde{\omega}б(\acute{\rho}) = 0$ $(б(\acute{\rho})\breve{u}\tilde{\omega} = 0)$, for $a, \acute{\rho}, \tilde{\omega} \in \mathfrak{R}$. A ring $\mathfrak{R}$ is L(R)- б-$\mathcal{S}$-ring if there exists a L(R)- $\mathcal{S}$-ring *endo* б of $\mathfrak{R}$. the concepts of an б-$\mathcal{S}$-ring is an extension of $\mathcal{S}$-rings and it is also a generalization of б-rg rings.

The ring notion was recently extended to include modules. A module $\widehat{\mathcal{M}}^{\mathfrak{R}}$ is called symmetric (For short, $\mathcal{S}$-module), if whenever $\breve{u}, \acute{\rho} \in \mathfrak{R}$, $m \in \widehat{\mathcal{M}}^{\mathfrak{R}}$ satisfy $m\breve{u}\acute{\rho} = 0$, then we have $m\acute{\rho}\breve{u} = 0$ ((Lambek, 1971) and (Raphael, 1975)). A module $\widehat{\mathcal{M}}^{\mathfrak{R}}$ is б-semi-commutative if, $m\breve{u} = 0$ implies $m\mathfrak{R}б(\breve{u}) = 0$, for $m \in \widehat{\mathcal{M}}^{\mathfrak{R}}$ *and* $\breve{u} \in \mathfrak{R}$. The module $\widehat{\mathcal{M}}^{\mathfrak{R}}$ is semi-commutative if it is $i_{\mathfrak{R}}$-semi-commutative. Buhphang and Rege in (Buhphang & Rege, 2002) examined the fundamental characteristics of semi-commutative modules. Agayev and Harmanci concentrated on semi-commutativity of subrings of matrix rings and carried out additional research on semi-commutative rings and modules in (Agayev & Harmancı, 2007).

Motivated to the above, this article is structured to introduce and define a new kind of rings named a R-б-$\mathcal{N}^{\mathcal{L}}$-$\mathcal{S}$ ring as a generalization of б-$\mathcal{S}$-rings and an extension of $\mathcal{N}^{\mathcal{L}}\mathcal{S}$-rings, and to explore and provide various characterizations, features and relations about this concept and to study its related properties. Additionally, we investigate the concept of right б-$\mathcal{N}^{\mathcal{L}}$-symmetric on some of ring extensions and localizations. This leads to a number of well-known outcomes as corollaries of our results. Then we extend the property of R-б-$\mathcal{N}^{\mathcal{L}}\mathcal{S}$ rings to modules by introducing the notion of right б-$\mathcal{N}^{\mathcal{L}}$-symmetric module which is a generalization of б-symmetric modules and extensions of symmetric modules. We examine the characteristics of right б-$\mathcal{N}^{\mathcal{L}}$-symmetric modules and their associated attributes, such as localizations and module extensions.

On б-$\mathcal{N}^{\mathcal{L}}$-Symmetric Rings:

The fundamental structure of б-$\mathcal{N}^{\mathcal{L}}$-$\mathcal{S}$ rings is examined in this section, along with a number of associated ring features. We begin with the following definition.

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Definition 2.1 An *endo* $\mathfrak{б}$ of a ring $\hat{\mathfrak{R}}$ is said to be left(L)-right(R) $\mathfrak{б}$-$\mathcal{N}^{\mathcal{L}}$-symmetric(For short, L-R-$\mathfrak{б}$-$\mathcal{N}^{\mathcal{L}}\mathcal{S}$-ring), if whenever $\breve{\upsilon}\acute{\rho}\tilde{\omega} = O$, for $\breve{\upsilon} \in \hat{\mathfrak{R}}$ and $\acute{\rho}, \tilde{\omega} \in \mathcal{N}^{\mathcal{L}}(\hat{\mathfrak{R}})$, *then* $\breve{\upsilon}\tilde{\omega}\mathfrak{б}(\acute{\rho}) = O(\mathfrak{б}(\acute{\rho})\breve{\upsilon}\tilde{\omega} = O)$. A ring $\hat{\mathfrak{R}}$ is L-R-$\mathfrak{б}$-$\mathcal{N}^{\mathcal{L}}\mathcal{S}$, if there exists a L-R $\mathcal{N}^{\mathcal{L}}\mathcal{S}$ *endo* $\mathfrak{б}$ of $\hat{\mathfrak{R}}$. Moreover, $\hat{\mathfrak{R}}$ is $\mathfrak{б}$-$\mathcal{N}^{\mathcal{L}}\mathcal{S}$-ring if it is both L-R- $\mathfrak{б}$ -$\mathcal{N}^{\mathcal{L}}$-$\mathcal{S}$-ring.

Remark 2.2:

Example 2.3 Suppose that a ring $\hat{\mathfrak{R}} = U_2(\bar{\mathcal{Z}}_4)$, then

$$\mathcal{N}^{\mathcal{L}}(\hat{\mathfrak{R}}) = \left\{\begin{pmatrix} ŧ & ř \\ O & ɉ \end{pmatrix} \mid ŧ, ɉ \in \{O, 2\}, ř \in \bar{\mathcal{Z}}_4\right\}.$$

(i) Let $\mathfrak{б}: \hat{\mathfrak{R}} \to \hat{\mathfrak{R}}$ be an *endo* defined by:

$$\mathfrak{б}\left(\begin{pmatrix} ŧ & ř \\ O & ɉ \end{pmatrix}\right) = \begin{pmatrix} ŧ & O \\ O & O \end{pmatrix}.$$

If $\check{Y}\check{U}\tilde{V} = O$ for $\check{Y} = \begin{pmatrix} ŧ & ř \\ O & ɉ \end{pmatrix} \in \hat{\mathfrak{R}}, \check{U} = \begin{pmatrix} \Upsilon & ƛ \\ O & ɦ \end{pmatrix}, \tilde{V} = \begin{pmatrix} ɣ & ƥ \\ O & ƕ \end{pmatrix} \in \mathcal{N}^{\mathcal{L}}(\hat{\mathfrak{R}})$, then we get $ŧ\Upsilon ɣ = O$ and so $ŧɣ\Upsilon = O$ since $\bar{\mathcal{Z}}_4$ is commutative. This yields $\check{Y}\tilde{V}\mathfrak{б}(\check{U}) = O$, and hence $\hat{\mathfrak{R}}$ is R-$\mathfrak{б}$-$\mathcal{N}^{\mathcal{L}}\mathcal{S}$-ring. For $\check{Y} = \begin{pmatrix} 1 & O \\ O & O \end{pmatrix} \in \hat{\mathfrak{R}}, \check{U} = \begin{pmatrix} 2 & 1 \\ O & 2 \end{pmatrix} = \tilde{V} \in \mathcal{N}^{\mathcal{L}}(\hat{\mathfrak{R}})$ with $\check{Y}\check{U}\tilde{V} = O$, we have $\mathfrak{б}(\check{U})\check{Y}\tilde{V} = \begin{pmatrix} O & 2 \\ O & O \end{pmatrix} \neq O$, and thus $\hat{\mathfrak{R}}$ is not L-$\mathfrak{б}$-$\mathcal{N}^{\mathcal{L}}\mathcal{S}$-ring.

(ii) Let $\mathbf{I}: \hat{\mathfrak{R}} \to \hat{\mathfrak{R}}$ be an *endo* defined by:

$$\mathbf{I}\left(\begin{pmatrix} ŧ & ř \\ O & ɉ \end{pmatrix}\right) = \begin{pmatrix} O & O \\ O & ɉ \end{pmatrix}.$$

By using the same technique as in (i), we may demonstrate that $\hat{\mathfrak{R}}$ is L-$\mathbf{I}$-$\mathcal{N}^{\mathcal{L}}$-$\mathcal{S}$-ring. However, $\hat{\mathfrak{R}}$ is not R-$\mathbf{I}$-$\mathcal{N}^{\mathcal{L}}$-$\mathcal{S}$-ring for $\check{Y}\check{U}\tilde{V} = O$ but $\check{Y}\tilde{V}\,\mathbf{I}\,(\check{U}) = \begin{pmatrix} O & 2 \\ O & O \end{pmatrix} \neq O$, and thus $\hat{\mathfrak{R}}$ is not R-$\mathbf{I}$-$\mathcal{N}^{\mathcal{L}}$-$\mathcal{S}$-ring.

Lemma 2.4 (1) For a ring $\hat{\mathfrak{R}}$, $\hat{\mathfrak{R}}$ is R-$\mathfrak{б}$-$\mathcal{N}^{\mathcal{L}}\mathcal{S}$-ring if and only if $\check{Y}\check{U}\tilde{V} = O$ implies $\check{Y}\tilde{V}\mathfrak{б}(\check{U}) = O$, for $\emptyset \neq \check{Y} \subseteq \hat{\mathfrak{R}}$ and $\emptyset \neq \check{U}, \emptyset \neq \tilde{V} \subseteq \mathcal{N}^{\mathcal{L}}(\hat{\mathfrak{R}})$.

(2) Consider $\hat{\mathfrak{R}}$ be a reversible ring. $\hat{\mathfrak{R}}$ is R-$\mathfrak{б}$-$\mathcal{N}^{\mathcal{L}}\mathcal{S}$-ring if and only if $\hat{\mathfrak{R}}$ is L-$\mathfrak{б}$-$\mathcal{N}^{\mathcal{L}}\mathcal{S}$-ring.

Proof. (1) It suffices to show that $\check{Y}\check{U}\tilde{V} = O$ for $\emptyset \neq \check{Y} \subseteq \hat{\mathfrak{R}}$ and $\emptyset \neq \check{U}, \emptyset \neq \tilde{V} \subseteq \mathcal{N}^{\mathcal{L}}(\hat{\mathfrak{R}})$, implies $\check{Y}\tilde{V}\mathfrak{б}(\check{U}) = O$, when $\hat{\mathfrak{R}}$ is right $\mathfrak{б}$-$\mathcal{N}^{\mathcal{L}}\mathcal{S}$-ring. Let $\check{Y}\check{U}\tilde{V} = O$, then $\breve{\upsilon}\acute{\rho}\tilde{\omega} = O$ for $\breve{\upsilon} \in \check{Y}$, $\acute{\rho} \in \check{U}$ and $\tilde{\omega} \in \tilde{V}$, and hence $\breve{\upsilon}\tilde{\omega}\mathfrak{б}(\acute{\rho}) = O$ by the condition. Thus $\check{Y}\tilde{V}\mathfrak{б}(\check{U}) = \sum_{\breve{\upsilon}\in\check{Y}, \acute{\rho}\in\check{U} \text{ and } \tilde{\omega}\in\tilde{V}} \breve{\upsilon}\tilde{\omega}\mathfrak{б}(\acute{\rho}) = O$.

(2) Let $\breve{\upsilon}\acute{\rho}\tilde{\omega} = O$ for $\breve{\upsilon} \in \hat{\mathfrak{R}}$ *and* $\acute{\rho}, \tilde{\omega} \in \mathcal{N}^{\mathcal{L}}(\hat{\mathfrak{R}})$. If $\hat{\mathfrak{R}}$ is R-$\mathfrak{б}$-$\mathcal{N}^{\mathcal{L}}\mathcal{S}$-ring, then $(\breve{\upsilon}\tilde{\omega})(\mathfrak{б}(\acute{\rho})) = O$, since $\hat{\mathfrak{R}}$ is reversible, we have $(\mathfrak{б}(\acute{\rho}))(\breve{\upsilon}\tilde{\omega}) = \mathfrak{б}(\acute{\rho})\breve{\upsilon}\tilde{\omega} = O$, and hence $\hat{\mathfrak{R}}$ is L-$\mathfrak{б}$-$\mathcal{N}^{\mathcal{L}}\mathcal{S}$-ring. The converse is similar. ∎

The condition "$\hat{\mathfrak{R}}$ is reversible" in (Proposition 2.4) is irremovable, as demonstrated by Example 2.3. While it is evident that all $\mathfrak{б}$-symmetric objects are $\mathfrak{б}$-$\mathcal{N}^{\mathcal{L}}\mathcal{S}$-ring, the following example shows that the converse is not true.

Example 2.5 Assume $\bar{\mathcal{Z}}_2$ is the ring of integer modulo 2, and $\hat{\mathfrak{R}} = \bar{\mathcal{Z}}_2 \oplus \bar{\mathcal{Z}}_2$. Using the standard addition and multiplication. Since $\mathcal{N}^{\mathcal{L}}(\hat{\mathfrak{R}}) = \{(O, O)\}$, $\hat{\mathfrak{R}}$ is $\mathfrak{б}$-$\mathcal{N}^{\mathcal{L}}\mathcal{S}$-ring. Now let $\mathfrak{б}: \hat{\mathfrak{R}} \to \hat{\mathfrak{R}}$ be defined by $\mathfrak{б}((\breve{\upsilon}, \acute{\rho})) = (\acute{\rho}, \breve{\upsilon})$. Then, for $\breve{\upsilon} = (1, O), \acute{\rho} = (O, 1), \tilde{\omega} = (1,1) \in \hat{\mathfrak{R}}$, $\breve{\upsilon}\acute{\rho}\tilde{\omega} = O$ but $\breve{\upsilon}\tilde{\omega}\,\mathfrak{б}(\acute{\rho}) = (1, O) \neq O$, and thus $\hat{\mathfrak{R}}$ is not an $\mathfrak{б}$-$\mathcal{S}$-ring. ∎

Consider $\hat{\mathfrak{R}}$ is a ring and $\emptyset \neq ɠ \subseteq \hat{\mathfrak{R}}$, $l_{R^*}(ɠ) = \{\tilde{\omega} \in \hat{\mathfrak{R}} \mid \tilde{\omega}ɠ = O\}$ is called the L-annihilator of $ɠ$ in $\hat{\mathfrak{R}}$. If $ɠ = \{\breve{\upsilon}\}$, then we write $l_{\hat{\mathfrak{R}}}(\breve{\upsilon})$ instead of $l_{\hat{\mathfrak{R}}}\{\breve{\upsilon}\}$.

Lemma 2.6 For a ring $\hat{\mathfrak{R}}$, then the following are equivalent for a nonzero *endo* $\mathfrak{б}$:

(1) $\hat{\mathfrak{R}}$ is R-$\mathfrak{б}$-$\mathcal{N}^{\mathcal{L}}\mathcal{S}$-ring;

(2) $l_{\hat{\mathfrak{R}}}(\acute{\rho}\tilde{\omega}) \subseteq l_{\hat{\mathfrak{R}}}(\tilde{\omega}\mathfrak{б}(\acute{\rho}))$, for any $\breve{\upsilon} \in \hat{\mathfrak{R}}$ and $\acute{\rho}, \tilde{\omega} \in \mathcal{N}^{\mathcal{L}}(\hat{\mathfrak{R}})$;

1. A ring $\hat{\mathfrak{R}}$ is $\mathcal{N}^{\mathcal{L}}\mathcal{S}$-ring if $\hat{\mathfrak{R}}$ is $1_{\hat{\mathfrak{R}}}$-$\mathcal{N}^{\mathcal{L}}$-symmetric, where $1_{\hat{\mathfrak{R}}}$ is the identity *endo*.
2. Every subring $\hat{S}$ with $\mathfrak{б}(\hat{S}) \subseteq \hat{S}$ of an $\mathfrak{б}$-$\mathcal{N}^{\mathcal{L}}\mathcal{S}$-ring is also $\mathfrak{б}$-$\mathcal{N}^{\mathcal{L}}\mathcal{S}$-ring.
3. $\hat{\mathfrak{R}}$, but the converse does not true (See (Kwak, 2007)Example 2.7(1)).
4. The concept of $\mathfrak{б}$-$\mathcal{N}^{\mathcal{L}}\mathcal{S}$-ring is not R-L- $\mathfrak{б}$-$\mathcal{N}^{\mathcal{L}}\mathcal{S}$-ring through the following example.

(3) $\check{Y}\check{U}\tilde{V} = O$ if and only if $\check{Y}\tilde{V}\mathfrak{б}(\check{U}) = O$, for any $\check{Y} \subseteq \hat{\mathfrak{R}}$ and $\check{U}, \tilde{V} \subseteq \mathcal{N}^{\mathcal{L}}(\hat{\mathfrak{R}})$;

(4) $l_{\hat{\mathfrak{R}}}(\check{U}\tilde{V}) \subseteq l_{\hat{\mathfrak{R}}}(\tilde{V}\mathfrak{б}(\check{U}))$, for any $\check{Y} \subseteq \hat{\mathfrak{R}}$ and $\check{U}, \tilde{V} \subseteq \mathcal{N}^{\mathcal{L}}(\hat{\mathfrak{R}})$.

Proof. (1) → (3). Suppose that $\check{Y}\check{U}\tilde{V} = O$ for $\check{Y} \subseteq \hat{\mathfrak{R}}$ and $\check{U}, \tilde{V} \subseteq \mathcal{N}^{\mathcal{L}}(\hat{\mathfrak{R}})$. For any $\breve{\upsilon} \in \check{Y}, \acute{\rho} \in \check{U}, \tilde{\omega} \in \tilde{V}$ Then $\breve{\upsilon}\acute{\rho}\tilde{\omega} = O$, and hence $\breve{\upsilon}\tilde{\omega}\mathfrak{б}(\acute{\rho}) = O$. Therefore $\check{Y}\tilde{V}\mathfrak{б}(\check{U}) = \{\sum \breve{\upsilon}_i\tilde{\omega}_i\mathfrak{б}(\acute{\rho}_i): \breve{\upsilon}_i \in \check{Y}, \acute{\rho}_i \in \check{U}, \tilde{\omega}_i \in \tilde{V}\} = O$.

The converse is obvious. (1) → (2) and (3) → (4) is clear. ∎

Lemma 2.7 The class of $\mathfrak{б}$-$\mathcal{N}^{\mathcal{L}}\mathcal{S}$-rings is closed under direct products.

Proof. Note that $\mathcal{N}^{\mathcal{L}}(\prod_{\Upsilon\in\Gamma}\hat{\mathfrak{R}}_\Upsilon) \subseteq \prod_{\Upsilon\in\Gamma}\mathcal{N}^{\mathcal{L}}(\hat{\mathfrak{R}}_\Upsilon)$ and $\mathfrak{б}_\Upsilon(\hat{\mathfrak{R}}_\Upsilon) \subseteq \hat{\mathfrak{R}}_\Upsilon$ for each $\Upsilon \in \Gamma$. Now, let $\check{Y}\check{U}\tilde{V} = 0$,where $\check{Y} = (\breve{\upsilon}_\Upsilon)_{\Upsilon\in\Gamma} \in \prod_{\Upsilon\in\Gamma}\hat{\mathfrak{R}}_\Upsilon$ and $\check{U} = (\acute{\rho}_\Upsilon)_{\Upsilon\in\Gamma}$, $\tilde{V} = (\tilde{\omega}_\Upsilon)_{\Upsilon\in\Gamma} \in \in \mathcal{N}^{\mathcal{L}}(\prod_{\Upsilon\in\Gamma}\hat{\mathfrak{R}}_\Upsilon)$. Thus for $\breve{\upsilon}_\Upsilon \in \hat{\mathfrak{R}}_\Upsilon$ and $\acute{\rho}_\Upsilon, \tilde{\omega}_\Upsilon \in \mathcal{N}^{\mathcal{L}}(\hat{\mathfrak{R}}_\Upsilon)$, $\breve{\upsilon}_\Upsilon\acute{\rho}_\Upsilon\tilde{\omega}_\Upsilon = 0$. Since $\hat{\mathfrak{R}}_\Upsilon$ is R-$\mathfrak{б}$-$\mathcal{N}^{\mathcal{L}}\mathcal{S}$-ring for each $\Upsilon \in \Gamma$, then $\breve{\upsilon}_\Upsilon\tilde{\omega}_\Upsilon\mathfrak{б}(\acute{\rho}_\Upsilon) = 0$ for each $\Upsilon \in \Gamma$. So we get $\check{Y}\tilde{V}\mathfrak{б}(\check{U}) = 0$. Therefore, the direct product $\prod_{\Upsilon\in\Gamma}\hat{\mathfrak{R}}_\Upsilon$ of $\hat{\mathfrak{R}}_\Upsilon$ is R-$\mathfrak{б}$-$\mathcal{N}^{\mathcal{L}}\mathcal{S}$-ring.

Recently, it was proven that if $\breve{\upsilon}, \acute{\rho} \in \hat{\mathfrak{R}}$, such that $\breve{\upsilon}\acute{\rho} = O \to \acute{\rho}\mathfrak{б}(\breve{\upsilon}) = O\ (\mathfrak{б}(\acute{\rho})\breve{\upsilon} = O)$, then $\mathfrak{б}$ is R(L) reversible, and the ring $\hat{\mathfrak{R}}$ is called R(L) $\mathfrak{б}$-reversible if there exist a R(L) reversible *endo* $\mathfrak{б}$ of $\hat{\mathfrak{R}}$. A ring $\hat{\mathfrak{R}}$ is $\mathfrak{б}$-reversible (Başer *et al*., 2009) if it is both L(R) $\mathfrak{б}$-reversible.

Theorem 2.8 Let $\hat{\mathfrak{R}}$ be a $\mathfrak{б}$-$\mathcal{N}^{\mathcal{L}}\mathcal{S}$-ring. Then we have the following.

1. For $\breve{\upsilon} \in \hat{\mathfrak{R}}$, $\acute{\rho}, \tilde{\omega} \in \mathcal{N}^{\mathcal{L}}(\hat{\mathfrak{R}})$ and $\breve{\upsilon}\acute{\rho} = O$, then $\breve{\upsilon}\tilde{\omega}\mathfrak{б}^n(\acute{\rho}) = O, \acute{\rho}\tilde{\omega}\mathfrak{б}^n(\breve{\upsilon}) = O$, and $\breve{\upsilon}\acute{\rho}\mathfrak{б}^n(\tilde{\omega}) = O, \forall n \in Z^+$. Consequently, $\hat{\mathfrak{R}}$ is right $\mathfrak{б}$-reversible ring.

2. Consider $\mathfrak{б}$ is a monomorphism of $\hat{\mathfrak{R}}$. Then we have the following.

i. $\hat{\mathfrak{R}}$ is $\mathcal{N}^{\mathcal{L}}$-symmetric ring,

ii. For $\breve{\upsilon} \in \hat{\mathfrak{R}}$, $\acute{\rho}, \tilde{\omega} \in \mathcal{N}^{\mathcal{L}}(\hat{\mathfrak{R}})$ and $\breve{\upsilon}\acute{\rho}\tilde{\omega} = O$, then $\mathfrak{б}^n(\breve{\upsilon})\acute{\rho}\tilde{\omega} = O$ and $\breve{\upsilon}\mathfrak{б}^n(\acute{\rho})\tilde{\omega} = O$, $\forall n \in Z^+$. Conversely, if $\mathfrak{б}^m(\breve{\upsilon})\acute{\rho}\tilde{\omega} = O, \breve{\upsilon}\mathfrak{б}^m(\acute{\rho})\tilde{\omega} = O$, or $\breve{\upsilon}\acute{\rho}\mathfrak{б}^m(\tilde{\omega}) = O$ for some $m \in Z^+$, then $\breve{\upsilon}\acute{\rho}\tilde{\omega} = O$.

Proof. The proof is similar to that of [(Kwak, 2007), Theorem2.5]. ∎

EXTENSIONS OF RIGHT $\mathfrak{б}$-$\mathcal{N}^{\mathcal{L}}$-SYMMETRIC RINGS :

In this section, we investigate the properly of right $\mathfrak{б}$-$\mathcal{N}^{\mathcal{L}}$-symmetric on some extensions of right $\mathfrak{б}$-$\mathcal{N}^{\mathcal{L}}$-symmetric. One may ask whether the following extensions $Mat_n(\hat{\mathfrak{R}}), U_n(\hat{\mathfrak{R}}), D_n(\hat{\mathfrak{R}}), T(\hat{\mathfrak{R}}, \hat{\mathfrak{R}})$ *and* $\hat{\mathfrak{R}}[\varkappa]$ are right $\mathfrak{б}$-$\mathcal{N}^{\mathcal{L}}$-symmetric, if $\hat{\mathfrak{R}}$ is right $\mathfrak{б}$-$\mathcal{N}^{\mathcal{L}}$-symmetric. According to this, many results were obtained. Consider an $n \times n$ upper triangular matrix ring, matrix ring over $\hat{\mathfrak{R}}$, denoted as $U_n(\hat{\mathfrak{R}}), Mat_n(\hat{\mathfrak{R}})$. Suppose that $D_n(\hat{\mathfrak{R}})$ represents the subring of $U_n(\hat{\mathfrak{R}})$ where all diagonal entries are the same.

For any red-ring $\hat{\mathfrak{R}}$, both $U_2(\hat{\mathfrak{R}})$ and $D_2(\hat{\mathfrak{R}})$ qualify as R-$\mathfrak{б}$-$\mathcal{N}^{\mathcal{L}}\mathcal{S}$-rings for any given *endo* $\mathfrak{б}$. However, the following counterexample demonstrates that there exists a red-ring $\hat{\mathfrak{R}}$ with an *endo* $\mathfrak{б}$ such that $Mat_n(\hat{\mathfrak{R}})$ does not satisfy the R-$\mathfrak{б}$-$\mathcal{N}^{\mathcal{L}}\mathcal{S}$-rings condition.

Example 3.1 An automorphism $\mathfrak{б}$ of $\bar{Z}_2$ defined by:
$O \to 1$ and $1 \to O$
Assume $\mathfrak{R} = Mat_2(\bar{Z}_2)$. Now for $\check{Y} = \begin{pmatrix} 1 & 1 \\ O & O \end{pmatrix} \in \mathfrak{R}$, and $\tilde{U} = \begin{pmatrix} O & 1 \\ O & O \end{pmatrix}, \tilde{V} = \begin{pmatrix} O & 1 \\ O & O \end{pmatrix} \in \mathcal{N}^{\mathcal{L}}(\mathfrak{R})$ we have $\check{Y}\tilde{U}\tilde{V} = O$ but $\check{Y}\tilde{V}\mathfrak{б}(\tilde{U}) = \begin{pmatrix} 1 & 1 \\ O & O \end{pmatrix}\begin{pmatrix} O & 1 \\ O & O \end{pmatrix}\begin{pmatrix} 1 & O \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} O & 1 \\ O & O \end{pmatrix}\begin{pmatrix} 1 & O \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ O & O \end{pmatrix} \neq O$. Therefore, $Mat_2(Z_2)$ is not $\mathfrak{б}$-$\mathcal{N}^{\mathcal{L}}\mathcal{S}$-ring.

The trivial extension of a ring $\mathfrak{R}$ by a $(\mathfrak{R},\mathfrak{R})$-bimodule $\widehat{\mathcal{M}}^{\mathfrak{R}}$is the ring $T(\mathfrak{R},\widehat{\mathcal{M}}) = \mathfrak{R}\oplus\widehat{\mathcal{M}}$, which can be obtained by the standard addition and multiplication as follows:

$$(r_1, m_1)(r_2, m_2) = (r_1r_2, r_1m_2 + m_1r_2).$$

This is isomorphic to the ring $\begin{pmatrix} \mathfrak{R} & \widehat{\mathcal{M}} \\ O & \mathfrak{R} \end{pmatrix}$ the usual matrix operations are used. For an *endo* $\mathfrak{б}$ of a ring $\mathfrak{R}$ and the trivial extension $T(\mathfrak{R},\mathfrak{R})$ of R°, $\mathfrak{б}: T(\mathfrak{R},\mathfrak{R}) \to T(\mathfrak{R},\mathfrak{R})$ defined by:

$$\mathfrak{б}\left(\begin{pmatrix} \breve{\upsilon} & \acute{\rho} \\ O & \breve{\upsilon} \end{pmatrix}\right) = \begin{pmatrix} \mathfrak{б}(\breve{\upsilon}) & \mathfrak{б}(\acute{\rho}) \\ O & \mathfrak{б}(\breve{\upsilon}) \end{pmatrix}$$

is an *endo* of $T(\mathfrak{R},\mathfrak{R})$. Since $T(\mathfrak{R}, O)$ is isomorphic to $\mathfrak{R}$.
The trivial extension of the red-ring is symmetric by [(Huh *et al.*, 2005), corollary 2.4]. However, for a R-$\mathfrak{б}$-$\mathcal{N}^{\mathcal{L}}\mathcal{S}$-ring $\mathfrak{R}$. $T(\mathfrak{R},\mathfrak{R})$ need not be a right $\mathfrak{б}$-$\mathcal{N}^{\mathcal{L}}\mathcal{S}$-ring by the following example.

Example 3.2 Suppose the R-$\mathfrak{б}$-$\mathcal{N}^{\mathcal{L}}\mathcal{S}$-ring $\mathfrak{R} = \left\{\begin{pmatrix} \breve{\upsilon} & \acute{\rho} \\ O & \breve{\upsilon} \end{pmatrix} \mid \breve{\upsilon}, \acute{\rho} \in \bar{Z}\right\}$. Assume $\mathfrak{б}: \mathfrak{R} \to \mathfrak{R}$ be an *endo* defined by $\mathfrak{б}\left(\begin{pmatrix} \breve{\upsilon} & \acute{\rho} \\ O & \breve{\upsilon} \end{pmatrix}\right) = \begin{pmatrix} \breve{\upsilon} & -\acute{\rho} \\ O & \breve{\upsilon} \end{pmatrix}$. Take $\mathfrak{T} = T(\mathfrak{R},\mathfrak{R})$, Let

$$A = \begin{pmatrix} \begin{pmatrix} 1 & O \\ O & 1 \end{pmatrix} & \begin{pmatrix} O & O \\ O & O \end{pmatrix} \\ \begin{pmatrix} O & O \\ O & O \end{pmatrix} & \begin{pmatrix} 1 & O \\ O & 1 \end{pmatrix} \end{pmatrix} \in \mathfrak{T}, B = \begin{pmatrix} \begin{pmatrix} O & 1 \\ O & O \end{pmatrix} & \begin{pmatrix} -1 & 1 \\ O & -1 \end{pmatrix} \\ \begin{pmatrix} O & O \\ O & O \end{pmatrix} & \begin{pmatrix} O & 1 \\ O & O \end{pmatrix} \end{pmatrix}, C = \begin{pmatrix} \begin{pmatrix} O & 1 \\ O & O \end{pmatrix} & \begin{pmatrix} 1 & 1 \\ O & 1 \end{pmatrix} \\ \begin{pmatrix} O & O \\ O & O \end{pmatrix} & \begin{pmatrix} O & 1 \\ O & O \end{pmatrix} \end{pmatrix} \in \mathcal{N}^{\mathcal{L}}(\mathfrak{T})$$

$ABC = O$ but $AC\ \mathfrak{б}(B) \neq O$. Thus $\mathfrak{T} = T(\mathfrak{R},\mathfrak{R})$ is not right $\mathfrak{б}$-$\mathcal{N}^{\mathcal{L}}\mathcal{S}$-ring.

Proposition 3.3 Consider $\mathfrak{R}$ is a red-ring, then $T(\mathfrak{R},\mathfrak{R})$ is a R-$\mathfrak{б}$-$\mathcal{N}^{\mathcal{L}}\mathcal{S}$-ring.
Proof. The proof is similar to that of [(Kwak, 2007), Proposition3.2]. ■

The following is an extension of the trivial extension $T(\mathfrak{R},\mathfrak{R})$ of the $\mathfrak{б}$-rg ring to a new ring:

$$\mathfrak{T}_n = \left\{\begin{pmatrix} \breve{\upsilon} & \breve{\upsilon}_{12} & \breve{\upsilon}_{13} & \cdots & \breve{\upsilon}_{1n} \\ O & \breve{\upsilon} & \breve{\upsilon}_{23} & \cdots & \breve{\upsilon}_{2n} \\ O & O & \breve{\upsilon} & \cdots & \breve{\upsilon}_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ O & O & O & \cdots & \breve{\upsilon} \end{pmatrix} : \breve{\upsilon}, \breve{\upsilon}_{ij} \in \mathfrak{R}\right\}$$

And,

$$\mathcal{N}^{\mathcal{L}}(\mathfrak{T}_n) = \left\{\begin{pmatrix} O & \breve{\upsilon}_{12} & \breve{\upsilon}_{13} & \cdots & \breve{\upsilon}_{1n} \\ O & O & \breve{\upsilon}_{23} & \cdots & \breve{\upsilon}_{2n} \\ O & O & O & \cdots & \breve{\upsilon}_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ O & O & O & \cdots & O \end{pmatrix} : a_{ij} \in \mathfrak{R}\right\}$$

The *endo* $\mathfrak{б}: \mathfrak{T}_n \to \mathfrak{T}_n$, defined by $\mathfrak{б}\left((\breve{\upsilon}_{ij})\right) = \left(\mathfrak{б}(\breve{\upsilon}_{ij})\right)$, is further extended to an *endo* $\mathfrak{б}$ of a ring $\mathfrak{R}$ for any $n \geq 3$. If $\mathfrak{R}$ is $\mathfrak{б}$-rg then $\mathfrak{T}_3$ is not a R-$\mathfrak{б}$-$\mathcal{N}^{\mathcal{L}}\mathcal{S}$-ring by [(Kwak, 2007), Example 3.4]. The following example shows that $\mathfrak{T}_n$ cannot be $\mathfrak{б}$-$\mathcal{N}^{\mathcal{L}}\mathcal{S}$-ring for any $n \geq 4$, even if $\mathfrak{R}$ is an $\mathfrak{б}$-rg ring.

Example 3.4 Consider $\mathfrak{б}$ is an *endo* of an $\mathfrak{б}$-rg ring $\mathfrak{R}$. Note that $\mathfrak{б}(e) = e$ for $e^2 = e \in \mathfrak{R}$. By [(Hong *et al.*, 2000), Proposition 5] In particular $\mathfrak{б}(1) = 1$.
Let $ABC = O$ for

$$A = \begin{pmatrix} O & 1 & -1 & O \\ O & O & O & O \\ O & O & O & O \\ O & O & O & O \end{pmatrix}, B = \begin{pmatrix} O & O & O & O \\ O & O & O & 1 \\ O & O & O & 1 \\ O & O & O & O \end{pmatrix} \in \mathcal{N}^{\mathcal{L}}(\mathfrak{R}),$$

$$C = \begin{pmatrix} O & O & O & O \\ O & O & 1 & O \\ O & O & O & O \\ O & O & O & O \end{pmatrix} \in \mathfrak{R}.$$

But we have,

$$AC\mathfrak{б}(B) = \begin{pmatrix} O & 1 & -1 & O \\ O & O & O & O \\ O & O & O & O \\ O & O & O & O \end{pmatrix}\begin{pmatrix} O & O & O & O \\ O & O & 1 & O \\ O & O & O & O \\ O & O & O & O \end{pmatrix}\begin{pmatrix} O & O & O & O \\ O & O & O & 1 \\ O & O & O & 1 \\ O & O & O & O \end{pmatrix} = \begin{pmatrix} O & O & 1 & O \\ O & O & O & O \\ O & O & O & O \\ O & O & O & O \end{pmatrix} \neq O$$

Thus $\mathfrak{T}_4$ is not a R-$\mathfrak{б}$-$\mathcal{N}^{\mathcal{L}}\mathcal{S}$-ring.

Theorem 3.5 Consider $\mathfrak{R}$ is a red-ring and $n \in \bar{Z}^+$. If $\mathfrak{R}$ is a R-$\mathfrak{б}$-$\mathcal{N}^{\mathcal{L}}\mathcal{S}$-ring with $\mathfrak{б}(1) = 1$, then $\mathfrak{R}[u]/ < u^n >$ is a R-$\mathfrak{б}$-$\mathcal{N}^{\mathcal{L}}\mathcal{S}$-ring, where $< u^n >$ is the ideal generated by u^n.
Proof. Suppose $\mathfrak{T} = \mathfrak{R}[u]/ < u^n >$ If $n = 1$, then $\mathfrak{T} \cong \mathfrak{R}$. If $n = 2$, then $\mathfrak{T} \cong T(\mathfrak{R},\mathfrak{R})$ is a right $\mathfrak{б}$-$\mathcal{N}^{\mathcal{L}}\mathcal{S}$-ring by Proposition 3.3, Now for $n \geq 3$ the prove is similar to the proof of [(Kwak, 2007), Theorem 3.8]. ■

From (Harmanci *et al.*, 2021), Consider $\mathfrak{R}$ is a ring and $\mathfrak{g}$ a subring of $\mathfrak{R}$ and $T(\mathfrak{R},\mathfrak{g}) = \{(r_1, r_2, \ldots, r_n, s, s, \ldots) \mid r_\imath \in \mathfrak{R}, s \in \mathfrak{g}, 1 \leq n, 1 \leq \imath \leq n, \imath, n \in \bar{Z}\}$. The operations of the ring $T(\mathfrak{R},\mathfrak{g})$ are twice addition and multiplication. We provide sufficient and necessary criteria for $T[\mathfrak{R},\mathfrak{g}]$ to be $\mathfrak{б}$-$\mathcal{N}^{\mathcal{L}}\mathcal{S}$-ring in the following proposition.

Proposition 3.6 Consider $\mathfrak{R}$ is a ring and $\mathfrak{g}$ is a subring of $\mathfrak{R}$. Then the following are equivalent:
(1) $T[\mathfrak{R},\mathfrak{g}]$ is R-$\mathfrak{б}$-$\mathcal{N}^{\mathcal{L}}\mathcal{S}$-ring;
(2) $\mathfrak{R}$ is R-$\mathfrak{б}$-$\mathcal{N}^{\mathcal{L}}\mathcal{S}$-ring.
Proof. (1) → (2) Let $\breve{\upsilon} \in \mathfrak{R}, \acute{\rho}, \tilde{\omega} \in \mathcal{N}^{\mathcal{L}}(\mathfrak{R})$ with $\breve{\upsilon}\acute{\rho}\tilde{\omega} = O$. Let $\check{Y} = (\breve{\upsilon}, O, O, O, \cdots) \in T[\mathfrak{R},\mathfrak{g}]$, $\tilde{O} = (\acute{\rho}, O, O, O, \cdots), \text{ß} = (\tilde{\omega}, O, O, O, \cdots) \in \mathcal{N}^{\mathcal{L}}(T[\mathfrak{R},\mathfrak{g}])$ and $\check{Y}\tilde{O}\text{ß} = O$. By(1), $\check{Y}\text{ß}\mathfrak{б}(\tilde{O}) = O$ in $T[\mathfrak{R},\mathfrak{g}]$. Hence $\breve{\upsilon}c\mathfrak{б}(\acute{\rho}) = O$ and so $\mathfrak{R}$ is R-$\mathfrak{б}$-$\mathcal{N}^{\mathcal{L}}\mathcal{S}$-ring,
(2) → (1) Assume that $\check{Y} = (\breve{\upsilon}_1, \breve{\upsilon}_2, \cdots, \breve{\upsilon}_n, s, s, \cdots) \in T[\mathfrak{R},\mathfrak{g}]$ and $\tilde{O} = (\acute{\rho}_1, \acute{\rho}_2, \cdots, \acute{\rho}_n, t, t, \cdots), \text{ß} = (\tilde{\omega}_1, \tilde{\omega}_2, \cdots, \tilde{\omega}_n, h, h, \cdots) \in \mathcal{N}^{\mathcal{L}}(T[\mathfrak{R},\mathfrak{g}])$ with $\check{Y}\tilde{O}\text{ß} = O$. Then all components of $\tilde{O}$ and ß are nilpotent in $\mathfrak{R}$. Since $\mathfrak{R}$ is R-$\mathfrak{б}$-$\mathcal{N}^{\mathcal{L}}\mathcal{S}$-ring, we obtain $\check{Y}\text{ß}\mathfrak{б}(\tilde{O}) = O$. Hence $T[\mathfrak{R},\mathfrak{g}]$ is R-$\mathfrak{б}$-$\mathcal{N}^{\mathcal{L}}\mathcal{S}$-ring. ■

The polynomial ring over a right $\mathcal{N}^{\mathcal{L}}$-symmetric is now examined to see if it is a R-$\mathfrak{б}$-$\mathcal{N}^{\mathcal{L}}\mathcal{S}$-ring. However, the following example shows that the answer is negative.

Example 3.7 Assume that $\bar{Z}_2$ is the field of integers modulo 2, and consider $\tilde{A} = \bar{Z}_2[\breve{\upsilon}_O, \breve{\upsilon}_1, \breve{\upsilon}_2, \acute{\rho}_O, \acute{\rho}_1, \acute{\rho}_2, \tilde{\omega}]$ is the free algebra of polynomials with zero constant term in non-commuting intermediates $\breve{\upsilon}_O, \breve{\upsilon}_1, \breve{\upsilon}_2, \acute{\rho}_O, \acute{\rho}_1, \acute{\rho}_2$ and $\tilde{\omega}$ over $\bar{Z}_2$. Define an automorphism $\mathfrak{б}$ of $\tilde{A}$ by :

$$\breve{\upsilon}_O, \breve{\upsilon}_1, \breve{\upsilon}_2, \acute{\rho}_O, \acute{\rho}_1, \acute{\rho}_2, \tilde{\omega} \to \acute{\rho}_O, \acute{\rho}_1, \acute{\rho}_2, \breve{\upsilon}_O, \breve{\upsilon}_1, \breve{\upsilon}_2, \tilde{\omega}$$

Take an ideal $\bar{I}$ in the ring $\bar{Z}_2 + \tilde{A}$, generated by the following elements:

$\breve{\upsilon}_O\acute{\rho}_O, \breve{\upsilon}_O\acute{\rho}_1+\breve{\upsilon}_1\acute{\rho}_O, \breve{\upsilon}_O\acute{\rho}_2+\breve{\upsilon}_1\acute{\rho}_1+\breve{\upsilon}_2\acute{\rho}_O, \breve{\upsilon}_1\acute{\rho}_2+\breve{\upsilon}_2\acute{\rho}_1, \breve{\upsilon}_2\acute{\rho}_2, \breve{\upsilon}_O\hat{r}\acute{\rho}_O, \breve{\upsilon}_2\hat{r}\breve{\upsilon}_2, \acute{\rho}_O\breve{\upsilon}_O, \acute{\rho}_O\breve{\upsilon}_1+\acute{\rho}_1\breve{\upsilon}_O, \acute{\rho}_O\breve{\upsilon}_2+\acute{\rho}_1\breve{\upsilon}_1+\acute{\rho}_2\breve{\upsilon}_O, \acute{\rho}_1\breve{\upsilon}_2+\acute{\rho}_2\breve{\upsilon}_1, \acute{\rho}_O\hat{r}\breve{\upsilon}_O, \acute{\rho}_2\hat{r}\breve{\upsilon}_2, (\breve{\upsilon}_O+\breve{\upsilon}_1+\breve{\upsilon}_2)\hat{r}(\acute{\rho}_O+\acute{\rho}_1+\acute{\rho}_2), (\acute{\rho}_O+\acute{\rho}_1+\acute{\rho}_2)\hat{r}(\breve{\upsilon}_O+\breve{\upsilon}_1+\breve{\upsilon}_2)$, and $\hat{r}_1\hat{r}_2\hat{r}_3\hat{r}_4$, where $\hat{r}, \hat{r}_1, \hat{r}_2, \hat{r}_3, \hat{r}_4 \in \tilde{A}$.

Now $\widehat{\mathfrak{R}} = (\bar{Z}_2 + \tilde{A})/\bar{I}$ is symmetric by [(Huh *et al.*, 2005),Example 3.1] and so a R-$\mathcal{N}^{\mathcal{L}}$-$\mathcal{S}$ring. By [(Mohammadi *et al.*, 2012), Example 3.6],
we have $\tilde{\omega} \in \widehat{\mathfrak{R}}[и]$ and $\breve{\upsilon}_O+\breve{\upsilon}_1 и+\breve{\upsilon}_2 и^2, \acute{\rho}_O+\acute{\rho}_1 и+\acute{\rho}_2 и^2 \in \mathcal{N}^{\mathcal{L}}(\widehat{\mathfrak{R}}[и])$. Now $\tilde{\omega}(\breve{\upsilon}_O+\breve{\upsilon}_1 и+\breve{\upsilon}_2 и^2)(\acute{\rho}_O+\acute{\rho}_1 и+\acute{\rho}_2 и^2) = (\tilde{\omega}\breve{\upsilon}_O+\tilde{\omega}\breve{\upsilon}_1 и+\tilde{\omega}\breve{\upsilon}_2 и^2)(\acute{\rho}_O+\acute{\rho}_1 и+\acute{\rho}_2 и^2) = \tilde{\omega}\breve{\upsilon}_O\acute{\rho}_O + \tilde{\omega}\breve{\upsilon}_O\acute{\rho}_1 и + \tilde{\omega}\breve{\upsilon}_O\acute{\rho}_2 и^2 + \tilde{\omega}\breve{\upsilon}_1\acute{\rho}_O и + \tilde{\omega}\acute{\rho}_1\breve{\upsilon}_1 и^2 + \tilde{\omega}\breve{\upsilon}_1\acute{\rho}_2 и^3 + \tilde{\omega}\breve{\upsilon}_2\acute{\rho}_O и^2 + \tilde{\omega}\breve{\upsilon}_2\acute{\rho}_1 и^3 + \tilde{\omega}\breve{\upsilon}_2\acute{\rho}_2 и^4 = \tilde{\omega}\breve{\upsilon}_O\acute{\rho}_O + (\tilde{\omega}\breve{\upsilon}_O\acute{\rho}_1 + \tilde{\omega}\breve{\upsilon}_1\acute{\rho}_O)и + (\tilde{\omega}\breve{\upsilon}_O\acute{\rho}_2 + \tilde{\omega}\breve{\upsilon}_1\acute{\rho}_1 + \tilde{\omega}\breve{\upsilon}_2\acute{\rho}_O)и^2 + (\tilde{\omega}\breve{\upsilon}_1\acute{\rho}_2 + \tilde{\omega}\breve{\upsilon}_2\acute{\rho}_1)и^3 + \tilde{\omega}\breve{\upsilon}_2\acute{\rho}_2 и^4 \in \bar{I}[и]$, but $\tilde{\omega}(\acute{\rho}_O+\acute{\rho}_1 и+\acute{\rho}_2 и^2)ɓ\big((\breve{\upsilon}_O+\breve{\upsilon}_1 и+\breve{\upsilon}_2 и^2)\big) = \tilde{\omega}(\acute{\rho}_O+\acute{\rho}_1 и+\acute{\rho}_2 и^2)(\acute{\rho}_O+\acute{\rho}_1 и+\acute{\rho}_2 и^2) = \tilde{\omega}\acute{\rho}_O^2 + \tilde{\omega}\acute{\rho}_O\acute{\rho}_1 и + \tilde{\omega}\acute{\rho}_0\acute{\rho}_2 и^2 + \tilde{\omega}\acute{\rho}_1\acute{\rho}_0 и + \tilde{\omega}\acute{\rho}_1^2 и^2 + \tilde{\omega}\acute{\rho}_1\acute{\rho}_2 и^3 + \tilde{\omega}\acute{\rho}_2\acute{\rho}_0 и^2 + \tilde{\omega}\acute{\rho}_2\acute{\rho}_1 и^3 + \tilde{\omega}\acute{\rho}_2^2 и^4 = \tilde{\omega}\acute{\rho}_O^2 + (\tilde{\omega}\acute{\rho}_O\acute{\rho}_1 + \tilde{\omega}\acute{\rho}_1\acute{\rho}_0)и + (\tilde{\omega}\acute{\rho}_0\acute{\rho}_2 + \tilde{\omega}\acute{\rho}_1^2 + \tilde{\omega}\acute{\rho}_2\acute{\rho}_0)и^2 + (\tilde{\omega}\acute{\rho}_1\acute{\rho}_2 + \tilde{\omega}\acute{\rho}_2\acute{\rho}_1)и^3 + \tilde{\omega}\acute{\rho}_2^2 и^4 \notin \bar{I}[и]$, because $\acute{\rho}_O^2, \tilde{\omega}\acute{\rho}_O\acute{\rho}_1 + \tilde{\omega}\acute{\rho}_1\acute{\rho}_0, \hat{c}\hat{\mathscr{b}}_O\hat{\mathscr{b}}_2 + \tilde{\omega}\acute{\rho}_0\acute{\rho}_2 + \tilde{\omega}\acute{\rho}_1^2 + \tilde{\omega}\acute{\rho}_2\acute{\rho}_0, \tilde{\omega}\acute{\rho}_1\acute{\rho}_2 + \tilde{\omega}\acute{\rho}_2\acute{\rho}_1, \tilde{\omega}\acute{\rho}_2^2 \notin \bar{I}$. Hence $\widehat{\mathfrak{R}}[и]$ is not a R-ɓ-$\mathcal{N}^{\mathcal{L}}\mathcal{S}$-ring.■

According to Rege and Chhawchharia (Rege&Chhawchharia,1997),a ring $\widehat{\mathfrak{R}}$ Armendariz exists if whenever any polynomials $\mathscr{f}(и) = \breve{\upsilon}_O + \breve{\upsilon}_1 и + \cdots + \breve{\upsilon}_m и^m, \mathscr{g}(и) = \acute{\rho}_O + \acute{\rho}_1 и + \cdots + \acute{\rho}_n и^n \in \widehat{\mathfrak{R}}[и]$ satisfy $\mathscr{f}(и)\mathscr{g}(и) = O$, then $\breve{\upsilon}_j\acute{\rho}_ɉ = O$ for each j and $ɉ$.
Since Armendariz was the first to demonstrate that a red-ring always satisfies this criterion, they used this terminology ([(Armendariz, 1974), Lemma1]). Assume $\widehat{\mathfrak{R}}$ is a ring with an *endo* ɓ. Recall that the map $\widehat{\mathfrak{R}}[и] \to \widehat{\mathfrak{R}}[и]$ by $\sum_{j=O}^{m} \breve{\upsilon}_j и^j \to \sum_{j=O}^{m} ɓ(\breve{\upsilon})и^j$.

Proposition 3.8 Suppose $\widehat{\mathfrak{R}}$ is an Armendariz ring then $\widehat{\mathfrak{R}}$ is R-ɓ-$\mathcal{N}^{\mathcal{L}}\mathcal{S}$-ring if and only if $\widehat{\mathfrak{R}}[и]$ is a R-ɓ-$\mathcal{N}^{\mathcal{L}}\mathcal{S}$-ring.
Proof. It also suffices to establish necessity. Let $\mathscr{f}(и) = \sum_{j=O}^{m} \breve{\upsilon}_j и^j \in \widehat{\mathfrak{R}}[и]$ and $\mathscr{g}(и) = \sum_{ɉ=O}^{n} \acute{\rho}_ɉ и^ɉ$, $\mathscr{h}(и) = \sum_{\check{r}=O}^{w} \tilde{\omega}_{\check{r}} и^{\check{r}} \in \mathcal{N}^{\mathcal{L}}(\widehat{\mathfrak{R}}[и])$ with $\mathscr{f}(и)\mathscr{g}(и)\mathscr{h}(и) = O$ and so $\breve{\upsilon}_j\acute{\rho}_ɉ\tilde{\omega}_{\check{r}} = O$ for all $j, ɉ$ and $\check{r}$. $\breve{\upsilon}_j\tilde{\omega}_{\check{r}}ɓ(\acute{\rho}_ɉ) = O$ since $\widehat{\mathfrak{R}}$ is Armendariz and a R-ɓ-$\mathcal{N}^{\mathcal{L}}\mathcal{S}$-ring. This yields $\mathscr{f}(и)\mathscr{h}(и)\, ɓ\big(\mathscr{g}(и)\big) = O$, therefore, $\widehat{\mathfrak{R}}[и]$ is a R-ɓ-$\mathcal{N}^{\mathcal{L}}\mathcal{S}$-ring.

Theorem 3.9 (1) For a ring $\widehat{\mathfrak{R}}$, if $\widehat{\mathfrak{R}}$ is ɓ-rg then $\widehat{\mathfrak{R}}$ is a R-ɓ-$\mathcal{N}^{\mathcal{L}}\mathcal{S}$-ring.
(2) If the skew polynomial ring $\widehat{\mathfrak{R}}[и; ɓ]$ of a ring $\widehat{\mathfrak{R}}$ is a $\mathcal{S}$-ring, then $\widehat{\mathfrak{R}}$ is a ɓ-$\mathcal{N}^{\mathcal{L}}\mathcal{S}$-ring.
Proof. (1) Consider $\widehat{\mathfrak{R}}$ is ɓ-rg. Note that any ɓ-rg ring is reduced and ɓ is a monomorphism by [(Marks, 2002), P.218]. We show that $\widehat{\mathfrak{R}}$ is R-ɓ-$\mathcal{N}^{\mathcal{L}}\mathcal{S}$-ring. Assume $\breve{\upsilon}\acute{\rho}\tilde{\omega} = O$ for $\breve{\upsilon} \in \widehat{\mathfrak{R}}$ and $\acute{\rho}, \tilde{\omega} \in \mathcal{N}^{\mathcal{L}}(\widehat{\mathfrak{R}})$. Then we obtain $\acute{\rho}\breve{\upsilon}\tilde{\omega} = O$, since $\widehat{\mathfrak{R}}$ is reduced (and so symmetric). Thus,
$\breve{\upsilon}\tilde{\omega}ɓ(\acute{\rho})ɓ\big(\breve{\upsilon}\tilde{\omega}ɓ(\acute{\rho})\big) = \breve{\upsilon}\tilde{\omega}ɓ(\acute{\rho}\breve{\upsilon}\tilde{\omega})ɓ^{\tilde{\omega}}(\acute{\rho}) = O$. Since $\widehat{\mathfrak{R}}$ is ɓ-rg, $\breve{\upsilon}\tilde{\omega}ɓ(\acute{\rho}) = O$ and thus $\widehat{\mathfrak{R}}$ is a R-ɓ-$\mathcal{N}^{\mathcal{L}}\mathcal{S}$-ring.
(2) Assume $\breve{\upsilon}\acute{\rho}\tilde{\omega} = O$ for $\breve{\upsilon}, \acute{\rho}, \tilde{\omega} \in \mathcal{N}^{\mathcal{L}}(\widehat{\mathfrak{R}})$. Let $ꝛ = \breve{\upsilon}, ꜱ = \acute{\rho}, ŧ = \tilde{\omega}x \in \widehat{\mathfrak{R}}[и; ɓ]$ Then $ꝛꜱŧ = \breve{\upsilon}\acute{\rho}\tilde{\omega}и = O \in \widehat{\mathfrak{R}}[и; ɓ]$, since $\widehat{\mathfrak{R}}[и; ɓ]$ is $\mathcal{S}$-ring, we get $O = ꝛŧꜱ = (\breve{\upsilon}\tilde{\omega})и\acute{\rho} = \breve{\upsilon}\tilde{\omega}ɓ(\acute{\rho})и$, and so $\breve{\upsilon}\tilde{\omega}ɓ(\acute{\rho}) = O$. Thus $\widehat{\mathfrak{R}}$ is a R-ɓ-$\mathcal{N}^{\mathcal{L}}\mathcal{S}$-ring. ■

The Dorroh extension(For short *DoEx*) of an algebra $\widehat{\mathfrak{R}}$ over a commutative ring $\hat{S}$, introduced by Dorroh in 1932(Dorroh, 1932), is a construction that enlarges $\widehat{\mathfrak{R}}$ by incorporating elements of $\widehat{\mathfrak{R}}$. It is defined as the Abelian group $\widehat{\mathcal{D}} = \widehat{\mathfrak{R}} \times \hat{S}$ with multiplication given by $(r_1, s_1)(r_2, s_2) = (r_1r_2 + s_1r_2 + s_2r_1, s_1s_2)$ for all $r_i \in \widehat{\mathfrak{R}}$ and $s_i \in \hat{S}$. This operation preserves the algebraic structure while introducing a direct interaction between elements of $\widehat{\mathfrak{R}}$ and $\hat{S}$. Additionally, any $\hat{S}$-linear *endo* ɓ of $\widehat{\mathfrak{R}}$ extends naturally to an S, S-algebra homomorphism $ɓ: \widehat{\mathcal{D}} \to \widehat{\mathcal{D}}$, defined by $ɓ(r, s) = (ɓ(r), s)$, applying ɓ to the first component while keeping the second component fixed.

Theorem 3.10 Consider $\widehat{\mathfrak{R}}$ is an algebra equipped with an *endo* ɓ and an identity element, defined over a commutative red-ring $\bar{\mathcal{Z}}$. Then $\widehat{\mathfrak{R}}$ is a R-ɓ-$\mathcal{N}^{\mathcal{L}}\mathcal{S}$-ring if and only if the *DoEx* $\widehat{\mathcal{D}}$ of $\widehat{\mathfrak{R}}$ by $\bar{\mathcal{Z}}$ is R-ɓ-$\mathcal{N}^{\mathcal{L}}\mathcal{S}$-ring.
Proof. It is clear that $\mathcal{N}^{\mathcal{L}}(\widehat{\mathcal{D}}) = (\mathcal{N}^{\mathcal{L}}(\widehat{\mathfrak{R}}), O)$. Since $\bar{\mathcal{Z}}$ is a commutative red-ring. Consider $(\breve{\upsilon}, O), (\acute{\rho}, O) \in \mathcal{N}^{\mathcal{L}}(\widehat{\mathcal{D}}(\widehat{\mathfrak{R}}, Z))$ and $(\breve{\upsilon}, \ddot{\varepsilon}) \in \widehat{\mathcal{D}}(\widehat{\mathfrak{R}}, Z)$ with $(\acute{\eta}, \ddot{\varepsilon})(\breve{\upsilon}, O)(\acute{\rho}, O) = \big((\acute{\eta} + \ddot{\varepsilon})\breve{\upsilon}, O\big)(\acute{\rho}, O) = \big((\acute{\eta} + \ddot{\varepsilon})\breve{\upsilon}\acute{\rho}, O\big)$. Thus $(\acute{\eta} + \ddot{\varepsilon})\breve{\upsilon}\acute{\rho} = O$, $\breve{\upsilon}, \acute{\rho} \in \mathcal{N}^{\mathcal{L}}(\widehat{\mathfrak{R}})$. Since $\widehat{\mathfrak{R}}$ is ɓ-$\mathcal{N}^{\mathcal{L}}\mathcal{S}$-ring, we get $\acute{\eta} + \ddot{\varepsilon} \in Z$, $(\acute{\eta} + \ddot{\varepsilon})\acute{\rho}ɓ(\breve{\upsilon}) = O$. So $(\acute{\eta}, \ddot{\varepsilon})(\acute{\rho}, O)ɓ\big((\breve{\upsilon}, O)\big) = O$. Thus $\widehat{\mathcal{D}}(\widehat{\mathfrak{R}}, Z)$ is $\bar{ɓ}$-$\mathcal{N}^{\mathcal{L}}\mathcal{S}$-ring.■

SOME LOCALIZATIONS OF RIGHT ɓ-$\mathcal{N}^{\mathcal{L}}$-SYMMETRIC RINGS:

Assume that ɓ is a monomorphism of the ring $\widehat{\mathfrak{R}}$. The construction of an over-ring of $\widehat{\mathfrak{R}}$. (A ring $\widehat{\mathfrak{R}}$ is an over ring of integral domain $\mathscr{g}$, if $\mathscr{g}$ is a subring of $\widehat{\mathfrak{R}}$ and $\widehat{\mathfrak{R}}$ is a subring of the field of fraction $\mathcal{Q}(\mathscr{g})$, the relationship $\mathscr{g} \subseteq \widehat{\mathfrak{R}} \subseteq \mathcal{Q}(\mathscr{g})$). As introduced by Jordan, is now under consideration (for more details, see (Jordan, 1982)). Define $\breve{Y}(\widehat{\mathfrak{R}}, ɓ)$ as the subset of the skew Laurent polynomial ring $\widehat{\mathfrak{R}}[и, и^{-1}; ɓ]$, consisting of elements of the form $и^{-n}\ddot{\varepsilon}и^n$ for $\ddot{\varepsilon} \in \widehat{\mathfrak{R}}$ and $n \geq O$. Notably, for $m \geq O$, the relation $и^{-m}\ddot{\varepsilon}и^m = ɓ^{-m}(\ddot{\varepsilon})$ hold for any $\ddot{\varepsilon} \in \widehat{\mathfrak{R}}$. This implies that for any $m \geq O$, the transformation follows the pattern:

$$и^{-n}\ddot{\varepsilon}и^n = и^{-(n+m)}ɓ^{-m}(\ddot{\varepsilon})и^{n+m}.$$

From this, it follows that $\breve{Y}(\widehat{\mathfrak{R}}, ɓ)$ forms a subring of $\widehat{\mathfrak{R}}[и, и^{-1}; ɓ]$, equipped with the natural operation:

$$(и^{-з}\ddot{\varepsilon}и^з)(и^{-\epsilon}\tilde{\eta}и^{\epsilon}) = и^{-(з+\epsilon)}ɓ^{\epsilon}(\ddot{\varepsilon})ɓ^{з}(\tilde{\eta})и^{з+\epsilon},$$

And,
$и^{-з}\ddot{\varepsilon}и^з + и^{-\epsilon}\tilde{\eta}и^{\epsilon} = и^{-(з+\epsilon)}\big(ɓ^{\epsilon}(\ddot{\varepsilon}) + ɓ^{з}(\tilde{\eta})\big)и^{з+\epsilon}, \forall \ddot{\varepsilon}, \tilde{\eta} \in \widehat{\mathfrak{R}}$ and $з, \epsilon \geq O$.
Notably, $\breve{Y}(\widehat{\mathfrak{R}}, ɓ)$ serves as an over-ring of $\widehat{\mathfrak{R}}$, and the mapping $\breve{Y}(\widehat{\mathfrak{R}}, ɓ) \to \breve{Y}(\widehat{\mathfrak{R}}, ɓ)$ defined by $и^{-з}\ddot{\varepsilon}и^з \to и^{-з}ɓ(\ddot{\varepsilon})и^з$, is an automorphism of $\breve{Y}(\widehat{\mathfrak{R}}, ɓ)$.
Jordan established that such an extension $\breve{Y}(\widehat{\mathfrak{R}}, ɓ)$ always exists for any given pair $(\widehat{\mathfrak{R}}, ɓ)$ (Jordan, 1982).
This is achieved using left localization of the skew polynomial $\widehat{\mathfrak{R}}[и, ɓ]$ with respect to the set of powers of $и$. This extension $\breve{Y}(\widehat{\mathfrak{R}}, ɓ)$ is commonly referred to as the Jordan extension of $\widehat{\mathfrak{R}}$ by ɓ.

Proposition 4.1 Consider $\widehat{\mathfrak{R}}$ is a ring with a monomorphism, then $\widehat{\mathfrak{R}}$ is R-ɓ-$\mathcal{N}^{\mathcal{L}}\mathcal{S}$-ring if and only if the Jordan extension $\breve{Y} = \breve{Y}(\widehat{\mathfrak{R}}, ɓ)$ is R-ɓ-$\mathcal{N}^{\mathcal{L}}\mathcal{S}$-ring.

Proof. If $\widehat{\mathfrak{R}}$ is R-ɓ-$\mathcal{N}^{\mathcal{L}}\mathcal{S}$-ring, then so is each subring $\mathscr{g}$ with $ɓ(\breve{Y}) \subseteq \breve{Y}$. Therefore, it is enough to demonstrate the necessity. Assume $\widehat{\mathfrak{R}}$ is ɓ-$\mathcal{N}^{\mathcal{L}}\mathcal{S}$-ring and $\breve{\upsilon}\acute{\rho}\tilde{\omega} = O$ where $\breve{\upsilon} = и^{-з}\ddot{\varepsilon}_1и^з \in \hat{\mathcal{A}}, \acute{\rho} = и^{-\epsilon}\ddot{\varepsilon}_2и^{\epsilon}, \tilde{\omega} = и^{-ғ}\ddot{\varepsilon}_3и^{ғ} \in \mathcal{N}^{\mathcal{L}}(\hat{\mathcal{A}})$ for $з, \epsilon, ғ > O$. Then $\ddot{\varepsilon}_1 \in \widehat{\mathfrak{R}}$ and $\ddot{\varepsilon}_2, \ddot{\varepsilon}_3 \in \mathcal{N}^{\mathcal{L}}(\widehat{\mathfrak{R}})$. From $\breve{\upsilon}\acute{\rho}\tilde{\omega} = O$, we get $ɓ^{ғ}(\ddot{\varepsilon}_1)ɓ^{\epsilon}(\ddot{\varepsilon}_2)ɓ^{з}(\ddot{\varepsilon}_3) = O$ and so $ɓ^{ғ}(\ddot{\varepsilon}_1)ɓ^{з}(\ddot{\varepsilon}_3)ɓ\big(ɓ^{\epsilon}(\ddot{\varepsilon}_2)\big) = ɓ^{ғ}(\ddot{\varepsilon}_1)ɓ^{з}(\ddot{\varepsilon}_3)ɓ^{\epsilon+1}(\ddot{\varepsilon}_2) = O$ by assumption. Hence $\breve{\upsilon}\tilde{\omega}ɓ(\acute{\rho}) = (и^{-з}\ddot{\varepsilon}_1и^з)(и^{-ғ}\ddot{\varepsilon}_3и^{ғ})ɓ(и^{-\epsilon}\ddot{\varepsilon}_2и^{\epsilon}) = (и^{-з}\ddot{\varepsilon}_1и^з)(и^{-ғ}\ddot{\varepsilon}_3и^{ғ})(и^{-\epsilon}ɓ(\ddot{\varepsilon}_2)и^{\epsilon}) = и^{-(з+ғ+\epsilon)}ɓ^{ғ}(\ddot{\varepsilon}_1)ɓ^{з}(\ddot{\varepsilon}_3)ɓ^{\epsilon+1}(\ddot{\varepsilon}_2)и^{+(з+ғ+\epsilon)} = O$.
Therefore, Jordan extension $\hat{\mathcal{A}}(\widehat{\mathfrak{R}}, ɓ)$ is right ɓ-$\mathcal{N}^{\mathcal{L}}\mathcal{S}$-ring. ■

Recall that the map $\mathfrak{R}[\varkappa,\varkappa^{-1}] \to \mathfrak{R}[\varkappa,\varkappa^{-1}]$ defined by $\sum_{i=-n}^{\infty} a_i\varkappa^i \to \sum_{i=-n}^{\infty} \mathfrak{б}(a_i)\varkappa^i$ is an *endo* of $\mathfrak{R}[\varkappa,\varkappa^{-1}]$ and the map obviously extends $\mathfrak{б}$.

Proposition 4.2 If $\mathfrak{R}$ is an Armendariz ring, then the following claims are equivalent:

(1) $\mathfrak{R}$ is a R-$б$-$\mathcal{N}^{\mathcal{L}}\mathcal{S}$-ring;

(2) $\mathfrak{R}[\varkappa]$ is a R-$б$-$\mathcal{N}^{\mathcal{L}}\mathcal{S}$-ring;

(3) $\mathfrak{R}[\varkappa,\varkappa^{-1}]$ is a R-$б$-$\mathcal{N}^{\mathcal{L}}\mathcal{S}$-ring.

Proof. (1) $\leftrightarrow$ (2) is proven in proposition 3.8

(2) $\leftrightarrow$ (3) Showing necessity is sufficient. Let $F(\varkappa) \in \mathfrak{R}[\varkappa,\varkappa^{-1}]$ and $\mathfrak{H}(\varkappa), \mathfrak{K}(\varkappa) \in \mathcal{N}^{\mathcal{L}}(\mathfrak{R}[\varkappa,\varkappa^{-1}])$ with $F(\varkappa)\mathfrak{H}(\varkappa)\mathfrak{K}(\varkappa) = 0$. Then $\exists n \in \bar{\mathcal{Z}}^+$ such that $F_1(\varkappa) = F(\varkappa)\varkappa^n \in \mathfrak{R}[\varkappa]$ and $\mathfrak{H}_1(\varkappa) = \mathfrak{H}(\varkappa)\varkappa^n$, $\mathfrak{K}_1(\varkappa) = \mathfrak{K}(\varkappa)\varkappa^n \in \mathcal{N}^{\mathcal{L}}(\mathfrak{R}[\varkappa])$ and so $F_1(\varkappa)\mathfrak{H}_1(\varkappa)\mathfrak{K}_1(\varkappa) = 0$. Since $\mathfrak{R}[\varkappa]$ is R-$б$-$\mathcal{N}^{\mathcal{L}}\mathcal{S}$-ring, we obtain $F_1(\varkappa)\mathfrak{K}_1(\varkappa)\, б(\mathfrak{H}_1(\varkappa)) = 0$. Hence $F(\varkappa)\mathfrak{K}(\varkappa)\, б(\mathfrak{H}(\varkappa)) = \varkappa^{-3n}F_1(\varkappa)\mathfrak{K}_1(\varkappa)\, б(\mathfrak{H}_1(\varkappa)) = 0$. Thus $\mathfrak{R}[\varkappa,\varkappa^{-1}]$ is a R-$б$-$\mathcal{N}^{\mathcal{L}}\mathcal{S}$-ring.

(3) $\to$ (2) and (3) $\to$ (1) are clear. ■

Proposition 4.3 Assume that $\mathfrak{R}$ is a ring and that $\bar{\mathcal{Z}}(\mathfrak{R})$ is an infinite subring with all of its nonzero elements regular in $\mathfrak{R}$. Then $\mathfrak{R}$ is R-$б$-$\mathcal{N}^{\mathcal{L}}\mathcal{S}$-ring if and only if $\mathfrak{R}[\varkappa]$ is R-$б$-$\mathcal{N}^{\mathcal{L}}\mathcal{S}$-ring if and only if $\mathfrak{R}[\varkappa;\varkappa^{-1}]$ is R-$б$-$\mathcal{N}^{\mathcal{L}}\mathcal{S}$-ring.

Proof. It is sufficient to demonstrate that, $\mathfrak{R}[\varkappa]$ is $б$-$\mathcal{N}^{\mathcal{L}}\mathcal{S}$-ring when so is $\mathfrak{R}$, $\mathfrak{R}[\varkappa]$is obtained as the subdirect product of an infinite collection of copies of $\mathfrak{R}$, as $\bar{\mathcal{Z}}(\mathfrak{R})$ comprises an infinite subring where each nonzero element is regular in $\mathfrak{R}$ according to the hypothesis. Thus $\mathfrak{R}[\varkappa]$ is $б$-$\mathcal{N}^{\mathcal{L}}\mathcal{S}$-ring because $\mathfrak{R}$ is $б$-$\mathcal{N}^{\mathcal{L}}\mathcal{S}$-ring by the assumption. ■

ON RIGHT $б$-$\mathcal{N}^{\mathcal{L}}$-SYMMETRIC MODULES:

This section extends the idea of a R-$б$-$\mathcal{N}^{\mathcal{L}}\mathcal{S}$-ring to modules by introducing the notion of a right $б$-$\mathcal{N}^{\mathcal{L}}$-symmetric module, which is an extension of symmetric modules and generalization of $б$-symmetric modules. Some of the well-established results which are obtained in section 3 and section 4 are generalized to right $б$-$\mathcal{N}^{\mathcal{L}}$-symmetric modules. We introduce the following definition first.

Definition 5.1 Assume $\mathfrak{R}$ is a ring and $б$ a nonzero *endo* of $\mathfrak{R}$. An $\mathfrak{R}$-module $\widehat{\mathcal{M}}^{\mathfrak{R}}$ is called a right $б$-$\mathcal{N}^{\mathcal{L}}$-symmetric modules (For short R-$б$-$\mathcal{N}^{\mathcal{L}}\mathcal{S}$-module) if whenever $mab = 0$ for $a, b \in \mathcal{N}^{\mathcal{L}}(\mathfrak{R})$ and $m \in \widehat{\mathcal{M}}^{\mathfrak{R}}$ implies $mbб(a) = 0$.

Example 5.2:

1. R-$1_{\mathfrak{R}}$-$\mathcal{N}^{\mathcal{L}}$-symmetric modules are exactly R-$б$-$\mathcal{N}^{\mathcal{L}}\mathcal{S}$-module.
2. For any commutative ring, any module $\widehat{\mathcal{M}}^{\mathfrak{R}}$ is an $б$-$\mathcal{N}^{\mathcal{L}}\mathcal{S}$-modules.
3. Let $\hat{\mathrm{D}}$ be a division ring, $\mathfrak{R} = \begin{bmatrix} \hat{\mathrm{D}} & \hat{\mathrm{D}} \\ 0 & \hat{\mathrm{D}} \end{bmatrix}$, and $\mathcal{A} = \begin{bmatrix} 0 & \hat{\mathrm{D}} \\ 0 & \hat{\mathrm{D}} \end{bmatrix}$. Then $\mathcal{A}^{\mathfrak{R}}$ is an $б$-$\mathcal{N}^{\mathcal{L}}\mathcal{S}$-module.
4. It is clear that $б$-symmetric modules are $б$-$\mathcal{N}^{\mathcal{L}}\mathcal{S}$-module but the converse implication is not true as we see in the following example.

Example 5.3 Let $\bar{\mathcal{Z}}$ be the ring of integers. We now consider the ring $\mathfrak{R} = \left\{\begin{pmatrix} \breve{\upsilon} & \acute{\rho} \\ 0 & \tilde{\omega} \end{pmatrix}; \breve{\upsilon}, \acute{\rho}, \tilde{\omega} \in \bar{\mathcal{Z}}\right\}$ and the $\mathfrak{R}$-module $\widehat{\mathcal{M}}^{\mathfrak{R}} = \left\{\begin{pmatrix} 0 & ч \\ ӈ & ƅ \end{pmatrix}; ч, ӈ, ƅ \in \bar{\mathcal{Z}}\right\}$ and $б$ an homomorphism defined on $\mathfrak{R}$ by $б\left(\begin{pmatrix} \breve{\upsilon} & \acute{\rho} \\ 0 & \tilde{\omega} \end{pmatrix}\right) = \begin{pmatrix} 0 & \acute{\rho} \\ 0 & 0 \end{pmatrix}$ where $\begin{pmatrix} \breve{\upsilon} & \acute{\rho} \\ 0 & \tilde{\omega} \end{pmatrix} \in \mathfrak{R}$. $\mathfrak{R}$ is R-$б$-$\mathcal{N}^{\mathcal{L}}\mathcal{S}$-module for $m = \begin{pmatrix} 0 & ч \\ ӈ & ƅ \end{pmatrix} \in \widehat{\mathcal{M}}^{\mathfrak{R}}$ and $\mathfrak{h}, \mathfrak{k} \in \mathcal{N}^{\mathcal{L}}(\mathfrak{R})$ where $\mathfrak{h} = \begin{pmatrix} 0 & \acute{\rho}_1 \\ 0 & 0 \end{pmatrix}$, $\mathfrak{k} = \begin{pmatrix} 0 & \acute{\rho}_2 \\ 0 & 0 \end{pmatrix}$ we have,

$$m\mathfrak{h}\mathfrak{k} = \begin{pmatrix} 0 & ч \\ ӈ & ƅ \end{pmatrix}\begin{pmatrix} 0 & \acute{\rho}_1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & \acute{\rho}_2 \\ 0 & 0 \end{pmatrix} = 0$$

Also,

$$m\mathfrak{k}б(\mathfrak{h}) = \begin{pmatrix} 0 & ч \\ ӈ & ƅ \end{pmatrix}\begin{pmatrix} 0 & \acute{\rho}_2 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & \acute{\rho}_1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = 0.$$

But $\widehat{\mathcal{M}}^{\mathfrak{R}}$ is not $б$-symmetric for $m = \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix} \in M$, $\mathfrak{h} = \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix}$, $\mathfrak{k} = \begin{pmatrix} 1 & 2 \\ 0 & 0 \end{pmatrix} \in \mathfrak{R}$, we have,

$$m\mathfrak{h}\mathfrak{k} = \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = 0$$

But, $m\mathfrak{k}б(\mathfrak{h}) = \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 2 \end{pmatrix} \neq 0.$

However, the converse is true if, $\widehat{\mathcal{M}}^{\mathfrak{R}}$ is an $б$-rg-module by the following Lemma.

Lemma 5.4 Let $\widehat{\mathcal{M}}^{\mathfrak{R}}$ be an $б$-rg-module, then the following are equivalent:

1. $\widehat{\mathcal{M}}^{\mathfrak{R}}$ is an $б$-symmetric module;
2. $\widehat{\mathcal{M}}^{\mathfrak{R}}$ is an $б$-$\mathcal{N}^{\mathcal{L}}$ symmetric module.

Proof. (1) $\Longrightarrow$ (2) It is clear.

(2) $\Longrightarrow$ (1) Let $m\acute{\rho}^2 = 0$, for $m \in \widehat{\mathcal{M}}^{\mathfrak{R}}$ and $\acute{\rho} \in \mathfrak{R}$. If $m = 0$, is trivial. Then $\acute{\rho}^2 = 0$ implies $\acute{\rho} \in \mathcal{N}^{\mathcal{L}}(\mathfrak{R})$, since $\widehat{\mathcal{M}}^{\mathfrak{R}}$ is $б$-$\mathcal{N}^{\mathcal{L}}$ symmetric. Hence $m\acute{\rho}^2 = 0$ implies $m\acute{\rho}\acute{\rho} = 0$ implies $m\acute{\rho}б(\acute{\rho}) = 0$, and since $\widehat{\mathcal{M}}^{\mathfrak{R}}$ is an $б$-rg-module implies that $m\acute{\rho} = 0$. Therefore, $\widehat{\mathcal{M}}^{\mathfrak{R}}$ is an $б$-red-module and by [(Agayev *et al.*, 2009), Theorem 2.1] $\widehat{\mathcal{M}}^{\mathfrak{R}}$ is an $б$-symmetric module. ■

Proposition 5.5 For a given *endo* of a ring $\mathfrak{R}$ and an $\mathfrak{R}$-module $\widehat{\mathcal{M}}^{\mathfrak{R}}$. The statements below are equivalent:

1. $\widehat{\mathcal{M}}^{\mathfrak{R}}$ is R-$б$-$\mathcal{N}^{\mathcal{L}}\mathcal{S}$-module,
2. $\ell_{\widehat{\mathcal{M}}^{\mathfrak{R}}}(\breve{\upsilon}(\acute{\rho})) \subseteq \ell_{\widehat{\mathcal{M}}^{\mathfrak{R}}}(\acute{\rho}б(\breve{\upsilon}))$, for any $\breve{\upsilon}, \acute{\rho} \in \mathcal{N}^{\mathcal{L}}(\mathfrak{R})$,
3. $\check{Y}\tilde{U}\tilde{V} = 0$ if and only if $\check{Y}\tilde{V}б(\tilde{U}) = 0$,for $\tilde{U}, \tilde{V} \subseteq \mathcal{N}^{\mathcal{L}}(\mathfrak{R})$ and $\check{Y} \subseteq \widehat{\mathcal{M}}^{\mathrm{R}^*}$,
4. $\ell_{\check{Y}}(\tilde{U}\tilde{V}) \subseteq \ell_{\check{Y}}(\tilde{V}б(\tilde{U}))$, for any $\tilde{U}, \tilde{V} \subseteq \mathcal{N}^{\mathcal{L}}(\mathfrak{R})$ and $\check{Y} \subseteq \widehat{\mathcal{M}}^{\mathfrak{R}}$.

Proof. (1) $\to$ (3) Suppose that $\check{Y}\tilde{U}\tilde{V} = 0$, for $\tilde{U}, \tilde{V} \subseteq \mathcal{N}^{\mathcal{L}}(\mathfrak{R})$ and $\check{Y} \subseteq \widehat{\mathcal{M}}^{\mathfrak{R}}$. Then $\breve{\upsilon}\acute{\rho}\tilde{\omega} = 0$ for any $\breve{\upsilon} \in \check{Y}, \acute{\rho} \in \tilde{U}$ and $\tilde{\omega} \in \tilde{V}$, and hence $\breve{\upsilon}\tilde{\omega}\, б(\acute{\rho}) = 0$. Therefore $\check{Y}\tilde{V}б(\tilde{U}) = \{\sum_{i=1} \breve{\upsilon}_i\tilde{\omega}_iб(\acute{\rho}_i)\,;\ \breve{\upsilon}_i \in \check{Y}, \acute{\rho}_i \in \tilde{U}$ and $\tilde{\omega}_i \in \tilde{V}\} = 0$. The converse is clear. (1) $\to$ (2) and (3) $\to$ (4) is obvious ■

Proposition 5.6 Suppose that $\mathfrak{R}$ is a ring and $б$ an *endo* of $\mathfrak{R}$ and $\mathcal{M}^{\mathfrak{R}}$ is an $\mathfrak{R}$-module. Then we have the following:

1. $m\acute{\rho}_1\acute{\rho}_2 \dots \acute{\rho}_{\varpi} = 0$ implies $m\acute{\rho}_{б(1)}\acute{\rho}_{б(2)} \dots \acute{\rho}_{б(\varpi)} = 0$ for each permutation $б$ of the set $\{1,2,\dots,\varpi\}$, where $\acute{\rho}_i \in \mathcal{N}^{\mathcal{L}}(\mathfrak{R})$ and $\varpi \in \bar{\mathcal{Z}}^+$.
2. $m\breve{\upsilon}_1\breve{\upsilon}_2 \dots \breve{\upsilon}_{\varpi} = 0$ if and only if $\widehat{\mathfrak{m}}б^{i_1}(\breve{\upsilon}_1)\, б^{i_2}(\breve{\upsilon}_2) \dots б^{i_{\varpi}}(\breve{\upsilon}_{\varpi}) = 0$ for any $i_1, i_2 \dots i_{\varpi} \in \bar{\mathcal{Z}}^+$.

Proof. The proof is similar to the proof of [(Agayev *et al.*, 2009), Proposition2.4]. ■

Proposition 5.7 Suppose $\mathfrak{R}$ is a ring and $б$ an *endo* of $\mathfrak{R}$ and $\mathfrak{R}$-module $\widehat{\mathcal{M}}^{\mathfrak{R}}$. Then we have the following:

1. The class of a R-$б$-$\mathcal{N}^{\mathcal{L}}\mathcal{S}$-modules is closed under submodules, and direct sums.
2. The direct product of R-$б$-$\mathcal{N}^{\mathcal{L}}\mathcal{S}$-modules is R-$б$-$\mathcal{N}^{\mathcal{L}}\mathcal{S}$-module.

3. If $ҿ$ is a central idempotent of a ring $\widehat{\mathfrak{R}}$ with $ϭ(ҿ) = ҿ$ and $ϭ(1-ҿ) = 1-ҿ$, then $\widehat{\mathcal{M}}^{ҿ\widehat{\mathfrak{R}}}$ and $\widehat{\mathcal{M}}^{(1-ҿ)\widehat{\mathfrak{R}}}$ are R-$ϭ$-$\mathcal{N}^{\mathcal{L}}\mathcal{S}$-module if and only if $\widehat{\mathcal{M}}^{\widehat{\mathfrak{R}}}$ is right $ϭ$-$\mathcal{N}^{\mathcal{L}}\mathcal{S}$-module.

Proof. (1) Depending on the definitions and algebraic structures, the proof is straightforward.

(2) Note that $\mathcal{N}^{\mathcal{L}}\left(\prod_{\mathfrak{f}\in I}\widehat{\mathfrak{R}}_{\mathfrak{f}}\right) \subseteq \prod_{\mathfrak{f}\in I}\mathcal{N}^{\mathcal{L}}\left(\widehat{\mathfrak{R}}_{\mathfrak{f}}\right)$ and $ϭ_{\mathfrak{f}}\left(\widehat{\mathfrak{R}}_{\mathfrak{f}}\right) \subseteq \widehat{\mathfrak{R}}_{\mathfrak{f}}$ for each $\mathfrak{f} \in I$. Suppose that $\widehat{\mathcal{M}}^{\widehat{\mathfrak{R}}_{\mathfrak{f}}}$ is $ϭ$-$\mathcal{N}^{\mathcal{L}}\mathcal{S}$-module for each $\mathfrak{f} \in I$ and let $\widehat{\mathcal{M}}\widehat{\mathcal{A}}\widehat{\mathcal{B}} = 0$ where, $\widehat{\mathcal{A}} = (\widehat{a}_{\mathfrak{f}})_{\mathfrak{f}\in I}$, $\widehat{\mathcal{B}} = (\widehat{b}_{\mathfrak{f}})_{\mathfrak{f}\in I} \in \mathcal{N}^{\mathcal{L}}\left(\prod_{\mathfrak{f}\in I}\widehat{\mathfrak{R}}_{\mathfrak{f}}\right)$ and $\widehat{\mathcal{M}} = (\mathrm{m}_{\mathfrak{f}})_{\mathfrak{f}\in I} \in \prod_{\mathfrak{f}\in I}\left(\widehat{\mathcal{M}}_{\mathfrak{f}}^{\widehat{\mathfrak{R}}_{\mathfrak{f}}}\right)$. Then $m_{\mathfrak{f}}\widehat{a}_{\mathfrak{f}}\widehat{b}_{\mathfrak{f}} = 0$ for each $\mathfrak{f} \in I$ and $\mathrm{m}_{\mathfrak{f}}\widehat{b}_{\mathfrak{f}}ϭ(\widehat{a}_{\mathfrak{f}}) = 0$ by hypothesis since $\widehat{a}_{\mathfrak{f}}, \widehat{b}_{\mathfrak{f}} \in \mathcal{N}^{\mathcal{L}}\left(\widehat{\mathfrak{R}}_{\mathfrak{f}}\right)$ and $\mathrm{m}_{\mathfrak{f}} \in \mathcal{M}_{\mathfrak{f}}^{\widehat{\mathfrak{R}}_{\mathfrak{f}}}$ for each $\mathfrak{f} \in I$. This implies $\widehat{\mathcal{M}}\widehat{\mathcal{B}}ϭ(\widehat{\mathcal{A}}) = 0$, entailing that the direct product $\prod_{\mathfrak{f}\in I}\widehat{\mathcal{M}}_{\mathfrak{f}}^{\widehat{\mathfrak{R}}_{\mathfrak{f}}}$ is R-$ϭ$-$\mathcal{N}^{\mathcal{L}}\mathcal{S}$-module.

(3) Establishing necessity is enough. Assume $\widehat{\mathcal{M}}^{ҿ\widehat{\mathfrak{R}}}$ and $\widehat{\mathcal{M}}^{(1-ҿ)\widehat{\mathfrak{R}}}$ are R-$ϭ$-$\mathcal{N}^{\mathcal{L}}\mathcal{S}$-modules. Consider $m\widehat{a}\widehat{b} = 0$, for $m \in \widehat{\mathcal{M}}^{\widehat{\mathfrak{R}}}$, and $\widehat{a}, \widehat{b} \in \mathcal{N}^{\mathcal{L}}\left(\widehat{\mathfrak{R}}_i\right)$, then $0 = ҿm\widehat{a}\widehat{b} = m(ҿ\widehat{a})\widehat{b}$. And $0 = (1-ҿ)m\widehat{a}\widehat{b} = m\left((1-ҿ)\widehat{a}\right)\widehat{b}$. By hypothesis, we get $0 = m\widehat{b}ϭ(ҿ\widehat{a})$ and $0 = m\widehat{b}ϭ(1-ҿ)\widehat{a}$,

$0 = m\widehat{b}ϭ(ҿ)ϭ(\widehat{a})$ and $0 = m\widehat{b}ϭ(1-ҿ)ϭ(\widehat{a})$,

$0 = m\widehat{b}ҿϭ(\widehat{a})$ and $0 = m\widehat{b}(1-ҿ)ϭ(\widehat{a})$,

$0 = m\widehat{b}ҿϭ(\widehat{a}) + \mathrm{m}\widehat{b}ϭ(\widehat{a}) - m\widehat{b}ҿϭ(\widehat{a})$,

$0 = m\widehat{b}ϭ(\widehat{a})$.

$\widehat{\mathcal{M}}^{\widehat{\mathfrak{R}}}$ is a R-$ϭ$-$\mathcal{N}^{\mathcal{L}}\mathcal{S}$-module. ■

According to (Lee & Zhou, 2004), the module $\widehat{\mathcal{M}}^{\widehat{\mathfrak{R}}}$ is said to be $ϭ$-reduced, if for each $m \in \widehat{\mathcal{M}}^{\widehat{\mathfrak{R}}}$ and each $\widehat{r} \in \widehat{\mathfrak{R}}$, with $m\widehat{r} = 0$, then $m\widehat{\mathfrak{R}} \cap \widehat{r}\widehat{\mathcal{M}} = 0$.

Lemma 5.8 ([(Raphael, 1975), Lemma 1.2]). Let $\widehat{\mathcal{M}}^{\widehat{\mathfrak{R}}}$ be an $\widehat{\mathfrak{R}}$-module. Then the following statements are equivalent:

1. $\widehat{\mathcal{M}}^{\widehat{\mathfrak{R}}}$ is $ϭ$-reduced;
2. The following statements are true: For each $m \in \widehat{\mathcal{M}}^{\widehat{\mathfrak{R}}}$ and $\widehat{r} \in \widehat{\mathfrak{R}}$,

a. $m\widehat{r} = 0 \to m\widehat{\mathfrak{R}}\widehat{r} = m\widehat{\mathfrak{R}}ϭ(\widehat{r}) = 0$;
b. $m\widehat{r}ϭ(\widehat{r}) = 0 \to m\widehat{r} = 0$;
c. $m\widehat{r}^2 = 0 \to m\widehat{r} = 0$.

If the module $\widehat{\mathcal{M}}^{\widehat{\mathfrak{R}}}$ is 1-red-module, it is referred to as reduced. Hence, a ring $\widehat{\mathfrak{R}}$ is a red-ring if and only if $\widehat{\mathfrak{R}}$ is is 1-red-module as an $\widehat{\mathfrak{R}}$-module $\widehat{\mathcal{M}}^{\widehat{\mathfrak{R}}}$.

Proposition 5.9 Every $ϭ$-reduced module is a R-$ϭ$-$\mathcal{N}^{\mathcal{L}}\mathcal{S}$-module.

Proof. Consider $m \in \widehat{\mathcal{M}}^{\widehat{\mathfrak{R}}}$ and $\breve{\upsilon}, \acute{\rho} \in \mathcal{N}^{\mathcal{L}}\left(\widehat{\mathfrak{R}}\right)$ with $m\breve{\upsilon}\acute{\rho} = 0$, we prove $m\acute{\rho}ϭ(\breve{\upsilon}) = 0$. We apply conditions of $ϭ$-reduced module in the process. Now $0 = m\breve{\upsilon}\acute{\rho} = m\breve{\upsilon}ϭ(\acute{\rho}) = 0$. Then, $mϭ(\acute{\rho})\breve{\upsilon}ϭ(\acute{\rho})\breve{\upsilon} = m(ϭ(\acute{\rho})\breve{\upsilon})ϭ(ϭ(\acute{\rho})\breve{\upsilon}) = mϭ(\acute{\rho})\breve{\upsilon} = mϭ(\acute{\rho})ϭ\left(ϭ(\breve{\upsilon})\right) = mϭ\left(\acute{\rho}ϭ(\breve{\upsilon})\right) = m\acute{\rho}ϭ(\breve{\upsilon})$. Hence $\widehat{\mathcal{M}}^{\widehat{\mathfrak{R}}}$ is a R-$ϭ$-$\mathcal{N}^{\mathcal{L}}\mathcal{S}$-module. ■

The following illustration shows that, in general, Proposition 5.9's converse is not true.

Example 5.10 Consider $\bar{\mathcal{Z}}_4$ denote the ring of integer modulo 4. Let the ring $\widehat{\mathfrak{R}} = \left\{\begin{pmatrix} \breve{\upsilon} & \acute{\rho} \\ 0 & \breve{\upsilon} \end{pmatrix} ; \breve{\upsilon}, \acute{\rho} \in \bar{\mathcal{Z}}_4\right\}$ and the $\widehat{\mathfrak{R}}$-module $\widehat{\mathcal{M}}^{\widehat{\mathfrak{R}}} = \left\{\begin{pmatrix} 0 & \text{ɋ} \\ \text{ɥ} & \text{ƅ} \end{pmatrix} ; \text{ɋ}, \text{ɥ}, \text{ƅ} \in \bar{\mathcal{Z}}_4\right\}$ and a homomorphism $ϭ: \widehat{\mathfrak{R}} \to \widehat{\mathfrak{R}}$ is defined by $ϭ\left(\begin{pmatrix} \breve{\upsilon} & \acute{\rho} \\ 0 & \breve{\upsilon} \end{pmatrix}\right) = \begin{pmatrix} \breve{\upsilon} & -\acute{\rho} \\ 0 & \breve{\upsilon} \end{pmatrix}$. $\widehat{\mathcal{M}}^{\widehat{\mathfrak{R}}}$ is R-$ϭ$-$\mathcal{N}^{\mathcal{L}}\mathcal{S}$-module but not $ϭ$-reduced.

For, if $m = \begin{pmatrix} 0 & 0 \\ 2 & 1 \end{pmatrix} \in \widehat{\mathcal{M}}^{\widehat{\mathfrak{R}}}$ and $\widehat{r} = \begin{pmatrix} 2 & 3 \\ 0 & 2 \end{pmatrix} \in \widehat{\mathfrak{R}}$. Then $m\widehat{r} = 0$ but $\begin{pmatrix} 0 & 0 \\ 0 & 2 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 2 & 0 \end{pmatrix}\begin{pmatrix} 2 & 3 \\ 0 & 2 \end{pmatrix} \in m\widehat{\mathfrak{R}} \cap \widehat{\mathcal{M}}\widehat{r} \neq 0$. Hence $\widehat{\mathcal{M}}^{\widehat{\mathfrak{R}}}$ is not $ϭ$-reduced.

Proposition 5.11 For a ring $\widehat{\mathfrak{R}}$ and $\widehat{\mathfrak{R}}$-module $\widehat{\mathcal{M}}^{\widehat{\mathfrak{R}}}$. Then the following conditions are equivalent,

i. $\widehat{\mathcal{M}}^{\widehat{\mathfrak{R}}}$ is R-$ϭ$-$\mathcal{N}^{\mathcal{L}}\mathcal{S}$-module.
ii. Each submodule of $\widehat{\mathcal{M}}^{\widehat{\mathfrak{R}}}$ is R-$ϭ$-$\mathcal{N}^{\mathcal{L}}\mathcal{S}$-module.
iii. Each finitely generated submodule of $\widehat{\mathcal{M}}^{\widehat{\mathfrak{R}}}$ is $ϭ$-$\mathcal{N}^{\mathcal{L}}\mathcal{S}$-module.
iv. Each cyclic submodule of $\widehat{\mathcal{M}}^{\widehat{\mathfrak{R}}}$ is R-$ϭ$-$\mathcal{N}^{\mathcal{L}}\mathcal{S}$-module.

Proof. It is a direct result of definitions and Proposition 3.6.

Theorem 5.12 Every flat module over an R-$ϭ$-$\mathcal{N}^{\mathcal{L}}\mathcal{S}$-ring is an R-$ϭ$-$\mathcal{N}^{\mathcal{L}}\mathcal{S}$-module.

Proof. Assume $\widehat{\mathcal{M}}^{\widehat{\mathfrak{R}}}$ be a flat module over the R-$ϭ$-$\mathcal{N}^{\mathcal{L}}\mathcal{S}$-ring $\widehat{\mathfrak{R}}$ and $0 \to \text{Ҕ} \to \text{Ғ} \to \widehat{\mathcal{M}}^{\widehat{\mathfrak{R}}} \to 0$ a short exact sequence with Ғ free $\widehat{\mathfrak{R}}$-module. By [(Lee & Zhou, 2004), Theorem 2.3] is a R-$ϭ$-$\mathcal{N}^{\mathcal{L}}\mathcal{S}$-module and we write $\widehat{\mathcal{M}}^{\widehat{\mathfrak{R}}} = \text{Ғ}/\text{Ҕ}$ and any element $\bar{y} = y + \text{Ҕ} \in \widehat{\mathcal{M}}^{\widehat{\mathfrak{R}}}$ for $y \in \text{Ғ}$. Let $\bar{y}\widehat{a}\widehat{b} = 0$ where $\bar{y} \in \widehat{\mathcal{M}}^{\widehat{\mathfrak{R}}}$ and $\widehat{a}, \widehat{b} \in \mathcal{N}^{\mathcal{L}}\left(\widehat{\mathfrak{R}}\right)$. Since $\widehat{\mathcal{M}}^{\widehat{\mathfrak{R}}}$ is flat there exists a homomorphism $\widehat{\star}: \text{Ғ} \to \text{Ҕ}$ such that $\widehat{\star}\left(y\widehat{a}\widehat{b}\right) = y\widehat{a}\widehat{b}$ Now set $u = \widehat{\star}(y) - y \in \text{Ғ}$. Then $u\widehat{a}\widehat{b} = 0$. Since Ғ is R-$ϭ$-$\mathcal{N}^{\mathcal{L}}\mathcal{S}$-module, $u\widehat{b}ϭ(\widehat{a}) = 0$. Then $\widehat{\star}\left(y\widehat{b}ϭ(\widehat{a})\right) = y\widehat{b}ϭ(\widehat{a})$. Since $\widehat{\star}(y) \in \text{Ҕ}$, we have $y\widehat{b}ϭ(\widehat{a}) \in \text{Ҕ}$. Therefore $\bar{y}\widehat{b}ϭ(\widehat{a}) = 0$. Therefore $\widehat{\mathcal{M}}^{\widehat{\mathfrak{R}}}$ is R-$ϭ$-$\mathcal{N}^{\mathcal{L}}\mathcal{S}$-module. ■

Proposition 5.13 Assume $\widehat{\mathfrak{R}}, \text{ʛ}$ are rings and $\vartheta: \widehat{\mathfrak{R}} \to \text{ʛ}$ be a ring *endo*. If $\widehat{\mathcal{M}}^{\text{ʛ}}$ is a right $\widehat{\mathfrak{R}}$-module, then $\widehat{\mathcal{M}}^{\widehat{\mathfrak{R}}}$is a right $\widehat{\mathfrak{R}}$-module via $mr = m\vartheta(r)$ for all $r \in \widehat{\mathfrak{R}}$ and $m \in \widehat{\mathcal{M}}^{\widehat{\mathfrak{R}}}$. Moreover, $\widehat{\mathcal{M}}^{\widehat{\mathfrak{R}}}$ is R-$ϭ$-$\mathcal{N}^{\mathcal{L}}\mathcal{S}$-module, if and only if $\widehat{\mathcal{M}}^{\text{ʛ}}$ is R-$ϭ$-$\mathcal{N}^{\mathcal{L}}\mathcal{S}$-module.

Proof. Let $\widehat{\mathcal{M}}^{\text{ʛ}}$ be an R-$ϭ$-$\mathcal{N}^{\mathcal{L}}\mathcal{S}$-module. Consider $\widehat{a}, \widehat{b} \in \mathcal{N}^{\mathcal{L}}\left(\widehat{\mathfrak{R}}\right)$ and $m \in \widehat{\mathcal{M}}^{\widehat{\mathfrak{R}}}$ Such that $m\widehat{a}\widehat{b} = 0$ Then $m\vartheta\left(\widehat{a}\widehat{b}\right) = m\vartheta(\widehat{a})\vartheta(\widehat{b}) = 0$. Since $\widehat{\mathcal{M}}^{\text{ʛ}}$ is R-$ϭ$-$\mathcal{N}^{\mathcal{L}}\mathcal{S}$-module, we have,

$$m\vartheta(\widehat{b})ϭ\left(\vartheta(\widehat{a})\right) = 0,$$
$$m\vartheta(\widehat{b})\vartheta(\widehat{a}) = 0,$$
$$m\vartheta\left(\widehat{b}ϭ(\widehat{a})\right) = 0.$$

Hence $\widehat{\mathcal{M}}^{\widehat{\mathfrak{R}}}$ is a $ϭ$-$\mathcal{N}^{\mathcal{L}}\mathcal{S}$-module.

Conversely. Assume that ϑ is onto and $\widehat{\mathcal{M}}^{\widehat{\mathfrak{R}}}$ is a R-$ϭ$-$\mathcal{N}^{\mathcal{L}}\mathcal{S}$-module. Let $\breve{\upsilon}, \acute{\rho} \in \mathcal{N}^{\mathcal{L}}(\text{ʛ})$ and $m \in \widehat{\mathcal{M}}^{\text{ʛ}}$ such that $m\breve{\upsilon}\acute{\rho} = 0$. Since ϑ is onto, there exists $\widehat{a}, \widehat{b} \in \mathcal{N}^{\mathcal{L}}\left(\widehat{\mathfrak{R}}\right)$ such that $\breve{\upsilon} = \vartheta(\widehat{a})$ and $\acute{\rho} = \vartheta(\widehat{b})$. Then $0 = m\vartheta(\widehat{a})\vartheta(\widehat{b}) = m\vartheta\left(\widehat{a}\widehat{b}\right) = m\widehat{a}\widehat{b}$. Since $\widehat{\mathcal{M}}^{\widehat{\mathfrak{R}}}$ is right $ϭ$-$\mathcal{N}^{\mathcal{L}}\mathcal{S}$-module, we have $0 = m\widehat{b}ϭ(\widehat{a})$. Hence $0 = m\vartheta\left(\widehat{b}ϭ(\widehat{a})\right) = 0 = m\vartheta(\widehat{b})ϭ\left(\vartheta(\widehat{a})\right) = m\acute{\rho}ϭ(\breve{\upsilon})$. Thus $\widehat{\mathcal{M}}^{\text{ʛ}}$ is R-$ϭ$-$\mathcal{N}^{\mathcal{L}}\mathcal{S}$-module. ■

Now we study the $\mathcal{N}^{\mathcal{L}}$-symmetric property on some module extensions and module localizations like $\widehat{\mathcal{M}}[\text{ӈ}]$, $\widehat{\mathcal{M}}[\text{ӈ}, \text{ӈ}^{-1}]$, $\widehat{\mathcal{M}}[\text{ӈ}, \text{ӈ}^{-1}; ϭ]$.

The following concepts were introduced by Lee and Zhou. For a module $\widehat{\mathcal{M}}$, We examine $\widehat{\mathcal{M}}[\text{ӈ}] = \left\{\sum_{i=0}^{s} m_i \text{ӈ}^i : s \geq 0, m_i \in \widehat{\mathcal{M}}\right\}$, $\widehat{\mathcal{M}}[\text{ӈ}]$ is an Abelian group under clearly addition operation. Additionally, the next, scalar product operation turns $\widehat{\mathcal{M}}[\text{ӈ}]$ into a right $\widehat{\mathfrak{R}}[\text{ӈ}]$-module:

For $m(\text{ӈ}) = \sum_{\sigma=0}^{s} m_\sigma \text{ӈ}^\sigma \in \widehat{\mathcal{M}}[\text{ӈ}]$ and $f(\text{ӈ}) = \sum_{\sigma=0}^{t} a_\sigma \text{ӈ}^\sigma \in \widehat{\mathfrak{R}}[\text{ӈ}]$,

$$m(\text{ӈ})f(\text{ӈ}) = \sum_{d=0}^{3+\epsilon}\left(\sum_{\sigma+\hat{\text{ọ}}=d} m_\sigma a_\sigma\right)x^d.$$

$\widehat{\mathcal{M}}[\varkappa]$ becomes a right module over $\widehat{\mathfrak{R}}[\varkappa]$ as a result of these operations. In the same way, the Laurent polynomial extension $\widehat{\mathcal{M}}[\varkappa,\varkappa^{-1}]$ becomes a right module over $\widehat{\mathfrak{R}}[\varkappa,\varkappa^{-1}]$ with a similar scalar product. Zhou and Lee (Lee & Zhou, 2004) also introduced notations for $\widehat{\mathcal{M}}$ module as,

$\widehat{\mathcal{M}}[\varkappa;\sigma] = \left\{\sum_{\sigma=0}^{p} m_\sigma \varkappa^\sigma \mid p \geq 0, m_\sigma \in \widehat{\mathcal{M}}\right\}$. Each of the above is abelian group underneath the addition condition. Furthermore, $\widehat{\mathcal{M}}[\varkappa;\sigma]$ is a module for $\widehat{\mathfrak{R}}[\varkappa;\sigma]$ under the product operation as:

$$m(\varkappa) = \sum_{\sigma=0}^{\mu} m_\sigma \varkappa^\sigma \in \widehat{\mathcal{M}}[\varkappa;\sigma],$$

$$f(\varkappa) = \sum_{\hat{o}=0}^{\sigma} f_{\hat{o}} \varkappa^{\hat{o}} \in \widehat{\mathfrak{R}}[\varkappa;\sigma]$$

$$m(\varkappa)f(\varkappa) = \sum_{d=0}^{\mu+\sigma}\left(\sum_{\sigma+\upsilon=d} m_\sigma\, \alpha^\sigma(f_\sigma)\right)\varkappa^d$$

In the same way, the skew Laurent polynomial module $\widehat{\mathcal{M}}[\varkappa,\varkappa^{-1};\sigma]$ transforms into a module on $\widehat{\mathfrak{R}}[\varkappa,\varkappa^{-1};\sigma]$.

Again, from (Lee & Zhou, 2004), module $\widehat{\mathcal{M}}$ is known as σ-Armendariz if the below conditions holds: (i) For $m \in \widehat{\mathcal{M}}$ and $a \in \widehat{\mathfrak{R}}, ma = 0$ for the case if $m\sigma(a) = 0$ (ii) any $m(\varkappa) = \sum_{\sigma=0}^{t} m_\sigma \varkappa^\sigma \in \widehat{\mathcal{M}}[\varkappa;\sigma]$ and $f(\varkappa) = \sum_{\hat{o}=0}^{n} a_{\hat{o}} \varkappa^{\hat{o}} \in \widehat{\mathfrak{R}}[\varkappa;\sigma], m(\varkappa)f(\varkappa) = 0$ imply $m_\sigma \sigma^\sigma(a_{\hat{o}}) = 0$ for all σ and $\hat{o}$. And then, Anderson and Camillo (Anderson & Camillo, 1999), extended the concept of Armendariz ring to Armendariz module, as follows: A $\widehat{\mathfrak{R}}$-module $\widehat{\mathcal{M}}^{\widehat{\mathfrak{R}}}$ is Armendariz when, if $m(\varkappa) = \sum_{\sigma=0}^{\mu} m_\sigma \varkappa^\sigma \in \widehat{\mathcal{M}}[\varkappa]$ and $g(\varkappa) = \sum_{\hat{o}=0}^{\sigma} a_{\hat{o}} \varkappa^{\hat{o}} \in \widehat{\mathfrak{R}}[\varkappa]$, such that $m(\varkappa)g(\varkappa) = 0$ implies $m_\sigma a_\upsilon = 0$ for all σ and $\hat{o}$. The Armendariz property is applicable for any finite product of polynomials. Clearly, $\widehat{\mathfrak{R}}$ is an Armendariz ring if and only if ${}_{\widehat{\mathfrak{R}}}\widehat{\mathfrak{R}}$ is an Armendariz $\widehat{\mathfrak{R}}$-module.

Theorem 5.14 Consider $\widehat{\mathcal{M}}^{\widehat{\mathfrak{R}}}$ is a σ-Armendariz module. Then, the statements that follow are equivalent:

1. $\widehat{\mathcal{M}}^{\widehat{\mathfrak{R}}}$ is R-σ-$\mathcal{N}^{\mathcal{L}}\mathcal{S}$-module;
2. $\widehat{\mathcal{M}}[\varkappa;\sigma]^{\widehat{\mathfrak{R}}[\varkappa;\sigma]}$ is R-σ-$\mathcal{N}^{\mathcal{L}}\mathcal{S}$-module;
3. $\widehat{\mathcal{M}}[\varkappa,\varkappa^{-1};\sigma]^{\widehat{\mathfrak{R}}[\varkappa,\varkappa^{-1};\sigma]}$ is R-σ-$\mathcal{N}^{\mathcal{L}}\mathcal{S}$-module.

Proof. It suffices to demonstrate that $1 \Rightarrow 3$. Let $m(\varkappa) = \sum_{\sigma=0}^{\infty} m_\sigma \varkappa^\sigma \in \widehat{\mathcal{M}}[\varkappa,\varkappa^{-1};\sigma]^{\widehat{\mathfrak{R}}[\varkappa,\varkappa^{-1};\sigma]}$ and $\mathfrak{A}(\varkappa) = \sum_{\hat{o}=0}^{\infty} a_{\hat{o}} \varkappa^{\hat{o}}, \mathfrak{B}(\varkappa) = \sum_{\mathrm{q}=0}^{\infty} m_{\mathrm{q}} \varkappa^{\mathrm{q}} \in \mathcal{N}^{\mathcal{L}}(\widehat{\mathfrak{R}}[\varkappa,\varkappa^{-1};\sigma])$. Then we obtain $a_{\hat{o}}, b_{\mathrm{q}} \in \mathcal{N}^{\mathcal{L}}(\widehat{\mathfrak{R}})$. Let $m(\varkappa)\mathfrak{A}(\varkappa)\,\mathfrak{B}(\varkappa) = 0$ this implies $m_\sigma a_{\hat{o}} b_{\mathrm{q}} = 0$ for all $\sigma, \hat{o}, \mathrm{q}$. Thus, by hypothesis $m_\sigma b_{\mathrm{q}} a_{\hat{o}} = 0$. Therefore $m(\varkappa)\mathfrak{B}(\varkappa)\,\mathfrak{A}(\varkappa) = 0$, and so $\widehat{\mathcal{M}}[\varkappa,\varkappa^{-1};\sigma]^{\widehat{\mathfrak{R}}[\varkappa,\varkappa^{-1};\sigma]}$ is a R-σ-$\mathcal{N}^{\mathcal{L}}\mathcal{S}$-module. ∎

Corollary 5.15 Consider $\widehat{\mathcal{M}}^{\widehat{\mathfrak{R}}}$ be an Armendariz module. Then the following are equivalent:

1. $\widehat{\mathcal{M}}^{\widehat{\mathfrak{R}}}$ is R-σ-$\mathcal{N}^{\mathcal{L}}\mathcal{S}$-module;
2. $\widehat{\mathcal{M}}[\varkappa]^{\widehat{\mathfrak{R}}[\varkappa]}$ is R-σ-$\mathcal{N}^{\mathcal{L}}\mathcal{S}$-module;
3. $\widehat{\mathcal{M}}[\varkappa,\varkappa^{-1}]^{\widehat{\mathfrak{R}}[\varkappa,\varkappa^{-1}]}$ is R-σ-$\mathcal{N}^{\mathcal{L}}\mathcal{S}$-module.

Proposition 5.16 Consider σ is an *endo* of a ring $\widehat{\mathfrak{R}}$ and $\widehat{\mathcal{M}}^{\widehat{\mathfrak{R}}}$is σ-reduced module. Then $\widehat{\mathcal{M}}^{\widehat{\mathfrak{R}}}$is R-$\sigma$-$\mathcal{N}^{\mathcal{L}}\mathcal{S}$-module over $\widehat{\mathfrak{R}}$ if and only if $\widehat{\mathcal{M}}^{\widehat{\mathfrak{R}}}[\varkappa]/\widehat{\mathcal{M}}^{\widehat{\mathfrak{R}}}[\varkappa](\varkappa^n)$ is R-σ-$\mathcal{N}^{\mathcal{L}}\mathcal{S}$-module over $\frac{\widehat{\mathfrak{R}}[\varkappa]}{<\varkappa^n>}$ for integer $n \geq 2$.

Proof. Let $\widehat{\mathcal{M}}^{\widehat{\mathfrak{R}}}$ is right σ-$\mathcal{N}^{\mathcal{L}}\mathcal{S}$-module with $pqh = 0$, where $\overline{\varkappa} = \varkappa + <\varkappa^n>$. Note that $a_\sigma b_{\hat{o}} c_{\mathrm{q}} \overline{\varkappa}^{\,i+j+k} = 0$, for each $\sigma, \hat{o}$ and q with $\sigma + \hat{o} + \mathrm{q} \geq n$. Therefore, it is sufficient to display the cases $\sigma + \hat{o} + \mathrm{q} \leq n-1$. Since $pqh = 0$, The following equations are available to us:

(1) $m_0 s_0 t_0 = 0,$

(2) $m_0 s_0 t_1 + m_0 s_1 t_0 + m_1 s_0 t_0 = 0,$

(3) $m_0 s_0 t_2 + m_0 s_1 t_1 + m_0 s_2 t_0 + m_1 s_0 t_1 + m_1 s_1 t_0 + m_2 s_0 t_0 = 0,$

$\vdots$

$(n-2)$ $m_0 s_0 t_{n-2} + m_0 s_1 t_{n-3} + \cdots + m_{n-3} s_1 t_0 + m_{n-2} s_0 t_0 = 0,$

$(n-1)$ $m_0 s_0 t_{n-1} + m_0 s_1 t_{n-2} + \cdots + m_{n-2} s_0 t_1 + m_{n-2} s_1 t_0 + m_{n-1} s_0 t_0 = 0.$

Since $\widehat{\mathcal{M}}^{\widehat{\mathfrak{R}}}$ is σ-reduced for any $m \in \widehat{\mathcal{M}}^{\widehat{\mathfrak{R}}}, a \in \widehat{\mathfrak{R}}, ma^2 = 0 \to ma = 0$, and each σ-reduced module is semi-commutative**.** These facts are used as follows:

Eq(1) and Eq(2) $\times\, s_0 t_0$ gives $m_1 (s_0 t_0)^2 = 0$, and so $m_1 s_0 t_0 = 0$ and $m_0 s_0 t_1 + m_0 s_1 t_0 = 0$, multiplying by $s_1 t_0$ gives $0 = m_0 s_1 (t_0{}^2) = m_0 s_1 t_0$, so we have, $m_0 s_0 t_1 = 0, m_0 s_1 t_0 = 0$ and $m_1 s_0 t_0 = 0$. From Eq(1),(2) and (3) $\times\, s_0 t_0$, we get $m_2 s_0 t_0 = 0$ and,

$m_0 s_0 t_2 + m_0 s_1 t_1 + m_0 s_2 t_0 + m_1 s_0 t_1 + m_1 s_1 t_0 = 0$, in a similar way. If we multiply the right side of Eq(3) by $s_1 t_0, s_0 t_1, s_2 t_0$ and $s_1 t_1$ respectively, then we obtain $m_1 s_1 t_0 = 0, m_1 s_0 t_1 = 0, m_0 s_2 t_0 = 0, m_0 s_1 t_1 = 0,$ and $m_0 s_0 t_2 = 0$ in turn Inductively we assume that $m_\sigma s_\upsilon t_k = 0$ where $\sigma + \hat{o} + \mathrm{q} = 0, 1, \ldots, (n-2)$. We apply the above method to Eq. $(n-1)$. First, the induction hypotheses and Eq. $(n-1)$ $\times\, s_0 t_0$ give $m_{n-1} s_0 t_0 = 0$ and,

$(n-1)$ $m_0 s_0 t_{n-1} + m_0 s_1 t_{n-2} + \cdots + m_{n-2} s_0 t_1 + m_{n-2} s_1 t_0 + m_{n-1} s_0 t_0 = 0.$

If we multiply Eq. $(n-1)$ on the right side by $s_1 t_0, s_0 t_1, \ldots,$ and $s_1 t_{n-2}$ respectively, then we obtain $m_{n-2} s_1 t_0 = 0, m_{n-2} s_0 t_1 = 0, \ldots, m_0 s_1 t_{n-2} = 0$ and so $m_0 s_0 t_{n-1} = 0$. In turn. This shows that $m_\sigma s_{\hat{o}} t_{\mathrm{q}} = 0$ for all $\sigma, \hat{o}$ and q with $\sigma + \hat{o} + \mathrm{q} = n-1$. Consequently, $m_\sigma s_{\hat{o}} t_{\mathrm{q}} = 0$ for all $\sigma, \hat{o}$ and q with $\sigma + \hat{o} \leq n-1$, and thus $m_\sigma t_{\mathrm{q}} \sigma^\sigma(s_{\hat{o}}) = 0, \forall \sigma \in Z^+$ by [(Kwak, 2007), Theorem 2.5(1)]. This yields $ph\bar{\sigma}(q) = 0$, and therefore $\widehat{\mathcal{M}}^{\widehat{\mathfrak{R}}}[\varkappa]/\widehat{\mathcal{M}}^{\widehat{\mathfrak{R}}}[\varkappa](\varkappa^n)$ is R-σ-$\mathcal{N}^{\mathcal{L}}\mathcal{S}$-module. ∎

If $ur = 0$ implies $r = 0$ for $r \in \widehat{\mathfrak{R}}$, then an element u of a ring $\widehat{\mathfrak{R}}$ is right regular. Regular indicates that it is both left and right regular (and so not a zero divisor), while left regular is defined similarly. Assume that $\widehat{\mathcal{M}}$ is a subset of $\widehat{\mathfrak{R}}$ that is multiplicatively closed and made up of central regular elements. Let σ be an automorphism of $\widehat{\mathfrak{R}}$ and consider $\sigma(m) = m, \forall m \in \widehat{\mathcal{M}}$. Then $\sigma(m^{-1}) = m^{-1}$ in $\widehat{\mathcal{M}}^{-1}\widehat{\mathfrak{R}}$ and the induced map $\sigma: \widehat{\mathcal{M}}^{-1}\widehat{\mathfrak{R}} \to \widehat{\mathcal{M}}^{-1}\widehat{\mathfrak{R}}$ defined by $\sigma(u^{-1}a) = u^{-1}\sigma(a)$ is also an automorphism.

Proposition 5.17 Consider a ring $\widehat{\mathfrak{R}}$ and a subset Ω of $\widehat{\mathfrak{R}}$ that is multiplicatively closed and consists of central regular elements. Then

(1) $\widehat{\mathfrak{R}}$ is a R-σ-$\mathcal{N}^{\mathcal{L}}\mathcal{S}$-ring if and only if is $\Omega^{-1}\widehat{\mathfrak{R}}$ is a R-σ-$\mathcal{N}^{\mathcal{L}}\mathcal{S}$-ring.

(2) A module $\widehat{\mathcal{M}}^{\widehat{\mathfrak{R}}}$is R-$\sigma$-$\mathcal{N}^{\mathcal{L}}\mathcal{S}$-module if and only if $\Omega^{-1}\widehat{\mathcal{M}}^{\Omega^{-1}\widehat{\mathfrak{R}}}$ is a R-σ-$\mathcal{N}^{\mathcal{L}}\mathcal{S}$-module.

Proof.(1) Assume $\chi\tau\kappa = 0$ with $\chi = \widehat{u}^{-1}\widehat{a}, \tau = \widehat{v}^{-1}\widehat{b}, \kappa = \widehat{w}^{-1}\widehat{c}$, $\widehat{u}, \widehat{v}, \widehat{w} \in \Omega$ and $\widehat{a} \in \mathrm{R}^\circ$, $\widehat{b}, \widehat{c} \in \mathcal{N}^{\mathcal{L}}(\widehat{\mathfrak{R}})$. Since Ω is included in the centre of $\widehat{\mathfrak{R}}$,

we have $0 = \chi\tau\kappa = \widehat{u}^{-1}\widehat{a}\widehat{v}^{-1}\widehat{b}\widehat{w}^{-1}\widehat{c} = (\widehat{u}^{-1}\widehat{v}^{-1}\widehat{w}^{-1})\widehat{a}\widehat{b}\widehat{c} = (\widehat{u}\widehat{v}\widehat{w})^{-1}\widehat{a}\widehat{b}\widehat{c}$ and so $s\widehat{a}\widehat{b}\widehat{c} = 0$ for some $s \in \Omega$. But $\widehat{\mathfrak{R}}$ is R-σ-$\mathcal{N}^{\mathcal{L}}\mathcal{S}$-ring by the condition, so $s\widehat{a}\widehat{c}\sigma(\widehat{b}) = 0$ and $s\chi\kappa\sigma(\tau) = s(\widehat{u}^{-1}\widehat{a})(\widehat{w}^{-1}\widehat{c})\sigma\left((\widehat{v}^{-1}\widehat{b})\right) = s(\widehat{u}\widehat{w}\widehat{v})^{-1}\widehat{a}\widehat{c}\sigma(\widehat{b}) = 0$.

Hence $\Omega^{-1}\widehat{\mathfrak{R}}$ is R-σ-$\mathcal{N}^{\mathcal{L}}\mathcal{S}$-ring.

(2) Since a submodule of a R-σ-$\mathcal{N}^{\mathcal{L}}\mathcal{S}$-module is likewise a R-$\sigma$-$\mathcal{N}^{\mathcal{L}}\mathcal{S}$-module, it is sufficient to verify the required condition. Assume that $\widehat{\mathcal{M}}^{\widehat{\mathfrak{R}}}$is a R-$\sigma$-$\mathcal{N}^{\mathcal{L}}\mathcal{S}$-module and $(\mathrm{q}^{-1}\mathfrak{m})(\mu^{-1}\breve{\upsilon})(\sigma^{-1}\acute{\rho}) = 0$ for $\mathrm{q}^{-1}\mathfrak{m} \in \Omega^{-1}\widehat{\mathcal{M}}^{\Omega^{-1}\widehat{\mathfrak{R}}}$ and $\mu^{-1}\breve{\upsilon}, \sigma^{-1}\acute{\rho} \in \mathcal{N}^{\mathcal{L}}(\Omega^{-1}\widehat{\mathfrak{R}})$ where $\mathfrak{m} \in \widehat{\mathcal{M}}^{\widehat{\mathfrak{R}}}$, $\breve{\upsilon}, \acute{\rho} \in \mathcal{N}^{\mathcal{L}}(\widehat{\mathfrak{R}})$. Since Ω is included in the centre of $\widehat{\mathfrak{R}}$, we have $0 = (\mathrm{q}^{-1}\mathfrak{m})(\mu^{-1}\breve{\upsilon})(\sigma^{-1}\acute{\rho}) = (\hat{s}\hat{t}\hat{r})^{-1}\mathfrak{m}\breve{\upsilon}\acute{\rho}$ and so $0 = \mathfrak{m}\breve{\upsilon}\acute{\rho}$. By

assumption $\mathfrak{m}\acute{\rho}\sigma(\breve{\upsilon}) = O$. Therefore $(\mathfrak{q}^{-1}\mathfrak{m})(\sigma^{-1}\acute{\rho})\sigma(\mu^{-1}\breve{\upsilon}) = (\mathfrak{q}^{-1}\mathfrak{m})(\sigma^{-1}\acute{\rho})(\mu^{-1}\sigma(\breve{\upsilon})) = O$. Hence $\Omega^{-1}\widehat{\mathcal{M}}^{\Omega^{-1}\widehat{\mathfrak{R}}}$ is a R-σ-$\mathcal{N}^{\mathcal{L}}\mathcal{S}$-module. ■

Corollary 5.18 (1) For a ring $\widehat{\mathfrak{R}}$, $\widehat{\mathfrak{R}}[\varkappa]$ is R-σ-$\mathcal{N}^{\mathcal{L}}\mathcal{S}$-ring if and only if is $\widehat{\mathfrak{R}}[\varkappa;\varkappa^{-1}]$ a R-σ-$\mathcal{N}^{\mathcal{L}}\mathcal{S}$-ring**.**

(2) For a $\widehat{\mathfrak{R}}$-module $\widehat{\mathcal{M}}^{\widehat{\mathfrak{R}}}$, $\widehat{\mathcal{M}}[\varkappa]^{\widehat{\mathfrak{R}}[x]}$ is R-σ-$\mathcal{N}^{\mathcal{L}}\mathcal{S}$-module if and only if $\widehat{\mathcal{M}}[x,x^{-1}]^{\widehat{\mathfrak{R}}[x,x^{-1}]}$ is a R-σ-$\mathcal{N}^{\mathcal{L}}\mathcal{S}$-module.

Proof (1). Consider $\Omega = \{1,\varkappa,\varkappa^2,\cdots\}$. Then clearly Ω is a multiplicatively closed subset of $\widehat{\mathfrak{R}}[\varkappa]$. Since $\widehat{\mathfrak{R}}[\varkappa;\varkappa^{-1}] = \Omega^{-1}\widehat{\mathfrak{R}}[\varkappa]$, it follows that $\widehat{\mathfrak{R}}[\varkappa;\varkappa^{-1}]$ is right σ-$\mathcal{N}^{\mathcal{L}}\mathcal{S}$-ring by proposition 5.17(1).

(2) It is evident from proposition 5.17(2). if $\Omega = \{1,\varkappa,\varkappa^2,\ldots\}$. Then Ω is a multiplicatively closed subset of $\widehat{\mathfrak{R}}[\varkappa]$ consisting of regular central element of $\widehat{\mathfrak{R}}[\varkappa]$. Since $\Omega^{-1}\widehat{\mathcal{M}}[\varkappa]^{\widehat{\mathfrak{R}}[\varkappa]} = \widehat{\mathcal{M}}[\varkappa,\varkappa^{-1}]^{\widehat{\mathfrak{R}}[\varkappa,\varkappa^{-1}]}$ and $\Omega^{-1}\widehat{\mathfrak{R}}[\varkappa] = \widehat{\mathfrak{R}}[\varkappa;\varkappa^{-1}]$. ■

$\bar{\mathrm{Q}}(\widehat{\mathfrak{R}})$ is *a classical right quotient* for $\widehat{\mathfrak{R}}$ if every regular element of $\widehat{\mathfrak{R}}$ is *invertible* in $\bar{\mathrm{Q}}$ and every element of $\bar{\mathrm{Q}}$ can be written in the form ab^{-1} with $a,b\in\widehat{\mathfrak{R}}$ and b regular.

A right *Ore* ring is a ring $\widehat{\mathfrak{R}}$ where, for any $a,b\in\widehat{\mathfrak{R}}$ with b being regular, $\exists a_1,b_1\in\widehat{\mathfrak{R}}$ with b_1 also regular, such that $ab_1 = ba_1$. It is well known that $\widehat{\mathfrak{R}}$ is a right *ore ring* if and only if its classical right quotient ring $\bar{\mathrm{Q}}(\widehat{\mathfrak{R}})$ exists. Now, suppose $\widehat{\mathfrak{R}}$ is a ring with the classical right quotient ring $\bar{\mathrm{Q}}(\widehat{\mathfrak{R}})$. Then any automorphism σ of $\widehat{\mathfrak{R}}$ extends to $\bar{\mathrm{Q}}(\widehat{\mathfrak{R}})$ by defining its action on fractions as $\sigma(ab^{-1}) = \sigma(a)(\sigma(b))^{-1}$ for all $a,b\in\widehat{\mathfrak{R}}$, provided that $\sigma(b)$ remains regular whenever b is a regular element in $\widehat{\mathfrak{R}}$.

Theorem 5.19 Consider $\widehat{\mathfrak{R}}$ is *Ore ring* with an *endo* σ of $\widehat{\mathfrak{R}}$ and $\bar{\mathrm{Q}}(\widehat{\mathfrak{R}})$ *is* the classical right quotient ring $\mathcal{NJ}$ ring of $\widehat{\mathfrak{R}}$. Then

(1) $\widehat{\mathfrak{R}}$ is a R-σ-$\mathcal{N}^{\mathcal{L}}\mathcal{S}$-ring if and only if $\bar{\mathrm{Q}}(\widehat{\mathfrak{R}})$is a R-$\sigma$-$\mathcal{N}^{\mathcal{L}}\mathcal{S}$-ring.

(2) $\widehat{\mathcal{M}}^{\widehat{\mathfrak{R}}}$ is a R-σ-$\mathcal{N}^{\mathcal{L}}\mathcal{S}$-module if and only if $\bar{\mathrm{Q}}(\widehat{\mathcal{M}})$ is a R-σ-$\mathcal{N}^{\mathcal{L}}\mathcal{S}$-module.

Proof. (1) Consider $\widehat{\mathfrak{R}}$ is a R-σ-$\mathcal{N}^{\mathcal{L}}\mathcal{S}$-ring. Assume $A = a\mu^{-1}\in\bar{\mathrm{Q}}(\widehat{\mathfrak{R}})$ and $B = bv^{-1}, C = c\omega^{-1}\in\mathcal{N}^{\mathcal{L}}(\bar{\mathrm{Q}}(\widehat{\mathfrak{R}}))$with $ABC = a\mu^{-1}\,bv^{-1}c\omega^{-1}$ where $a,\mu\in\widehat{\mathfrak{R}}$ and $b,v,c,\omega\in\mathcal{N}^{\mathcal{L}}(\widehat{\mathfrak{R}})$ with μ,v,ω regular. Let $\bar{\mathrm{Q}}(\widehat{\mathfrak{R}})$be an $\mathcal{NJ}$ ring Then $\widehat{\mathfrak{R}}$ is $\mathcal{NJ}$ and so $b,c\in\mathcal{N}^{\mathcal{L}}(\widehat{\mathfrak{R}})$. $\exists\ c_1,b_1\in\widehat{\mathfrak{R}}$ with b_1 regular such that $bc_1 = cb_1$ and $c_1b_1^{-1} = b^{-1}c$. Now $\exists\ \mu_1,b_1\in\widehat{\mathfrak{R}}$ with μ_1 regular such that $b\mu_1 = \mu b_1$, $\mu^{-1}b = b_1\mu_1^{-1}$. Hence $ABC = a\mu^{-1}bv^{-1}c\omega^{-1} = ab_1\mu_1^{-1}v^{-1}c\omega^{-1} = O$. Let I and J be the ideals in $\bar{\mathrm{Q}}(\widehat{\mathfrak{R}})$, generated by B and C within $\mathcal{N}^{\mathcal{L}}(\bar{\mathrm{Q}}(\widehat{\mathfrak{R}}))$, respectively. Then each of I and J are $\mathcal{N}^{\mathcal{L}}$ with $b = Bv\in I$, $c = C\omega\in J$, Since $\widehat{\mathfrak{R}}$ is right Ore, for $c,v\in\mathcal{N}^{\mathcal{L}}(\widehat{\mathfrak{R}})\ \exists\ c_1,v_1\in\mathcal{N}^{\mathcal{L}}(\widehat{\mathfrak{R}})$ with v_1 regular such that $cv_1 = vc_1$, $v^{-1}c = c_1v_1^{-1}$ .Here note that $c_1\in\mathcal{N}^{\mathcal{L}}(\widehat{\mathfrak{R}})$. Indeed, $vc_1 = cv_1\in J$ and so $c_1 = v^{-1}(vc_1)\in J$. So $ABC = ab_1\mu_1^{-1}c_1v_1^{-1}\,\omega^{-1} = O$.

Similarly, also there exists $c_2\in\mathcal{N}^{\mathcal{L}}(\widehat{\mathfrak{R}})$ and $\mu_2\in\widehat{\mathfrak{R}}$ with μ_2 regular such that $c_1\mu_2 = \mu_1c_2, \mu_1^{-1}c_1 = c_2\mu_2^{-1}$, Thus, we obtain that $ABC = ab_1c_2\,\mu_2^{-1}v_1^{-1}\omega^{-1} = O$ and hence $ab_1c_2 = O$. This implies $O = ab_1c_2\mu = a\mu b_1c_2 = ab\mu_1c_2 = abc_2\mu_1$, and $O = abc_2 = abc_2\mu_1 = ab\mu_1c_2 = abc_1\mu_1$. So we have $O = abc_1 = abc_1v = abvc_1 = abcv$. It follows that $ac\sigma(b) = O$, since $\widehat{\mathfrak{R}}$ is a R-σ-$\mathcal{N}^{\mathcal{L}}\mathcal{S}$-ring.

Similar, there exists $c_3,b_2,\omega_2,b_4\in\mathcal{N}^{\mathcal{L}}(\widehat{\mathfrak{R}})$ and $\mu_3,\mu_4\in\widehat{\mathfrak{R}}$ with μ_3,ω_2,μ_4 regular such that $c\mu_3 = \mu c_3$, $\mu^{-1}c = c_3\mu_3^{-1}$, $b\omega_2 = \omega b_2, \omega^{-1}b = b_2{\omega_2}^{-1}, b_2\mu_4 = \mu_3b_4$, and, $AC\sigma(B) = ac_3{\mu_3}^{-1}\omega^{-1}\sigma(b)\sigma(v)^{-1} = ac_3\mu_3^{-1}b_2\omega_2^{-1}\sigma(v)^{-1} = ac_3b_4{\mu_4}^{-1}\omega_2^{-1}\sigma(v)^{-1}$. Form $ac\sigma(b) = O$. We have $O = ac\sigma(b)\omega_2 = ac\omega b_2 = acb_2\omega$, and hence $O = acb_2 = acb_2\mu_4 = ac\mu_3b_4 = acb_4\mu_3$. It follows that, $O = acb_4 = acb_4\mu_3 = ac\mu_3b_4 = a\mu c_3b_4 = ac_3b_4\mu$, and hence $ac_3b_4 = O$. Now we have $AC\sigma(B) = O$, therefore $\bar{\mathrm{Q}}(\widehat{\mathfrak{R}})$ is a R-σ-$\mathcal{N}^{\mathcal{L}}\mathcal{S}$-ring.

(2) Assume that $\widehat{\mathcal{M}}^{\widehat{\mathfrak{R}}}$ is a R-σ-$\mathcal{N}^{\mathcal{L}}\mathcal{S}$-module. Let $A = a\mu^{-1}\in\bar{\mathrm{Q}}(\widehat{\mathcal{M}})$ and $B = bv^{-1}, C = c\omega^{-1}\in\mathcal{N}^{\mathcal{L}}(\bar{\mathrm{Q}}(\widehat{\mathfrak{R}}))$ with $ABC = a\mu^{-1}\,bv^{-1}c\omega^{-1}$ where $a,\mu\in\widehat{\mathcal{M}}^{\widehat{\mathfrak{R}}}$ and $b,v,c,\omega\in\mathcal{N}^{\mathcal{L}}(\widehat{\mathfrak{R}})$ with μ,v,ω regular. Let $\bar{\mathrm{Q}}(\widehat{\mathfrak{R}})$ be an $\mathcal{NJ}$ ring Then $\widehat{\mathfrak{R}}$ is $\mathcal{NJ}$ and so $b,c\in\mathcal{N}^{\mathcal{L}}(\widehat{\mathfrak{R}})$. then $\exists\ c_1,b_1\in\widehat{\mathfrak{R}}$ with b_1 regular such that $bc_1 = cb_1$ and $c_1b_1^{-1} = b^{-1}c$. Now $\exists\ \mu_1\in\widehat{\mathcal{M}}^{\widehat{\mathfrak{R}}}, b_1\in\widehat{\mathfrak{R}}$ with μ_1 regular such that $b\mu_1 = \mu b_1$, $\mu^{-1}b = b_1\mu_1^{-1}$. Hence $ABC = a\mu^{-1}bv^{-1}c\omega^{-1} = ab_1\mu_1^{-1}v^{-1}c\omega^{-1} = O$. Let I and J be the ideals in $\bar{\mathrm{Q}}(\widehat{\mathfrak{R}})$, generated by B and C within $\mathcal{N}^{\mathcal{L}}(\bar{\mathrm{Q}}(\widehat{\mathfrak{R}}))$, respectively. Then each of I and J are $\mathcal{N}^{\mathcal{L}}$ with $b = Bv\in I$ and $c = C\omega\in J$, Since $\widehat{\mathfrak{R}}$ is right Ore, for $c,v\in\mathcal{N}^{\mathcal{L}}(\widehat{\mathfrak{R}})\ \exists\ c_1,v_1\in\mathcal{N}^{\mathcal{L}}(\widehat{\mathfrak{R}})$ with v_1 regular such that $cv_1 = vc_1$, $v^{-1}c = c_1v_1^{-1}$ .Here note that $c_1\in\mathcal{N}^{\mathcal{L}}(\widehat{\mathfrak{R}})$. Indeed, $vc_1 = cv_1\in J$ and so $c_1 = v^{-1}(vc_1)\in J$. So $ABC = ab_1\mu_1^{-1}c_1v_1^{-1}\,\omega^{-1} = O$.

Similarly, also $\exists\ c_2\in\mathcal{N}^{\mathcal{L}}(\widehat{\mathfrak{R}})$ and $\mu_2\in\widehat{\mathcal{M}}^{\widehat{\mathfrak{R}}}$ with μ_2 regular such that $c_1\mu_2 = \mu_1c_2, \mu_1^{-1}c_1 = c_2\mu_2^{-1}$, Thus, we obtain that $ABC = ab_1c_2\,\mu_2^{-1}v_1^{-1}\omega^{-1} = O$ and hence $ab_1c_2 = O$. This implies $O = ab_1c_2\mu = a\mu b_1c_2 = ab\mu_1c_2 = abc_2\mu_1$, and $O = abc_2 = abc_2\mu_1 = ab\mu_1c_2 = abc_1\mu_1$. So we have $O = abc_1 = abc_1v = abvc_1 = abcv$. It follows that $ac\sigma(b) = O$, since $\widehat{\mathcal{M}}^{\widehat{\mathfrak{R}}}$ is a R-σ-$\mathcal{N}^{\mathcal{L}}\mathcal{S}$-module.

Similar, $\exists c_3,b_2,\omega_2,b_4\in\mathcal{N}^{\mathcal{L}}(\widehat{\mathfrak{R}})$ and $\mu_3,\mu_4\in\widehat{\mathcal{M}}^{\widehat{\mathfrak{R}}}$ with μ_3,ω_2,μ_4 regular such that $c\mu_3 = \mu c_3$, $\mu^{-1}c = c_3\mu_3^{-1}$, $b\omega_2 = \omega b_2, \omega^{-1}b = b_2{\omega_2}^{-1}, b_2\mu_4 = \mu_3b_4$, and, $AC\sigma(B) = ac_3{\mu_3}^{-1}\omega^{-1}\sigma(b)\sigma(v)^{-1} = ac_3\mu_3^{-1}b_2\omega_2^{-1}\sigma(v)^{-1} = ac_3b_4{\mu_4}^{-1}\omega_2^{-1}\sigma(v)^{-1}$. Form $ac\sigma(b) = O$. We have $O = ac\sigma(b)\omega_2 = ac\omega b_2 = acb_2\omega$, and hence $O = acb_2 = acb_2\mu_4 = ac\mu_3b_4 = acb_4\mu_3$. It follows that, $O = acb_4 = acb_4\mu_3 = ac\mu_3b_4 = a\mu c_3b_4 = ac_3b_4\mu$, and hence $ac_3b_4 = O$. Now we have $AC\sigma(B) = O$, therefore $\bar{\mathrm{Q}}(\widehat{\mathcal{M}})$ is a R-σ-$\mathcal{N}^{\mathcal{L}}\mathcal{S}$-module. ■

CONCLUSION

This article introduced the concept right σ -$\mathcal{N}^{\mathcal{L}}$ symmetric rings and then extends it to right σ-$\mathcal{N}^{\mathcal{L}}$ symmetric modules, which serve as generalizations of both σ-symmetric rings and σ-symmetric modules. Several results were founded as the characterization of σ-$\mathcal{N}^{\mathcal{L}}$-symmetric rings in section 2, also for σ-$\mathcal{N}^{\mathcal{L}}$-symmetric modules in section 5. In addition to that we investigated the concept of an σ-$\mathcal{N}^{\mathcal{L}}$-symmetric rings on some of ring extensions and localizations in section 3 and 4, also for σ-$\mathcal{N}^{\mathcal{L}}$-symmetric modules in section. As a proposal for a future work, the following questions are presented;

1. Are all right σ-$\mathcal{N}^{\mathcal{L}}$-symmetric rings and σ-$\mathcal{N}^{\mathcal{L}}$-symmetric modules necessarily non-commutative?
2. Is there a relationship between σ-$\mathcal{N}^{\mathcal{L}}$-symmetric module and σ-semi-commutative?
3. Are there a class of modules which are $\mathcal{N}^{\mathcal{L}}$-symmetric over their endomorphism?

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