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ON NIL-SYMMETRIC RINGS AND MODULES SKEWED BY RING ENDOMORPHISM

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ABSTRACT:

The symmetric property plays an important role in non-commutative ring theory and module theory. In this paper, we study the symmetric property with one element of the ring $\mathfrak{R}$ and two nilpotent elements of $\mathfrak{R}$ skewed by ring endomorphism $\mathfrak{б}$ on rings, introducing the concept of a right $\mathfrak{б}$-$\mathcal{N}^{\mathcal{L}}$-symmetric ring and extend the concept of right $\mathfrak{б}$-$\mathcal{N}^{\mathcal{L}}$-symmetric rings to modules by introducing another concept called the right $\mathfrak{б}$-$\mathcal{N}^{\mathcal{L}}$-symmetric module which is a generalization of $\mathfrak{б}$-symmetric modules. According to this, we examine the characterization of a right $\mathfrak{б}$-$\mathcal{N}^{\mathcal{L}}$-symmetric ring and a right $\mathfrak{б}$-$\mathcal{N}^{\mathcal{L}}$-symmetric module and their related properties including ring and explore their connections to other classes of rings and modules. Furthermore, we investigate the concept of $\mathfrak{б}$-$\mathcal{N}^{\mathcal{L}}$-symmetric on some ring extensions and localizations like $\mathfrak{R}[ʉ]$, $\mathfrak{R}[ʉ, ʉ^{-1}]$, Dorroh extension, Jordan extension and module localizations like $\Omega^{-1}\widehat{\mathcal{M}}^{\Omega^{-1}\mathfrak{R}}$.

KEYWORDS: Reduced-Ring, Symmetric Ring, Flat Module, $\mathfrak{б}$-Reduced Module, Polynomial Module.

1. INTRODUCTION

Every ring in this study has a unique identity, and every module that is investigated is a unital module. $\bar{\mathcal{Z}}, \bar{\mathcal{Z}}_n$ and $\mathcal{N}^{\mathcal{L}}(\mathfrak{R})$ denotes the ring of integers, integers modulo n and the set of nilpotent elements in $\mathfrak{R}$, respectively. Furthermore, $1_{\mathfrak{R}}$, $\mathfrak{б}$, $\widehat{\mathcal{M}}^{\mathfrak{R}}$ denote the identity endomorphism, an endomorphism of an arbitrary ring $\mathfrak{R}$ (For short, *endo*) and right $\mathfrak{R}$-module respectively. $\ell_m\ (\mathfrak{R}) = \{m \in \widehat{\mathcal{M}} : m\mathfrak{R} = 0\}$ is the left annihilator of $\mathfrak{R}$ in $\widehat{\mathcal{M}}$.

A ring $\mathfrak{R}$ is reduced (For short red-ring), if it has no nonzero nilpotent elements. However, if $\breve{ʋ}\mathfrak{б}(\breve{ʋ}) = 0$ implies $\breve{ʋ} = 0$ for $\breve{ʋ} \in \mathfrak{R}$, then *endo* $\mathfrak{б}$ of the ring $\mathfrak{R}$ is said to be rigid (For short, rg-ring endo) (Krempa, 1996). If there is a rg-ring *endo* $\mathfrak{б}$ of ring $\mathfrak{R}$, then $\mathfrak{R}$ is said to be $\mathfrak{б}$-rigid ring (For short, $\mathfrak{б}$-rg-ring) (Suarez H., *et al*., 2024). Note that, $\mathfrak{б}$-rg-rings are red-rings by [(Hong *et al*., 2000), Proposition 5]. and any rg-ring *endo* of a ring is a monomorphism. Cohn introduced a ring $\mathfrak{R}$ as *reversible*, if whenever $\breve{ʋ}ṕ = 0$, then $ṕ\breve{ʋ} = 0$, for $\breve{ʋ}, ṕ \in \mathfrak{R}$ (Cohn, 1999). Lembek referred to a ring $\mathfrak{R}$ as symmetric (For short, $\mathcal{S}$-ring), if whenever $\breve{ʋ}ṕ\tilde{ϖ} = 0$, then $\breve{ʋ}\tilde{ϖ}ṕ = 0$, for $\breve{ʋ}, ṕ, \tilde{ϖ} \in \mathfrak{R}$ (Lambek, 1971). According to [(Shin, 1973), Lemma 1.1], every red-ring is symmetric; however, the convers does not true in general [(Anderson & Camillo, 1999), Example 11.5]. Although, it is clear that $\mathcal{S}$-rings are reversible and commutative rings are symmetric, the convers of each of them does not true in general [(Anderson & Camillo, 1999), Example 1.5 and 11.5] and [(Marks, 2002), Example 5 and 7]. As an extension of $\mathcal{S}$-rings and a specific instance of $\mathcal{N}^{\mathcal{L}}$-semi-commutative rings, Chakraborty and Das presented the idea of $\mathcal{N}^{\mathcal{L}}$-symmetric rings in (Chakraborty & Das, 2014). A ring $\mathfrak{R}$ is right(R) (left(L)) $\mathcal{N}^{\mathcal{L}}$-symmetric (For short, R(L)-$\mathcal{N}^{\mathcal{L}}\mathcal{S}$-ring), if for $\breve{ʋ} \in \mathfrak{R}$, and $ṕ, \tilde{ϖ} \in \mathcal{N}^{\mathcal{L}}(\mathfrak{R})$ with $\breve{ʋ}ṕ\tilde{ϖ} = 0\ (\tilde{ϖ}\breve{ʋ}ṕ = 0)$, then $\breve{ʋ}\tilde{ϖ}ṕ = 0$. A ring is $\mathcal{N}^{\mathcal{L}}\mathcal{S}$-ring if it is both L(R) $\mathcal{N}^{\mathcal{L}}\mathcal{S}$-ring.

The concept of an $\mathfrak{б}$-symmetric ring was first proposed by Kwak, T. K. in 2007, as an extension of $\mathcal{S}$-rings and a generalization of $\mathfrak{б}$-rg rings. In (Kwak, 2007) an *endo* $\mathfrak{б}$ of a ring $\mathfrak{R}$ is called L(R)-$\mathfrak{б}$-symmetric ring(For short, $\mathfrak{б}$-$\mathcal{S}$-ring), if $\breve{ʋ}ṕ\tilde{ϖ} = 0$ imply $\breve{ʋ}\tilde{ϖ}\mathfrak{б}(ṕ) = 0\ (\mathfrak{б}(ṕ)\breve{ʋ}\tilde{ϖ} = 0)$, for $a, ṕ, \tilde{ϖ} \in \mathfrak{R}$. A ring $\mathfrak{R}$ is L(R)- $\mathfrak{б}$-$\mathcal{S}$-ring if there exists a L(R)- $\mathcal{S}$-ring *endo* $\mathfrak{б}$ of $\mathfrak{R}$. the concepts of an $\mathfrak{б}$-$\mathcal{S}$-ring is an extension of $\mathcal{S}$-rings and it is also a generalization of $\mathfrak{б}$-rg rings.

The ring notion was recently extended to include modules. A module $\widehat{\mathcal{M}}^{\mathfrak{R}}$ is called symmetric (For short, $\mathcal{S}$-module), if whenever $\breve{ʋ}, ṕ \in \mathfrak{R}$, $m \in \widehat{\mathcal{M}}^{\mathfrak{R}}$ satisfy $m\breve{ʋ}ṕ = 0$, then we have $mṕ\breve{ʋ} = 0$ ((Lambek, 1971) and (Raphael, 1975)). A module $\widehat{\mathcal{M}}^{\mathfrak{R}}$ is $\mathfrak{б}$-semi-commutative if, $m\breve{ʋ} = 0$ implies $m\mathfrak{R}\mathfrak{б}(\breve{ʋ}) = 0$, for $m \in \widehat{\mathcal{M}}^{\mathfrak{R}}$ *and* $\breve{ʋ} \in \mathfrak{R}$. The module $\widehat{\mathcal{M}}^{\mathfrak{R}}$ is semi-commutative if it is $i_{\mathfrak{R}}$-semi-commutative. Buhphang and Rege in (Buhphang & Rege, 2002) examined the fundamental characteristics of semi-commutative modules. Agayev and Harmanci concentrated on semi-commutativity of subrings of matrix rings and carried out additional research on semi-commutative rings and modules in (Agayev & Harmancı, 2007).

Motivated to the above, this article is structured to introduce and define a new kind of rings named a R-$\mathfrak{б}$-$\mathcal{N}^{\mathcal{L}}$-$\mathcal{S}$ ring as a generalization of $\mathfrak{б}$-$\mathcal{S}$-rings and an extension of $\mathcal{N}^{\mathcal{L}}\mathcal{S}$-rings, and to explore and provide various characterizations, features and relations about this concept and to study its related properties. Additionally, we investigate the concept of right $\mathfrak{б}$-$\mathcal{N}^{\mathcal{L}}$-symmetric on some of ring extensions and localizations. This leads to a number of well-known outcomes as corollaries of our results. Then we extend the property of R-$\mathfrak{б}$-$\mathcal{N}^{\mathcal{L}}\mathcal{S}$ rings to modules by introducing the notion of right $\mathfrak{б}$-$\mathcal{N}^{\mathcal{L}}$-symmetric module which is a generalization of $\mathfrak{б}$-symmetric modules and extensions of symmetric modules. We examine the characteristics of right $\mathfrak{б}$-$\mathcal{N}^{\mathcal{L}}$-symmetric modules and their associated attributes, such as localizations and module extensions.

On $\mathfrak{б}$-$\mathcal{N}^{\mathcal{L}}$-Symmetric Rings:

The fundamental structure of $\mathfrak{б}$-$\mathcal{N}^{\mathcal{L}}$-$\mathcal{S}$ rings is examined in this section, along with a number of associated ring features. We begin with the following definition.

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Definition 2.1 An *endo* б of a ring $\hat{\mathfrak{R}}$ is said to be left(L)-right(R) б-$\mathcal{N}^{\mathcal{L}}$-symmetric(For short, L-R-б-$\mathcal{N}^{\mathcal{L}}\mathcal{S}$-ring), if whenever $ŭṗῶ = 0$, for $ŭ \in \hat{\mathfrak{R}}$ and $ṗ, ῶ \in \mathcal{N}^{\mathcal{L}}(\hat{\mathfrak{R}})$, *then* $ŭῶб(ṗ) = 0(б(ṗ)ŭῶ = 0)$. A ring $\hat{\mathfrak{R}}$ is L-R-б-$\mathcal{N}^{\mathcal{L}}\mathcal{S}$, if there exists a L-R $\mathcal{N}^{\mathcal{L}}\mathcal{S}$ *endo* б of $\hat{\mathfrak{R}}$. Moreover, $\hat{\mathfrak{R}}$ is б-$\mathcal{N}^{\mathcal{L}}\mathcal{S}$-ring if it is both L-R- б -$\mathcal{N}^{\mathcal{L}}$-$\mathcal{S}$-ring.

Remark 2.2:

Example 2.3 Suppose that a ring $\hat{\mathfrak{R}} = U_2(\bar{\mathcal{Z}}_4)$, then

$$\mathcal{N}^{\mathcal{L}}(\hat{\mathfrak{R}}) = \left\{\begin{pmatrix} ŧ & ř \\ 0 & ɉ \end{pmatrix} \mid ŧ, ɉ \in \{0,2\}, ř \in \bar{\mathcal{Z}}_4\right\}.$$

(i) Let $б: \hat{\mathfrak{R}} \to \hat{\mathfrak{R}}$ be an *endo* defined by:

$$б\left(\begin{pmatrix} ŧ & ř \\ 0 & ɉ \end{pmatrix}\right) = \begin{pmatrix} ŧ & 0 \\ 0 & 0 \end{pmatrix}.$$

If $ŸŰṼ = 0$ for $Ÿ = \begin{pmatrix} ŧ & ř \\ 0 & ɉ \end{pmatrix} \in \hat{\mathfrak{R}}, Ű = \begin{pmatrix} ɤ & ƛ \\ 0 & ɦ \end{pmatrix}, Ṽ = \begin{pmatrix} ɣ & ƥ \\ 0 & ƕ \end{pmatrix} \in \mathcal{N}^{\mathcal{L}}(\hat{\mathfrak{R}})$, then we get $ŧɤɣ = 0$ and so $ŧɣɤ = 0$ since $\bar{\mathcal{Z}}_4$ is commutative. This yields $ŸṼб(Ű) = 0$, and hence $\hat{\mathfrak{R}}$ is R-б-$\mathcal{N}^{\mathcal{L}}\mathcal{S}$-ring. For $Ÿ = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix} \in \hat{\mathfrak{R}}, Ű = \begin{pmatrix} 2 & 1 \\ 0 & 2 \end{pmatrix} = Ṽ \in \mathcal{N}^{\mathcal{L}}(\hat{\mathfrak{R}})$ with $ŸŰṼ = 0$, we have $б(Ű)ŸṼ = \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix} \neq 0$, and thus $\hat{\mathfrak{R}}$ is not L-б-$\mathcal{N}^{\mathcal{L}}\mathcal{S}$-ring.

(ii) Let $ꟾ: \hat{\mathfrak{R}} \to \hat{\mathfrak{R}}$ be an *endo* defined by:

$$ꟾ\left(\begin{pmatrix} ŧ & ř \\ 0 & ɉ \end{pmatrix}\right) = \begin{pmatrix} 0 & 0 \\ 0 & ɉ \end{pmatrix}.$$

By using the same technique as in (i), we may demonstrate that $\hat{\mathfrak{R}}$ is L-ꟾ-$\mathcal{N}^{\mathcal{L}}$-$\mathcal{S}$-ring. However, $\hat{\mathfrak{R}}$ is not R-ꟾ-$\mathcal{N}^{\mathcal{L}}$-$\mathcal{S}$-ring for $ŸŰṼ = 0$ but $ŸṼ\,ꟾ\,(Ű) = \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix} \neq 0$, and thus $\hat{\mathfrak{R}}$ is not R-ꟾ-$\mathcal{N}^{\mathcal{L}}$-$\mathcal{S}$-ring.

Lemma 2.4 (1) For a ring $\hat{\mathfrak{R}}$, $\hat{\mathfrak{R}}$ is R-б-$\mathcal{N}^{\mathcal{L}}\mathcal{S}$-ring if and only if $ŸŰṼ = 0$ implies $ŸṼб(Ű) = 0$, for $\emptyset \neq Ÿ \subseteq \hat{\mathfrak{R}}$ and $\emptyset \neq Ű, \emptyset \neq Ṽ \subseteq \mathcal{N}^{\mathcal{L}}(\hat{\mathfrak{R}})$.

(2) Consider $\hat{\mathfrak{R}}$ be a reversible ring. $\hat{\mathfrak{R}}$ is R-б-$\mathcal{N}^{\mathcal{L}}\mathcal{S}$-ring if and only if $\hat{\mathfrak{R}}$ is L-б-$\mathcal{N}^{\mathcal{L}}\mathcal{S}$-ring.

Proof. (1) It suffices to show that $ŸŰṼ = 0$ for $\emptyset \neq Ÿ \subseteq \hat{\mathfrak{R}}$ and $\emptyset \neq Ű, \emptyset \neq Ṽ \subseteq \mathcal{N}^{\mathcal{L}}(\hat{\mathfrak{R}})$, implies $ŸṼб(Ű) = 0$, when $\hat{\mathfrak{R}}$ is right б-$\mathcal{N}^{\mathcal{L}}\mathcal{S}$-ring. Let $ŸŰṼ = 0$, then $ŭṗῶ = 0$ for $ŭ \in Ÿ, ṗ \in Ű$ and $ῶ \in Ṽ$, and hence $ŭῶб(ṗ) = 0$ by the condition. Thus $ŸṼб(Ű) = \sum_{ŭ\in Ÿ, ṗ \in Ű \text{ and } ῶ \in Ṽ} ŭῶб(ṗ) = 0$.

(2) Let $ŭṗῶ = 0$ for $ŭ \in \hat{\mathfrak{R}}$ *and* $ṗ, ῶ \in \mathcal{N}^{\mathcal{L}}(\hat{\mathfrak{R}})$. If $\hat{\mathfrak{R}}$ is R-б-$\mathcal{N}^{\mathcal{L}}\mathcal{S}$-ring, then $(ŭῶ)(б(ṗ)) = 0$, since $\hat{\mathfrak{R}}$ is reversible, we have $(б(ṗ))(ŭῶ) = б(ṗ)ŭῶ = 0$, and hence $\hat{\mathfrak{R}}$ is L-б-$\mathcal{N}^{\mathcal{L}}\mathcal{S}$-ring. The converse is similar. ■

The condition "$\hat{\mathfrak{R}}$ is reversible" in (Proposition 2.4) is irremovable, as demonstrated by Example 2.3. While it is evident that all б-symmetric objects are б-$\mathcal{N}^{\mathcal{L}}\mathcal{S}$-ring, the following example shows that the converse is not true.

Example 2.5 Assume $\bar{\mathcal{Z}}_2$ is the ring of integer modulo 2, and $\hat{\mathfrak{R}} = \bar{\mathcal{Z}}_2 \oplus \bar{\mathcal{Z}}_2$. Using the standard addition and multiplication. Since $\mathcal{N}^{\mathcal{L}}(\hat{\mathfrak{R}}) = \{(0,0)\}$, $\hat{\mathfrak{R}}$ is б-$\mathcal{N}^{\mathcal{L}}\mathcal{S}$-ring. Now let $б: \hat{\mathfrak{R}} \to \hat{\mathfrak{R}}$ be defined by $б((ŭ, ṗ)) = (ṗ, ŭ)$. Then, for $ŭ = (1,0), ṗ = (0,1), ῶ = (1,1) \in \hat{\mathfrak{R}}$, $ŭṗῶ = 0$ but $ŭῶ\,б(ṗ) = (1,0) \neq 0$, and thus $\hat{\mathfrak{R}}$ is not an б-$\mathcal{S}$-ring. ■

Consider $\hat{\mathfrak{R}}$ is a ring and $\emptyset \neq ɠ \subseteq \hat{\mathfrak{R}}$, $l_{R^{\circ}}(ɠ) = \{ῶ \in \hat{\mathfrak{R}} \mid ῶɠ = 0\}$ is called the L-annihilator of ɠ in $\hat{\mathfrak{R}}$. If $ɠ = \{ŭ\}$, then we write $l_{\hat{\mathfrak{R}}}(ŭ)$ instead of $l_{\hat{\mathfrak{R}}}\{ŭ\}$.

Lemma 2.6 For a ring $\hat{\mathfrak{R}}$, then the following are equivalent for a nonzero *endo* б:

(1) $\hat{\mathfrak{R}}$ is R-б-$\mathcal{N}^{\mathcal{L}}\mathcal{S}$-ring;

(2) $l_{\hat{\mathfrak{R}}}(ṗῶ) \subseteq l_{\hat{\mathfrak{R}}}(ῶб(ṗ))$, for any $ŭ \in \hat{\mathfrak{R}}$ and $ṗ, ῶ \in \mathcal{N}^{\mathcal{L}}(\hat{\mathfrak{R}})$;

1. A ring $\hat{\mathfrak{R}}$ is $\mathcal{N}^{\mathcal{L}}\mathcal{S}$-ring if $\hat{\mathfrak{R}}$ is $1_{\hat{\mathfrak{R}}}$-$\mathcal{N}^{\mathcal{L}}$-symmetric, where $1_{\hat{\mathfrak{R}}}$ is the identity *endo*.
2. Every subring $\hat{S}$ with $б(\hat{S}) \subseteq \hat{S}$ of an б-$\mathcal{N}^{\mathcal{L}}\mathcal{S}$-ring is also б-$\mathcal{N}^{\mathcal{L}}\mathcal{S}$-ring.
3. $\hat{\mathfrak{R}}$, but the converse does not true (See (Kwak, 2007)Example 2.7(1)).
4. The concept of б-$\mathcal{N}^{\mathcal{L}}\mathcal{S}$-ring is not R-L- б-$\mathcal{N}^{\mathcal{L}}\mathcal{S}$-ring through the following example.

(3) $ŸŰṼ = 0$ if and only if $ŸṼб(Ű) = 0$, for any $Ÿ \subseteq \hat{\mathfrak{R}}$ and $Ű, Ṽ \subseteq \mathcal{N}^{\mathcal{L}}(\hat{\mathfrak{R}})$;

(4) $l_{\hat{\mathfrak{R}}}(ŰṼ) \subseteq l_{\hat{\mathfrak{R}}}(Ṽб(Ű))$, for any $Ÿ \subseteq \hat{\mathfrak{R}}$ and $Ű, Ṽ \subseteq \mathcal{N}^{\mathcal{L}}(\hat{\mathfrak{R}})$.

Proof. (1) → (3). Suppose that $ŸŰṼ = 0$ for $Ÿ \subseteq \hat{\mathfrak{R}}$ and $Ű, Ṽ \subseteq \mathcal{N}^{\mathcal{L}}(\hat{\mathfrak{R}})$. For any $ŭ \in Ÿ, ṗ \in Ű, ῶ \in Ṽ$ Then $ŭṗῶ = 0$, and hence $ŭῶб(ṗ) = 0$. Therefore $ŸṼб(Ű) = \{\sum ŭ_i ῶ_i б(ṗ_i): ŭ_i \in Ÿ, ṗ_i \in Ű, ῶ_i \in Ṽ\} = 0$.

The converse is obvious. (1) → (2) and (3) → (4) is clear. ■

Lemma 2.7 The class of б-$\mathcal{N}^{\mathcal{L}}\mathcal{S}$-rings is closed under direct products.

Proof. Note that $\mathcal{N}^{\mathcal{L}}(\prod_{ɤ\in\Gamma}\hat{\mathfrak{R}}_ɤ) \subseteq \prod_{ɤ\in\Gamma}\mathcal{N}^{\mathcal{L}}(\hat{\mathfrak{R}}_ɤ)$ and $б_ɤ(\hat{\mathfrak{R}}_ɤ) \subseteq \hat{\mathfrak{R}}_ɤ$ for each $ɤ \in \Gamma$. Now, let $ŸŰṼ = 0$,where $Ÿ = (ŭ_ɤ)_{ɤ\in\Gamma} \in \prod_{ɤ\in\Gamma}\hat{\mathfrak{R}}_ɤ$ and $Ű = (ṗ_ɤ)_{ɤ\in\Gamma}$, $Ṽ = (ῶ_ɤ)_{ɤ\in\Gamma} \in\in \mathcal{N}^{\mathcal{L}}(\prod_{ɤ\in\Gamma}\hat{\mathfrak{R}}_ɤ)$. Thus for $ŭ_ɤ \in \hat{\mathfrak{R}}_ɤ$ and $ṗ_ɤ, ῶ_ɤ \in \mathcal{N}^{\mathcal{L}}(\hat{\mathfrak{R}}_ɤ)$, $ŭ_ɤṗ_ɤῶ_ɤ = 0$. Since $\hat{\mathfrak{R}}_ɤ$ is R-б-$\mathcal{N}^{\mathcal{L}}\mathcal{S}$-ring for each $ɤ \in \Gamma$, then $ŭ_ɤῶ_ɤб(ṗ_ɤ) = 0$ for each $ɤ \in \Gamma$. So we get $ŸṼб(Ű) = 0$. Therefore, the direct product $\prod_{ɤ\in\Gamma}\hat{\mathfrak{R}}_ɤ$ of $\hat{\mathfrak{R}}_ɤ$ is R-б-$\mathcal{N}^{\mathcal{L}}\mathcal{S}$-ring.

Recently, it was proven that if $ŭ, ṗ \in \hat{\mathfrak{R}}$, such that $ŭṗ = 0 \to ṗб(ŭ) = 0\ (б(ṗ)ŭ = 0)$, then б is R(L) reversible, and the ring $\hat{\mathfrak{R}}$ is called R(L) б-reversible if there exist a R(L) reversible *endo* б of $\hat{\mathfrak{R}}$. A ring $\hat{\mathfrak{R}}$ is б-reversible (Başer *et al*., 2009) if it is both L(R) б-reversible.

Theorem 2.8 Let $\hat{\mathfrak{R}}$ be a б-$\mathcal{N}^{\mathcal{L}}\mathcal{S}$-ring. Then we have the following.

1. For $ŭ \in \hat{\mathfrak{R}}$, $ṗ, ῶ \in \mathcal{N}^{\mathcal{L}}(\hat{\mathfrak{R}})$ and $ŭṗ = 0$, then $ŭῶб^n(ṗ) = 0, ṗῶб^n(ŭ) = 0$, and $ŭṗб^n(ῶ) = 0, \forall n \in Z^+$. Consequently, $\hat{\mathfrak{R}}$ is right б-reversible ring.

2. Consider б is a monomorphism of $\hat{\mathfrak{R}}$. Then we have the following.

i. $\hat{\mathfrak{R}}$ is $\mathcal{N}^{\mathcal{L}}$-symmetric ring,

ii. For $ŭ \in \hat{\mathfrak{R}}$, $ṗ, ῶ \in \mathcal{N}^{\mathcal{L}}(\hat{\mathfrak{R}})$ and $ŭṗῶ = 0$, then $б^n(ŭ)ṗῶ = 0$ and $ŭб^n(ṗ)ῶ = 0$, $\forall n \in Z^+$. Conversely, if $б^m(ŭ)ṗῶ = 0, ŭб^m(ṗ)ῶ = 0$, or $ŭṗб^m(ῶ) = 0$ for some $m \in Z^+$, then $ŭṗῶ = 0$.

Proof. The proof is similar to that of [(Kwak, 2007), Theorem2.5]. ■

EXTENSIONS OF RIGHT б-$\mathcal{N}^{\mathcal{L}}$-SYMMETRIC RINGS :

In this section, we investigate the properly of right б-$\mathcal{N}^{\mathcal{L}}$-symmetric on some extensions of right б-$\mathcal{N}^{\mathcal{L}}$-symmetric. One may ask whether the following extensions $Mat_n(\hat{\mathfrak{R}}), U_n(\hat{\mathfrak{R}}), D_n(\hat{\mathfrak{R}}), T(\hat{\mathfrak{R}}, \hat{\mathfrak{R}})$ *and* $\hat{\mathfrak{R}}[ӄ]$ are right б-$\mathcal{N}^{\mathcal{L}}$-symmetric, if $\hat{\mathfrak{R}}$ is right б-$\mathcal{N}^{\mathcal{L}}$-symmetric. According to this, many results were obtained. Consider an $n \times n$ upper triangular matrix ring, matrix ring over $\hat{\mathfrak{R}}$, denoted as $U_n(\hat{\mathfrak{R}}), Mat_n(\hat{\mathfrak{R}})$. Suppose that $D_n(\hat{\mathfrak{R}})$ represents the subring of $U_n(\hat{\mathfrak{R}})$ where all diagonal entries are the same.

For any red-ring $\hat{\mathfrak{R}}$, both $U_2(\hat{\mathfrak{R}})$ and $D_2(\hat{\mathfrak{R}})$ qualify as R-б-$\mathcal{N}^{\mathcal{L}}\mathcal{S}$-rings for any given *endo* б. However, the following counterexample demonstrates that there exists a red-ring $\hat{\mathfrak{R}}$ with an *endo* б such that $Mat_n(\hat{\mathfrak{R}})$ does not satisfy the R-б-$\mathcal{N}^{\mathcal{L}}\mathcal{S}$-rings condition.

Example 3.1 An automorphism σ of $\bar{Z}_2$ defined by:
$O \to 1$ and $1 \to O$
Assume $\hat{\mathfrak{R}} = Mat_2(\bar{Z}_2)$. Now for $\check{Y} = \begin{pmatrix} 1 & 1 \\ O & O \end{pmatrix} \in \hat{\mathfrak{R}}$, and $\check{U} = \begin{pmatrix} O & 1 \\ O & O \end{pmatrix}, \check{V} = \begin{pmatrix} O & 1 \\ O & O \end{pmatrix} \in \mathcal{N}^{\mathcal{L}}(\hat{\mathfrak{R}})$ we have $\check{Y}\check{U}\check{V} = O$ but $\check{Y}\check{V}\sigma(\check{U}) = \begin{pmatrix} 1 & 1 \\ O & O \end{pmatrix}\begin{pmatrix} O & 1 \\ O & O \end{pmatrix}\begin{pmatrix} 1 & O \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} O & 1 \\ O & O \end{pmatrix}\begin{pmatrix} 1 & O \\ 1 & 1 \end{pmatrix} = \begin{pmatrix} 1 & 1 \\ O & O \end{pmatrix} \neq O$. Therefore, $Mat_2(Z_2)$ is not σ-$\mathcal{N}^{\mathcal{L}}\mathcal{S}$-ring.

The trivial extension of a ring $\hat{\mathfrak{R}}$ by a $(\hat{\mathfrak{R}}, \hat{\mathfrak{R}})$-bimodule $\hat{\mathcal{M}}^{\hat{\mathfrak{R}}}$is the ring $T(\hat{\mathfrak{R}}, \hat{\mathcal{M}}) = \hat{\mathfrak{R}} \oplus \hat{\mathcal{M}}$, which can be obtained by the standard addition and multiplication as follows:

$$(r_1, m_1)(r_2, m_2) = (r_1 r_2, r_1 m_2 + m_1 r_2).$$

This is isomorphic to the ring $\begin{pmatrix} \hat{\mathfrak{R}} & \hat{\mathcal{M}} \\ O & \hat{\mathfrak{R}} \end{pmatrix}$ the usual matrix operations are used. For an *endo* σ of a ring $\hat{\mathfrak{R}}$ and the trivial extension $T(\hat{\mathfrak{R}}, \hat{\mathfrak{R}})$ of R°, $\sigma: T(\hat{\mathfrak{R}}, \hat{\mathfrak{R}}) \to T(\hat{\mathfrak{R}}, \hat{\mathfrak{R}})$ defined by:

$$\sigma\left(\begin{pmatrix} \check{v} & \acute{\rho} \\ O & \check{v} \end{pmatrix}\right) = \begin{pmatrix} \sigma(\check{v}) & \sigma(\acute{\rho}) \\ O & \sigma(\check{v}) \end{pmatrix}$$

is an *endo* of $T(\hat{\mathfrak{R}}, \hat{\mathfrak{R}})$. Since $T(\hat{\mathfrak{R}}, O)$ is isomorphic to $\hat{\mathfrak{R}}$.
The trivial extension of the red-ring is symmetric by [(Huh *et al.*, 2005), corollary 2.4]. However, for a R-σ-$\mathcal{N}^{\mathcal{L}}\mathcal{S}$-ring $\hat{\mathfrak{R}}$. $T(\hat{\mathfrak{R}}, \hat{\mathfrak{R}})$ need not be a right σ-$\mathcal{N}^{\mathcal{L}}\mathcal{S}$-ring by the following example.

Example 3.2 Suppose the R-σ-$\mathcal{N}^{\mathcal{L}}\mathcal{S}$-ring $\hat{\mathfrak{R}} = \left\{\begin{pmatrix} \check{v} & \acute{\rho} \\ O & \check{v} \end{pmatrix} \mid \check{v}, \acute{\rho} \in \bar{Z}\right\}$. Assume $\sigma: \hat{\mathfrak{R}} \to \hat{\mathfrak{R}}$ be an *endo* defined by $\sigma\left(\begin{pmatrix} \check{v} & \acute{\rho} \\ O & \check{v} \end{pmatrix}\right) = \begin{pmatrix} \check{v} & -\acute{\rho} \\ O & \check{v} \end{pmatrix}$. Take $\mathfrak{T} = T(\hat{\mathfrak{R}}, \hat{\mathfrak{R}})$, Let
$A = \begin{pmatrix} \begin{pmatrix} 1 & O \\ O & 1 \end{pmatrix} & \begin{pmatrix} O & O \\ O & O \end{pmatrix} \\ \begin{pmatrix} O & O \\ O & O \end{pmatrix} & \begin{pmatrix} 1 & O \\ O & 1 \end{pmatrix} \end{pmatrix} \in \mathfrak{T}, B = \begin{pmatrix} \begin{pmatrix} O & 1 \\ O & O \end{pmatrix} & \begin{pmatrix} -1 & 1 \\ O & -1 \end{pmatrix} \\ \begin{pmatrix} O & O \\ O & O \end{pmatrix} & \begin{pmatrix} O & 1 \\ O & O \end{pmatrix} \end{pmatrix}, C = \begin{pmatrix} \begin{pmatrix} O & 1 \\ O & O \end{pmatrix} & \begin{pmatrix} 1 & 1 \\ O & 1 \end{pmatrix} \\ \begin{pmatrix} O & O \\ O & O \end{pmatrix} & \begin{pmatrix} O & 1 \\ O & O \end{pmatrix} \end{pmatrix} \in \mathcal{N}^{\mathcal{L}}(\mathfrak{T})$
$ABC = O$ but $AC\,\sigma(B) \neq O$. Thus $\mathfrak{T} = T(\hat{\mathfrak{R}}, \hat{\mathfrak{R}})$ is not right σ-$\mathcal{N}^{\mathcal{L}}\mathcal{S}$-ring.

Proposition 3.3 Consider $\hat{\mathfrak{R}}$ is a red-ring, then $T(\hat{\mathfrak{R}}, \hat{\mathfrak{R}})$ is a R-σ-$\mathcal{N}^{\mathcal{L}}\mathcal{S}$-ring.
Proof. The proof is similar to that of [(Kwak, 2007), Proposition3.2]. ■

The following is an extension of the trivial extension $T(\hat{\mathfrak{R}}, \hat{\mathfrak{R}})$ of the σ-rg ring to a new ring:

$$\mathfrak{T}_n = \left\{\begin{pmatrix} \check{v} & \check{v}_{12} & \check{v}_{13} & \cdots & \check{v}_{1n} \\ O & \check{v} & \check{v}_{23} & \cdots & \check{v}_{2n} \\ O & O & \check{v} & \cdots & \check{v}_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ O & O & O & \cdots & \check{v} \end{pmatrix} : \check{v}, \check{v}_{ij} \in \hat{\mathfrak{R}}\right\}$$

And,

$$\mathcal{N}^{\mathcal{L}}(\mathfrak{T}_n) = \left\{\begin{pmatrix} O & \check{v}_{12} & \check{v}_{13} & \cdots & \check{v}_{1n} \\ O & O & \check{v}_{23} & \cdots & \check{v}_{2n} \\ O & O & O & \cdots & \check{v}_{3n} \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ O & O & O & \cdots & O \end{pmatrix} : a_{ij} \in \hat{\mathfrak{R}}\right\}$$

The *endo* $\sigma: \mathfrak{T}_n \to \mathfrak{T}_n$, defined by $\sigma\left((\check{v}_{ij})\right) = \left(\sigma(\check{v}_{ij})\right)$, is further extended to an *endo* σ of a ring $\hat{\mathfrak{R}}$ for any $n \geq 3$. If $\hat{\mathfrak{R}}$ is σ-rg then $\mathfrak{T}_3$ is not a R-σ-$\mathcal{N}^{\mathcal{L}}\mathcal{S}$-ring by [(Kwak, 2007), Example 3.4]. The following example shows that $\mathfrak{T}_n$ cannot be σ-$\mathcal{N}^{\mathcal{L}}\mathcal{S}$-ring for any $n \geq 4$, even if $\hat{\mathfrak{R}}$ is an σ-rg ring.

Example 3.4 Consider σ is an *endo* of an σ-rg ring $\hat{\mathfrak{R}}$. Note that $\sigma(e) = e$ for $e^2 = e \in \hat{\mathfrak{R}}$. By [(Hong *et al.*, 2000), Proposition 5] In particular $\sigma(1) = 1$.
Let $ABC = O$ for

$$A = \begin{pmatrix} O & 1 & -1 & O \\ O & O & O & O \\ O & O & O & O \\ O & O & O & O \end{pmatrix}, B = \begin{pmatrix} O & O & O & O \\ O & O & O & 1 \\ O & O & O & 1 \\ O & O & O & O \end{pmatrix} \in \mathcal{N}^{\mathcal{L}}(\hat{\mathfrak{R}}),$$

$$C = \begin{pmatrix} O & O & O & O \\ O & O & 1 & O \\ O & O & O & O \\ O & O & O & O \end{pmatrix} \in \hat{\mathfrak{R}}.$$

But we have,

$$AC\sigma(B) = \begin{pmatrix} O & 1 & -1 & O \\ O & O & O & O \\ O & O & O & O \\ O & O & O & O \end{pmatrix}\begin{pmatrix} O & O & O & O \\ O & O & 1 & O \\ O & O & O & O \\ O & O & O & O \end{pmatrix}\begin{pmatrix} O & O & O & O \\ O & O & O & 1 \\ O & O & O & 1 \\ O & O & O & O \end{pmatrix} = \begin{pmatrix} O & O & 1 & O \\ O & O & O & O \\ O & O & O & O \\ O & O & O & O \end{pmatrix} \neq O$$

Thus $\mathfrak{T}_4$ is not a R-σ-$\mathcal{N}^{\mathcal{L}}\mathcal{S}$-ring.

Theorem 3.5 Consider $\hat{\mathfrak{R}}$ is a red-ring and $n \in \bar{Z}^+$. If $\hat{\mathfrak{R}}$ is a R-σ-$\mathcal{N}^{\mathcal{L}}\mathcal{S}$-ring with $\sigma(1) = 1$, then $\hat{\mathfrak{R}}[\varkappa]/ < \varkappa^n >$ is a R-σ-$\mathcal{N}^{\mathcal{L}}\mathcal{S}$-ring, where $< \varkappa^n >$ is the ideal generated by $\varkappa^n$.
Proof. Suppose $\mathfrak{T} = \hat{\mathfrak{R}}[\varkappa]/ < \varkappa^n >$ If $n = 1$, then $\mathfrak{T} \cong \hat{\mathfrak{R}}$. If $n = 2$, then $\mathfrak{T} \cong T(\hat{\mathfrak{R}}, \hat{\mathfrak{R}})$ is a right σ-$\mathcal{N}^{\mathcal{L}}\mathcal{S}$-ring by Proposition 3.3, Now for $n \geq 3$ the prove is similar to the proof of [(Kwak, 2007), Theorem 3.8]. ■

From (Harmanci *et al.*, 2021), Consider $\hat{\mathfrak{R}}$ is a ring and $\mathfrak{g}$ a subring of $\hat{\mathfrak{R}}$ and $T(\hat{\mathfrak{R}}, \mathfrak{g}) = \{(r_1, r_2, \ldots, r_n, s, s, \ldots) \mid r_\imath \in \hat{\mathfrak{R}}, s \in \mathfrak{g}, 1 \leq n, 1 \leq \imath \leq n, \imath, n \in \bar{Z}\}$. The operations of the ring $T(\hat{\mathfrak{R}}, \mathfrak{g})$ are twice addition and multiplication. We provide sufficient and necessary criteria for $T[\hat{\mathfrak{R}}, \mathfrak{g}]$ to be σ-$\mathcal{N}^{\mathcal{L}}\mathcal{S}$-ring in the following proposition.

Proposition 3.6 Consider $\hat{\mathfrak{R}}$ is a ring and $\mathfrak{g}$ is a subring of $\hat{\mathfrak{R}}$. Then the following are equivalent:
(1) $T[\hat{\mathfrak{R}}, \mathfrak{g}]$ is R-σ-$\mathcal{N}^{\mathcal{L}}\mathcal{S}$-ring;
(2) $\hat{\mathfrak{R}}$ is R-σ-$\mathcal{N}^{\mathcal{L}}\mathcal{S}$-ring.
Proof. (1) → (2) Let $\check{v} \in \hat{\mathfrak{R}}, \acute{\rho}, \tilde{\omega} \in \mathcal{N}^{\mathcal{L}}(\hat{\mathfrak{R}})$ with $\check{v}\,\acute{\rho}\tilde{\omega} = O$. Let $\check{Y} = (\check{v}, O, O, O, \cdots) \in T[\hat{\mathfrak{R}}, \mathfrak{g}]$, $\tilde{O} = (\acute{\rho}, O, O, O, \cdots), \beta = (\tilde{\omega}, O, O, O, \cdots) \in \mathcal{N}^{\mathcal{L}}(T[\hat{\mathfrak{R}}, \mathfrak{g}])$ and $\check{Y}\tilde{O}\beta = O$. By(1), $\check{Y}\beta\sigma(\tilde{O}) = O$ in $T[\hat{\mathfrak{R}}, \mathfrak{g}]$. Hence $\check{v}c\sigma(\acute{\rho}) = O$ and so $\hat{\mathfrak{R}}$ is R-σ-$\mathcal{N}^{\mathcal{L}}\mathcal{S}$-ring,
(2) → (1) Assume that $\check{Y} = (\check{v}_1, \check{v}_2, \cdots, \check{v}_n, s, s, \cdots) \in T[\hat{\mathfrak{R}}, \mathfrak{g}]$ and $\tilde{O} = (\acute{\rho}_1, \acute{\rho}_2, \cdots, \acute{\rho}_n, t, t, \cdots), \beta = (\tilde{\omega}_1, \tilde{\omega}_2, \cdots, \tilde{\omega}_n, h, h, \cdots) \in \mathcal{N}^{\mathcal{L}}(T[\hat{\mathfrak{R}}, \mathfrak{g}])$ with $\check{Y}\tilde{O}\beta = O$. Then all components of $\tilde{O}$ and β are nilpotent in $\hat{\mathfrak{R}}$. Since $\hat{\mathfrak{R}}$ is R-σ-$\mathcal{N}^{\mathcal{L}}\mathcal{S}$-ring, we obtain $\check{Y}\beta\sigma(\tilde{O}) = O$. Hence $T[\hat{\mathfrak{R}}, \mathfrak{g}]$ is R-σ-$\mathcal{N}^{\mathcal{L}}\mathcal{S}$-ring. ■

The polynomial ring over a right $\mathcal{N}^{\mathcal{L}}$-symmetric is now examined to see if it is a R-σ-$\mathcal{N}^{\mathcal{L}}\mathcal{S}$-ring. However, the following example shows that the answer is negative.

Example 3.7 Assume that $\bar{Z}_2$ is the field of integers modulo 2, and consider $\tilde{A} = \bar{Z}_2[\check{v}_O, \check{v}_1, \check{v}_2, \acute{\rho}_O, \acute{\rho}_1, \acute{\rho}_2, \tilde{\omega}]$ is the free algebra of polynomials with zero constant term in non-commuting intermediates $\check{v}_O, \check{v}_1, \check{v}_2, \acute{\rho}_O, \acute{\rho}_1, \acute{\rho}_2$ and $\tilde{\omega}$ over $\bar{Z}_2$. Define an automorphism σ of $\tilde{A}$ by :

$$\check{v}_O, \check{v}_1, \check{v}_2, \acute{\rho}_O, \acute{\rho}_1, \acute{\rho}_2, \tilde{\omega} \to \acute{\rho}_O, \acute{\rho}_1, \acute{\rho}_2, \check{v}_O, \check{v}_1, \check{v}_2, \tilde{\omega}$$

Take an ideal $\bar{I}$ in the ring $\bar{Z}_2 + \tilde{A}$, generated by the following elements:

$\breve{u}_O\acute{\rho}_O, \breve{u}_O\acute{\rho}_1 + \breve{u}_1\acute{\rho}_O, \breve{u}_O\acute{\rho}_2 + \breve{u}_1\acute{\rho}_1 + \breve{u}_2\acute{\rho}_O, \breve{u}_1\acute{\rho}_2 + \breve{u}_2\acute{\rho}_1, \breve{u}_2\acute{\rho}_2, \breve{u}_O\hat{r}\acute{\rho}_O, \breve{u}_2\hat{r}\breve{u}_2, \acute{\rho}_O\breve{u}_O, \acute{\rho}_O\breve{u}_1 + \acute{\rho}_1\breve{u}_O, \acute{\rho}_O\breve{u}_2 + \acute{\rho}_1\breve{u}_1 + \acute{\rho}_2\breve{u}_O, \acute{\rho}_1\breve{u}_2 + \acute{\rho}_2\breve{u}_1, \acute{\rho}_O\hat{r}\breve{u}_O, \acute{\rho}_2\hat{r}\breve{u}_2, (\breve{u}_O + \breve{u}_1 + \breve{u}_2)\hat{r}(\acute{\rho}_O + \acute{\rho}_1 + \acute{\rho}_2), (\acute{\rho}_O + \acute{\rho}_1 + \acute{\rho}_2)\hat{r}(\breve{u}_O + \breve{u}_1 + \breve{u}_2)$, and $\hat{r}_1\hat{r}_2\hat{r}_3\hat{r}_4$, where $\hat{r}, \hat{r}_1, \hat{r}_2, \hat{r}_3, \hat{r}_4 \in \tilde{A}$.

Now $\mathfrak{R} = (\bar{Z}_2 + \tilde{A})/\bar{I}$ is symmetric by [(Huh *et al.*, 2005),Example 3.1] and so a R-$\mathcal{N}^{\mathcal{L}}$-$\mathcal{S}$ring. By [(Mohammadi *et al.*, 2012), Example 3.6],

we have $\tilde{\omega} \in \mathfrak{R}[и]$ and $\breve{u}_O + \breve{u}_1 и + \breve{u}_2 и^2, \acute{\rho}_O + \acute{\rho}_1 и + \acute{\rho}_2 и^2 \in \mathcal{N}^{\mathcal{L}}(\mathfrak{R}[и])$. Now $\tilde{\omega}(\breve{u}_O + \breve{u}_1 и + \breve{u}_2 и^2)(\acute{\rho}_O + \acute{\rho}_1 и + \acute{\rho}_2 и^2) = (\tilde{\omega}\breve{u}_O + \tilde{\omega}\breve{u}_1 и + \tilde{\omega}\breve{u}_2 и^2)(\acute{\rho}_O + \acute{\rho}_1 и + \acute{\rho}_2 и^2) = \tilde{\omega}\breve{u}_O\acute{\rho}_O + \tilde{\omega}\breve{u}_O\acute{\rho}_1 и + \tilde{\omega}\breve{u}_O\acute{\rho}_2 и^2 + \tilde{\omega}\breve{u}_1\acute{\rho}_O и + \tilde{\omega}\acute{\rho}_1\breve{u}_1 и^2 + \tilde{\omega}\breve{u}_1\acute{\rho}_2 и^3 + \tilde{\omega}\breve{u}_2\acute{\rho}_O и^2 + \tilde{\omega}\breve{u}_2\acute{\rho}_1 и^3 + \tilde{\omega}\breve{u}_2\acute{\rho}_2 и^4 = \tilde{\omega}\breve{u}_O\acute{\rho}_O + (\tilde{\omega}\breve{u}_O\acute{\rho}_1 + \tilde{\omega}\breve{u}_1\acute{\rho}_O)и + (\tilde{\omega}\breve{u}_O\acute{\rho}_2 + \tilde{\omega}\breve{u}_1\acute{\rho}_1 + \tilde{\omega}\breve{u}_2\acute{\rho}_O)и^2 + (\tilde{\omega}\breve{u}_1\acute{\rho}_2 + \tilde{\omega}\breve{u}_2\acute{\rho}_1)и^3 + \tilde{\omega}\breve{u}_2\acute{\rho}_2 и^4 \in \bar{I}[и]$, but $\tilde{\omega}(\acute{\rho}_O + \acute{\rho}_1 и + \acute{\rho}_2 и^2)ϭ((\breve{u}_O + \breve{u}_1 и + \breve{u}_2 и^2)) = \tilde{\omega}(\acute{\rho}_O + \acute{\rho}_1 и + \acute{\rho}_2 и^2)(\acute{\rho}_O + \acute{\rho}_1 и + \acute{\rho}_2 и^2) = \tilde{\omega}\acute{\rho}_O^2 + \tilde{\omega}\acute{\rho}_O\acute{\rho}_1 и + \tilde{\omega}\acute{\rho}_0\acute{\rho}_2 и^2 + \tilde{\omega}\acute{\rho}_1\acute{\rho}_0 и + \tilde{\omega}\acute{\rho}_1^2 и^2 + \tilde{\omega}\acute{\rho}_1\acute{\rho}_2 и^3 + \tilde{\omega}\acute{\rho}_2\acute{\rho}_0 и^2 + \tilde{\omega}\acute{\rho}_2\acute{\rho}_1 и^3 + \tilde{\omega}\acute{\rho}_2^2 и^4 = \tilde{\omega}\acute{\rho}_O^2 + (\tilde{\omega}\acute{\rho}_O\acute{\rho}_1 + \tilde{\omega}\acute{\rho}_1\acute{\rho}_0)и + (\tilde{\omega}\acute{\rho}_0\acute{\rho}_2 + \tilde{\omega}\acute{\rho}_1^2 + \tilde{\omega}\acute{\rho}_2\acute{\rho}_0)и^2 + (\tilde{\omega}\acute{\rho}_1\acute{\rho}_2 + \tilde{\omega}\acute{\rho}_2\acute{\rho}_1)и^3 + \tilde{\omega}\acute{\rho}_2^2 и^4 \notin \bar{I}[и]$, because $\acute{\rho}_O^2, \tilde{\omega}\acute{\rho}_O\acute{\rho}_1 + \tilde{\omega}\acute{\rho}_1\acute{\rho}_0, \hat{c}\hat{b}_O\hat{b}_2 + \tilde{\omega}\acute{\rho}_0\acute{\rho}_2 + \tilde{\omega}\acute{\rho}_1^2 + \tilde{\omega}\acute{\rho}_2\acute{\rho}_0, \tilde{\omega}\acute{\rho}_1\acute{\rho}_2 + \tilde{\omega}\acute{\rho}_2\acute{\rho}_1, \tilde{\omega}\acute{\rho}_2^2 \notin \bar{I}$. Hence $\mathfrak{R}[и]$ is not a R-ϭ-$\mathcal{N}^{\mathcal{L}}\mathcal{S}$-ring.■

According to Rege and Chhawchharia (Rege&Chhawchharia,1997),a ring $\mathfrak{R}$ Armendariz exists if whenever any polynomials $f(и) = \breve{u}_O + \breve{u}_1 и + \cdots + \breve{u}_m и^m, g(и) = \acute{\rho}_O + \acute{\rho}_1 и + \cdots + \acute{\rho}_n и^n \in \mathfrak{R}[и]$ satisfy $f(и)g(и) = O$, then $\breve{u}_j\acute{\rho}_j = O$ for each j and j.

Since Armendariz was the first to demonstrate that a red-ring always satisfies this criterion, they used this terminology ([(Armendariz, 1974), Lemma1]). Assume $\mathfrak{R}$ is a ring with an *endo* ϭ. Recall that the map $\mathfrak{R}[и] \to \mathfrak{R}[и]$ by $\sum_{j=O}^{m} \breve{u}_j и^j \to \sum_{j=O}^{m} ϭ(\breve{u})и^j$.

Proposition 3.8 Suppose $\mathfrak{R}$ is an Armendariz ring then $\mathfrak{R}$ is R-ϭ-$\mathcal{N}^{\mathcal{L}}\mathcal{S}$-ring if and only if $\mathfrak{R}[и]$ is a R-ϭ-$\mathcal{N}^{\mathcal{L}}\mathcal{S}$-ring.

Proof. It also suffices to establish necessity. Let $f(и) = \sum_{j=O}^{m} \breve{u}_j и^j \in \mathfrak{R}[и]$ and $g(и) = \sum_{j=O}^{n} \acute{\rho}_j и^j$, $h(и) = \sum_{ř=O}^{w} \tilde{\omega}_ř и^ř \in \mathcal{N}^{\mathcal{L}}(\mathfrak{R}[и])$ with $f(и)g(и)h(и) = O$ and so $\breve{u}_j\acute{\rho}_j\tilde{\omega}_ř = O$ for all j, j and ř. $\breve{u}_j\tilde{\omega}_ř ϭ(\acute{\rho}_j) = O$ since $\mathfrak{R}$ is Armendariz and a R-ϭ-$\mathcal{N}^{\mathcal{L}}\mathcal{S}$-ring. This yields $f(и)h(и)\,ϭ(g(и)) = O$, therefore, $\mathfrak{R}[и]$ is a R-ϭ-$\mathcal{N}^{\mathcal{L}}\mathcal{S}$-ring.

Theorem 3.9 (1) For a ring $\mathfrak{R}$, if $\mathfrak{R}$ is ϭ-rg then $\mathfrak{R}$ is a R-ϭ-$\mathcal{N}^{\mathcal{L}}\mathcal{S}$-ring.

(2) If the skew polynomial ring $\mathfrak{R}[и; ϭ]$ of a ring $\mathfrak{R}$ is a $\mathcal{S}$-ring, then $\mathfrak{R}$ is a ϭ-$\mathcal{N}^{\mathcal{L}}\mathcal{S}$-ring.

Proof. (1) Consider $\mathfrak{R}$ is ϭ-rg. Note that any ϭ-rg ring is reduced and ϭ is a monomorphism by [(Marks, 2002), P.218]. We show that $\mathfrak{R}$ is R-ϭ-$\mathcal{N}^{\mathcal{L}}\mathcal{S}$-ring. Assume $\breve{u}\acute{\rho}\tilde{\omega} = O$ for $\breve{u} \in \mathfrak{R}$ and $\acute{\rho}, \tilde{\omega} \in \mathcal{N}^{\mathcal{L}}(\mathfrak{R})$. Then we obtain $\acute{\rho}\breve{u}\tilde{\omega} = O$, since $\mathfrak{R}$ is reduced (and so symmetric). Thus,

$\breve{u}\tilde{\omega}ϭ(\acute{\rho})ϭ(\breve{u}\tilde{\omega}ϭ(\acute{\rho})) = \breve{u}\tilde{\omega}ϭ(\acute{\rho}\breve{u}\tilde{\omega})ϭ^{\tilde{\omega}}(\acute{\rho}) = O$. Since $\mathfrak{R}$ is ϭ-rg, $\breve{u}\tilde{\omega}ϭ(\acute{\rho}) = O$ and thus $\mathfrak{R}$ is a R-ϭ-$\mathcal{N}^{\mathcal{L}}\mathcal{S}$-ring.

(2) Assume $\breve{u}\acute{\rho}\tilde{\omega} = O$ for $\breve{u}, \acute{\rho}, \tilde{\omega} \in \mathcal{N}^{\mathcal{L}}(\mathfrak{R})$. Let $r = \breve{u}, s = \acute{\rho}, t = \tilde{\omega}x \in \mathfrak{R}[и; ϭ]$ Then $rst = \breve{u}\acute{\rho}\tilde{\omega}и = O \in \mathfrak{R}[и; ϭ]$, since $\mathfrak{R}[и; ϭ]$ is $\mathcal{S}$-ring, we get $O = rts = (\breve{u}\tilde{\omega})и\acute{\rho} = \breve{u}\tilde{\omega}ϭ(\acute{\rho})и$, and so $\breve{u}\tilde{\omega}ϭ(\acute{\rho}) = O$. Thus $\mathfrak{R}$ is a R-ϭ-$\mathcal{N}^{\mathcal{L}}\mathcal{S}$-ring. ■

The Dorroh extension(For short *DoEx*) of an algebra $\mathfrak{R}$ over a commutative ring $\hat{S}$, introduced by Dorroh in 1932(Dorroh, 1932), is a construction that enlarges $\mathfrak{R}$ by incorporating elements of $\mathfrak{R}$. It is defined as the Abelian group $\hat{\mathcal{D}} = \mathfrak{R} \times \hat{S}$ with multiplication given by $(r_1, s_1)(r_2, s_2) = (r_1 r_2 + s_1 r_2 + s_2 r_1, s_1 s_2)$ for all $r_i \in \mathfrak{R}$ and $s_i \in \hat{S}$. This operation preserves the algebraic structure while introducing a direct interaction between elements of $\mathfrak{R}$ and $\hat{S}$. Additionally, any $\hat{S}$-linear *endo* ϭ of $\mathfrak{R}$ extends naturally to an S, S-algebra homomorphism $ϭ: \hat{\mathcal{D}} \to \hat{\mathcal{D}}$, defined by $ϭ(r, s) = (ϭ(r), s)$, applying ϭ to the first component while keeping the second component fixed.

Theorem 3.10 Consider $\mathfrak{R}$ is an algebra equipped with an *endo* ϭ and an identity element, defined over a commutative red-ring $\bar{Z}$. Then $\mathfrak{R}$ is a R-ϭ-$\mathcal{N}^{\mathcal{L}}\mathcal{S}$-ring if and only if the *DoEx* $\hat{\mathcal{D}}$ of $\mathfrak{R}$ by $\bar{Z}$ is R-ϭ-$\mathcal{N}^{\mathcal{L}}\mathcal{S}$-ring.

Proof. It is clear that $\mathcal{N}^{\mathcal{L}}(\hat{\mathcal{D}}) = (\mathcal{N}^{\mathcal{L}}(\mathfrak{R}), O)$. Since $\bar{Z}$ is a commutative red-ring. Consider $(\breve{u}, O), (\acute{\rho}, O) \in \mathcal{N}^{\mathcal{L}}(\hat{\mathcal{D}}(\mathfrak{R}, Z))$ and $(\breve{u}, \ddot{\varepsilon}) \in \hat{\mathcal{D}}(\mathfrak{R}, Z)$ with $(\ddot{\eta}, \ddot{\varepsilon})(\breve{u}, O)(\acute{\rho}, O) = ((\ddot{\eta} + \ddot{\varepsilon})\breve{u}, O)(\acute{\rho}, O) = ((\ddot{\eta} + \ddot{\varepsilon})\breve{u}\acute{\rho}, O)$. Thus $(\ddot{\eta} + \ddot{\varepsilon})\breve{u}\acute{\rho} = O$, $\breve{u}, \acute{\rho} \in \mathcal{N}^{\mathcal{L}}(\mathfrak{R})$. Since $\mathfrak{R}$ is ϭ-$\mathcal{N}^{\mathcal{L}}\mathcal{S}$-ring, we get $\ddot{\eta} + \ddot{\varepsilon} \in Z$, $(\ddot{\eta} + \ddot{\varepsilon})\acute{\rho}ϭ(\breve{u}) = O$. So $(\ddot{\eta}, \ddot{\varepsilon})(\acute{\rho}, O)ϭ((\breve{u}, O)) = O$. Thus $\hat{\mathcal{D}}(\mathfrak{R}, Z)$ is $\bar{ϭ}$-$\mathcal{N}^{\mathcal{L}}\mathcal{S}$-ring.■

SOME LOCALIZATIONS OF RIGHT ϭ-$\mathcal{N}^{\mathcal{L}}$-SYMMETRIC RINGS:

Assume that ϭ is a monomorphism of the ring $\mathfrak{R}$. The construction of an over-ring of $\mathfrak{R}$. (A ring $\mathfrak{R}$ is an over ring of integral domain g, if g is a subring of $\mathfrak{R}$ and $\mathfrak{R}$ is a subring of the field of fraction $Q(g)$, the relationship $g \subseteq \mathfrak{R} \subseteq Q(g)$). As introduced by Jordan, is now under consideration (for more details, see (Jordan, 1982)). Define $\check{Y}(\mathfrak{R}, ϭ)$ as the subset of the skew Laurent polynomial ring $\mathfrak{R}[и, и^{-1}; ϭ]$, consisting of elements of the form $и^{-n}\ddot{\varepsilon}и^n$ for $\ddot{\varepsilon} \in \mathfrak{R}$ and $n \geq O$. Notably, for $m \geq O$, the relation $и^{-m}\ddot{\varepsilon}и^m = ϭ^{-m}(\ddot{\varepsilon})$ hold for any $\ddot{\varepsilon} \in \mathfrak{R}$. This implies that for any $m \geq O$, the transformation follows the pattern:

$$и^{-n}\ddot{\varepsilon}и^n = и^{-(n+m)}ϭ^{-m}(\ddot{\varepsilon})и^{n+m}.$$

From this, it follows that $\check{Y}(\mathfrak{R}, ϭ)$ forms a subring of $\mathfrak{R}[и, и^{-1}; ϭ]$, equipped with the natural operation:

$$(и^{-з}\ddot{\varepsilon}и^з)(и^{-\epsilon}\ddot{\eta}и^{\epsilon}) = и^{-(з+\epsilon)}ϭ^{\epsilon}(\ddot{\varepsilon})ϭ^{з}(\ddot{\eta})и^{з+\epsilon},$$

And,

$и^{-з}\ddot{\varepsilon}и^з + и^{-\epsilon}\ddot{\eta}и^{\epsilon} = и^{-(з+\epsilon)}(ϭ^{\epsilon}(\ddot{\varepsilon}) + ϭ^{з}(\ddot{\eta}))и^{з+\epsilon}, \forall \ddot{\varepsilon}, \ddot{\eta} \in \mathfrak{R}$ and $з, \epsilon \geq O$.

Notably, $\check{Y}(\mathfrak{R}, ϭ)$ serves as an over-ring of $\mathfrak{R}$, and the mapping $\check{Y}(\mathfrak{R}, ϭ) \to \check{Y}(\mathfrak{R}, ϭ)$ defined by $и^{-з}\ddot{\varepsilon}и^з \to и^{-з}ϭ(\ddot{\varepsilon})и^з$, is an automorphism of $\check{Y}(\mathfrak{R}, ϭ)$.

Jordan established that such an extension $\check{Y}(\mathfrak{R}, ϭ)$ always exists for any given pair $(\mathfrak{R}, ϭ)$ (Jordan, 1982).

This is achieved using left localization of the skew polynomial $\mathfrak{R}[и, ϭ]$ with respect to the set of powers of и. This extension $\check{Y}(\mathfrak{R}, ϭ)$ is commonly referred to as the Jordan extension of $\mathfrak{R}$ by ϭ.

Proposition 4.1 Consider $\mathfrak{R}$ is a ring with a monomorphism, then $\mathfrak{R}$ is R-ϭ-$\mathcal{N}^{\mathcal{L}}\mathcal{S}$-ring if and only if the Jordan extension $\check{Y} = \check{Y}(\mathfrak{R}, ϭ)$ is R-ϭ-$\mathcal{N}^{\mathcal{L}}\mathcal{S}$-ring.

Proof. If $\mathfrak{R}$ is R-ϭ-$\mathcal{N}^{\mathcal{L}}\mathcal{S}$-ring, then so is each subring g with $ϭ(\check{Y}) \subseteq \check{Y}$. Therefore, it is enough to demonstrate the necessity. Assume $\mathfrak{R}$ is ϭ-$\mathcal{N}^{\mathcal{L}}\mathcal{S}$-ring and $\breve{u}\acute{\rho}\tilde{\omega} = O$ where $\breve{u} = и^{-з}\ddot{\varepsilon}_1 и^з \in \mathcal{A}$, $\acute{\rho} = и^{-\epsilon}\ddot{\varepsilon}_2 и^{\epsilon}$, $\tilde{\omega} = и^{-f}\ddot{\varepsilon}_3 и^f \in \mathcal{N}^{\mathcal{L}}(\mathcal{A})$ for $з, \epsilon, f > O$. Then $\ddot{\varepsilon}_1 \in \mathfrak{R}$ and $\ddot{\varepsilon}_2, \ddot{\varepsilon}_3 \in \mathcal{N}^{\mathcal{L}}(\mathfrak{R})$. From $\breve{u}\acute{\rho}\tilde{\omega} = O$, we get $ϭ^f(\ddot{\varepsilon}_1)ϭ^{\epsilon}(\ddot{\varepsilon}_2)ϭ^з(\ddot{\varepsilon}_3) = O$ and so $ϭ^f(\ddot{\varepsilon}_1)ϭ^з(\ddot{\varepsilon}_3)ϭ(ϭ^{\epsilon}(\ddot{\varepsilon}_2)) = ϭ^f(\ddot{\varepsilon}_1)ϭ^з(\ddot{\varepsilon}_3)ϭ^{\epsilon+1}(\ddot{\varepsilon}_2) = O$ by assumption. Hence $\breve{u}\tilde{\omega}ϭ(\acute{\rho}) = (и^{-з}\ddot{\varepsilon}_1 и^з)(и^{-f}\ddot{\varepsilon}_3 и^f)ϭ(и^{-\epsilon}\ddot{\varepsilon}_2 и^{\epsilon}) = (и^{-з}\ddot{\varepsilon}_1 и^з)(и^{-f}\ddot{\varepsilon}_3 и^f)(и^{-\epsilon}ϭ(\ddot{\varepsilon}_2)и^{\epsilon}) = и^{-(з+f+\epsilon)}ϭ^f(\ddot{\varepsilon}_1)ϭ^з(\ddot{\varepsilon}_3)ϭ^{\epsilon+1}(\ddot{\varepsilon}_2)и^{+(з+f+\epsilon)} = O$.

Therefore, Jordan extension $\mathcal{A}(\mathfrak{R}, ϭ)$ is right ϭ-$\mathcal{N}^{\mathcal{L}}\mathcal{S}$-ring. ■

Recall that the map $\mathfrak{R}[и, и^{-1}] \to \mathfrak{R}[и, и^{-1}]$ defined by $\sum_{i=-n}^{\infty} a_i и^i \to \sum_{i=-n}^{\infty}$ б$(a_i)и^i$ is an *endo* of $\mathfrak{R}[и, и^{-1}]$ and the map obviously extends б.

Proposition 4.2 If $\mathfrak{R}$ is an Armendariz ring, then the following claims are equivalent:

(1) $\mathfrak{R}$ is a R-б-$\mathcal{N}^{\mathcal{L}}\mathcal{S}$-ring;

(2) $\mathfrak{R}[и]$ is a R-б-$\mathcal{N}^{\mathcal{L}}\mathcal{S}$-ring;

(3) $\mathfrak{R}[и, и^{-1}]$ is a R-б-$\mathcal{N}^{\mathcal{L}}\mathcal{S}$-ring.

Proof. (1) ↔ (2) is proven in proposition 3.8

(2) ↔ (3) Showing necessity is sufficient. Let $Ƒ(и) \in \mathfrak{R}[и, и^{-1}]$ and $Ҕ(и), Ҝ(и) \in \mathcal{N}^{\mathcal{L}}(\mathfrak{R}[и, и^{-1}])$ with $Ƒ(и)Ҕ(и)Ҝ(и) = 0$. Then $\exists n \in \bar{Z}^{+}$ such that $\hat{f}_1(и) = Ƒ(и)и^n \in \mathfrak{R}[и]$ and $Ҕ_1(и) = Ҕ(и)и^n$, $Ҝ_1(и) = Ҝ(и)и^n \in \mathcal{N}^{\mathcal{L}}(\mathfrak{R}[и])$ and so $Ƒ_1(и)Ҕ_1(и)Ҝ_1(и) = 0$. Since $\mathfrak{R}[и]$ is R-б-$\mathcal{N}^{\mathcal{L}}\mathcal{S}$-ring, we obtain $Ƒ_1(и)Ҝ_1(и)$ б$(Ҕ_1(и)) = 0$. Hence $Ƒ(и)Ҝ(и)$ б$(Ҕ(и)) = и^{-3n}Ƒ_1(и)Ҝ_1(и)$ б$(Ҕ_1(и)) = 0$. Thus $\mathfrak{R}[и, и^{-1}]$ is a R-б-$\mathcal{N}^{\mathcal{L}}\mathcal{S}$-ring.

(3) → (2) and (3) → (1) are clear. ■

Proposition 4.3 Assume that $\mathfrak{R}$ is a ring and that $\bar{Z}(\mathfrak{R})$ is an infinite subring with all of its nonzero elements regular in $\mathfrak{R}$. Then $\mathfrak{R}$ is R-б-$\mathcal{N}^{\mathcal{L}}\mathcal{S}$-ring if and only if $\mathfrak{R}[и]$ is R-б-$\mathcal{N}^{\mathcal{L}}\mathcal{S}$-ring if and only if $\mathfrak{R}[и; и^{-1}]$ is R-б-$\mathcal{N}^{\mathcal{L}}\mathcal{S}$-ring.

Proof. It is sufficient to demonstrate that, $\mathfrak{R}[и]$ is б-$\mathcal{N}^{\mathcal{L}}\mathcal{S}$-ring when so is $\mathfrak{R}$, $\mathfrak{R}[и]$is obtained as the subdirect product of an infinite collection of copies of $\mathfrak{R}$, as $\bar{Z}(\mathfrak{R})$ comprises an infinite subring where each nonzero element is regular in $\mathfrak{R}$ according to the hypothesis. Thus $\mathfrak{R}[и]$ is б-$\mathcal{N}^{\mathcal{L}}\mathcal{S}$-ring because $\mathfrak{R}$ is б-$\mathcal{N}^{\mathcal{L}}\mathcal{S}$-ring by the assumption. ■

ON RIGHT б-$\mathcal{N}^{\mathcal{L}}$-SYMMETRIC MODULES:

This section extends the idea of a R-б-$\mathcal{N}^{\mathcal{L}}\mathcal{S}$-ring to modules by introducing the notion of a right б-$\mathcal{N}^{\mathcal{L}}$-symmetric module, which is an extension of symmetric modules and generalization of б-symmetric modules. Some of the well-established results which are obtained in section 3 and section 4 are generalized to right б-$\mathcal{N}^{\mathcal{L}}$-symmetric modules. We introduce the following definition first.

Definition 5.1 Assume $\mathfrak{R}$ is a ring and б a nonzero *endo* of $\mathfrak{R}$. An $\mathfrak{R}$-module $\widehat{\mathcal{M}}^{\mathfrak{R}}$ is called a right б-$\mathcal{N}^{\mathcal{L}}$-symmetric modules (For short R-б-$\mathcal{N}^{\mathcal{L}}\mathcal{S}$-module) if whenever $mab = 0$ for $a, b \in \mathcal{N}^{\mathcal{L}}(\mathfrak{R})$ and $m \in \widehat{\mathcal{M}}^{\mathfrak{R}}$ implies mbб$(a) = 0$.

Example 5.2:

1. R-$1_{\mathfrak{R}}$-$\mathcal{N}^{\mathcal{L}}$-symmetric modules are exactly R-б-$\mathcal{N}^{\mathcal{L}}\mathcal{S}$-module.
2. For any commutative ring, any module $\widehat{\mathcal{M}}^{\mathfrak{R}}$ is an б-$\mathcal{N}^{\mathcal{L}}\mathcal{S}$-modules.
3. Let $\hat{D}$ be a division ring, $\mathfrak{R} = \begin{bmatrix} \hat{D} & \hat{D} \\ 0 & \hat{D} \end{bmatrix}$, and $\mathcal{A} = \begin{bmatrix} 0 & \hat{D} \\ 0 & \hat{D} \end{bmatrix}$. Then $\mathcal{A}^{\mathfrak{R}}$ is an б-$\mathcal{N}^{\mathcal{L}}\mathcal{S}$-module.
4. It is clear that б-symmetric modules are б-$\mathcal{N}^{\mathcal{L}}\mathcal{S}$-module but the converse implication is not true as we see in the following example.

Example 5.3 Let $\bar{Z}$ be the ring of integers. We now consider the ring $\mathfrak{R} = \left\{\begin{pmatrix} ŭ & ṕ \\ 0 & \tilde{ω} \end{pmatrix} ; ŭ, ṕ, \tilde{ω} \in \bar{Z}\right\}$ and the $\mathfrak{R}$-module $\widehat{\mathcal{M}}^{\mathfrak{R}} = \left\{\begin{pmatrix} 0 & ɥ \\ ɥ & ƅ \end{pmatrix} ; ɥ, ɥ, ƅ \in \bar{Z}\right\}$ and б an homomorphism defined on $\mathfrak{R}$ by б$\left(\begin{pmatrix} ŭ & ṕ \\ 0 & \tilde{ω} \end{pmatrix}\right) = \begin{pmatrix} 0 & ṕ \\ 0 & 0 \end{pmatrix}$ where $\begin{pmatrix} ŭ & ṕ \\ 0 & \tilde{ω} \end{pmatrix} \in \mathfrak{R}$. $\mathfrak{R}$ is R-б-$\mathcal{N}^{\mathcal{L}}\mathcal{S}$-module for $m = \begin{pmatrix} 0 & ɥ \\ ɥ & ƅ \end{pmatrix} \in \widehat{\mathcal{M}}^{\mathfrak{R}}$ and $\hat{h}, \hat{k} \in \mathcal{N}^{\mathcal{L}}(\mathfrak{R})$ where $\hat{h} = \begin{pmatrix} 0 & ṕ_1 \\ 0 & 0 \end{pmatrix}$, $\hat{k} = \begin{pmatrix} 0 & ṕ_2 \\ 0 & 0 \end{pmatrix}$ we have,

$$m\hat{h}\hat{k} = \begin{pmatrix} 0 & ɥ \\ ɥ & ƅ \end{pmatrix}\begin{pmatrix} 0 & ṕ_1 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & ṕ_2 \\ 0 & 0 \end{pmatrix} = 0$$

Also,

$$m\hat{k}б(\hat{h}) = \begin{pmatrix} 0 & ɥ \\ ɥ & ƅ \end{pmatrix}\begin{pmatrix} 0 & ṕ_2 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & ṕ_1 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = 0.$$

But $\widehat{\mathcal{M}}^{\mathfrak{R}}$ is not б-symmetric for $m = \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix} \in M$, $\hat{h} = \begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix}$, $\hat{k} = \begin{pmatrix} 1 & 2 \\ 0 & 0 \end{pmatrix} \in \mathfrak{R}$, we have,

$$m\hat{h}\hat{k} = \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} = 0$$

But, $m\hat{k}б(\hat{h}) = \begin{pmatrix} 0 & 0 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 0 & 0 \end{pmatrix}\begin{pmatrix} 0 & 2 \\ 0 & 0 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 0 & 2 \end{pmatrix} \neq 0.$

However, the converse is true if, $\widehat{\mathcal{M}}^{\mathfrak{R}}$ is an б-rg-module by the following Lemma.

Lemma 5.4 Let $\widehat{\mathcal{M}}^{\mathfrak{R}}$ be an б-rg-module, then the following are equivalent:

1. $\widehat{\mathcal{M}}^{\mathfrak{R}}$ is an б-symmetric module;
2. $\widehat{\mathcal{M}}^{\mathfrak{R}}$ is an б-$\mathcal{N}^{\mathcal{L}}$ symmetric module.

Proof. (1) ⟹ (2) It is clear.

(2) ⟹ (1) Let $mṕ^2 = 0$, for $m \in \widehat{\mathcal{M}}^{\mathfrak{R}}$ and $ṕ \in \mathfrak{R}$. If $m = 0$, is trivial. Then $ṕ^2 = 0$ implies $ṕ \in \mathcal{N}^{\mathcal{L}}(\mathfrak{R})$, since $\widehat{\mathcal{M}}^{\mathfrak{R}}$ is б-$\mathcal{N}^{\mathcal{L}}$ symmetric. Hence $mṕ^2 = 0$ implies $mṕṕ = 0$ implies $mṕ$б$(ṕ) = 0$, and since $\widehat{\mathcal{M}}^{\mathfrak{R}}$ is an б-rg-module implies that $mṕ = 0$. Therefore, $\widehat{\mathcal{M}}^{\mathfrak{R}}$ is an б-red-module and by [(Agayev *et al.*, 2009), Theorem 2.1] $\widehat{\mathcal{M}}^{\mathfrak{R}}$ is an б-symmetric module. ■

Proposition 5.5 For a given *endo* of a ring $\mathfrak{R}$ and an $\mathfrak{R}$-module $\widehat{\mathcal{M}}^{\mathfrak{R}}$. The statements below are equivalent:

1. $\widehat{\mathcal{M}}^{\mathfrak{R}}$ is R-б-$\mathcal{N}^{\mathcal{L}}\mathcal{S}$-module,
2. $\ell_{\widehat{\mathcal{M}}^{\mathfrak{R}}}(ŭ(ṕ)) \subseteq \ell_{\widehat{\mathcal{M}}^{\mathfrak{R}}}(ṕб(ŭ))$, for any $ŭ, ṕ \in \mathcal{N}^{\mathcal{L}}(\mathfrak{R})$,
3. $ŸŨ'\tilde{V} = 0$ if and only if $Ÿ\tilde{V}б(Ũ') = 0$,for $Ũ', \tilde{V} \subseteq \mathcal{N}^{\mathcal{L}}(\mathfrak{R})$ and $Ÿ \subseteq \widehat{\mathcal{M}}^{R^{\circ}}$,
4. $\ell_{Ÿ}(Ũ'\tilde{V}) \subseteq \ell_{Ÿ}(\tilde{V}б(Ũ'))$, for any $Ũ', \tilde{V} \subseteq \mathcal{N}^{\mathcal{L}}(\mathfrak{R})$ and $Ÿ \subseteq \widehat{\mathcal{M}}^{\mathfrak{R}}$.

Proof. (1) → (3) Suppose that $ŸŨ'\tilde{V} = 0$, for $Ũ', \tilde{V} \subseteq \mathcal{N}^{\mathcal{L}}(\mathfrak{R})$ and $Ÿ \subseteq \widehat{\mathcal{M}}^{\mathfrak{R}}$. Then $ŭṕ\tilde{ω} = 0$ for any $ŭ \in Ÿ, ṕ \in Ũ'$ and $\tilde{ω} \in \tilde{V}$, and hence $ŭ\tilde{ω}$ б$(ṕ) = 0$. Therefore $Ÿ\tilde{V}б(Ũ') = \{\sum_{i=1} ŭ_i\tilde{ω}_i б(ṕ_i)\ ;\ ŭ_i \in Ÿ, ṕ_i \in Ũ'$ and $\tilde{ω}_i \in \tilde{V}\} = 0$. The converse is clear. (1) → (2) and (3) → (4) is obvious ■

Proposition 5.6 Suppose that $\mathfrak{R}$ is a ring and б an *endo* of $\mathfrak{R}$ and $\mathcal{M}^{\mathfrak{R}}$ is an $\mathfrak{R}$-module. Then we have the following:

1. $mṕ_1ṕ_2 \dots ṕ_{\varpi} = 0$ implies $mṕ_{б(1)}ṕ_{б(2)} \dots ṕ_{б(\varpi)} = 0$ for each permutation б of the set $\{1,2,\dots,\varpi\}$, where $ṕ_i \in \mathcal{N}^{\mathcal{L}}(\mathfrak{R})$ and $\varpi \in \bar{Z}^{+}$.
2. $mŭ_1ŭ_2 \dots ŭ_{\varpi} = 0$ if and only if $\hat{m}б^{i_1}(ŭ_1)\, б^{i_2}(ŭ_2) \dots б^{i_{\varpi}}(ŭ_{\varpi}) = 0$ for any $i_1, i_2 \dots i_{\varpi} \in \bar{Z}^{+}$.

Proof. The proof is similar to the proof of [(Agayev *et al.*, 2009), Proposition2.4]. ■

Proposition 5.7 Suppose $\mathfrak{R}$ is a ring and б an *endo* of $\mathfrak{R}$ and $\mathfrak{R}$-module $\widehat{\mathcal{M}}^{\mathfrak{R}}$. Then we have the following:

1. The class of a R-б-$\mathcal{N}^{\mathcal{L}}\mathcal{S}$-modules is closed under submodules, and direct sums.
2. The direct product of R-б-$\mathcal{N}^{\mathcal{L}}\mathcal{S}$-modules is R-б-$\mathcal{N}^{\mathcal{L}}\mathcal{S}$-module.

3. If ę is a central idempotent of a ring $\mathfrak{R}$ with $Ϭ(ę) = ę$ and $Ϭ(1 - ę) = 1 - ę$, then $\widehat{\mathcal{M}}^{ę\mathfrak{R}}$ and $\widehat{\mathcal{M}}^{(1-ę)\mathfrak{R}}$ are R-Ϭ-$\mathcal{N}^{\mathcal{L}}\mathcal{S}$-module if and only if $\widehat{\mathcal{M}}^{\mathfrak{R}}$ is right Ϭ-$\mathcal{N}^{\mathcal{L}}\mathcal{S}$-module.

Proof. (1) Depending on the definitions and algebraic structures, the proof is straightforward.

(2) Note that $\mathcal{N}^{\mathcal{L}}\left(\prod_{ɾ\in I}\mathfrak{R}_ɾ\right) \subseteq \prod_{ɾ\in I}\mathcal{N}^{\mathcal{L}}(\mathfrak{R}_ɾ)$ and $Ϭ_ɾ(\mathfrak{R}_ɾ) \subseteq \mathfrak{R}_ɾ$ for each $ɾ \in I$. Suppose that $\widehat{\mathcal{M}}^{\mathfrak{R}_ɾ}$ is Ϭ-$\mathcal{N}^{\mathcal{L}}\mathcal{S}$-module for each $ɾ \in I$ and let $\widehat{\mathcal{M}}\mathcal{A}\mathcal{B} = 0$ where, $\mathcal{A} = (\hat{a}_ɾ)_{ɾ\in I}$, $\mathcal{B} = (\hat{b}_ɾ)_{ɾ\in I} \in \mathcal{N}^{\mathcal{L}}(\prod_{ɾ\in I}\mathfrak{R}_ɾ)$ and $\widehat{\mathcal{M}} = (m_ɾ)_{ɾ\in I} \in \prod_{ɾ\in I}\left(\widehat{\mathcal{M}}_ɾ^{\mathfrak{R}_ɾ}\right)$. Then $m_ɾ\hat{a}_ɾ\hat{b}_ɾ = 0$ for each $ɾ \in I$ and $m_ɾ\hat{b}_ɾϬ(\hat{a}_ɾ) = 0$ by hypothesis since $\hat{a}_ɾ, \hat{b}_ɾ \in \mathcal{N}^{\mathcal{L}}(\mathfrak{R}_ɾ)$ and $m_ɾ \in \mathcal{M}_ɾ^{\mathfrak{R}_ɾ}$ for each $ɾ \in I$. This implies $\widehat{\mathcal{M}}\mathcal{B}Ϭ(\mathcal{A}) = 0$, entailing that the direct product $\prod_{ɾ\in I}\widehat{\mathcal{M}}_ɾ^{\mathfrak{R}_ɾ}$ is R-Ϭ-$\mathcal{N}^{\mathcal{L}}\mathcal{S}$-module.

(3) Establishing necessity is enough. Assume $\widehat{\mathcal{M}}^{ę\mathfrak{R}}$ and $\widehat{\mathcal{M}}^{(1-ę)\mathfrak{R}}$ are R-Ϭ-$\mathcal{N}^{\mathcal{L}}\mathcal{S}$-modules. Consider $m\hat{a}\hat{b} = 0$, for $m \in \widehat{\mathcal{M}}^{\mathfrak{R}}$, and $\hat{a}, \hat{b} \in \mathcal{N}^{\mathcal{L}}(\mathfrak{R}_i)$, then $0 = ęm\hat{a}\hat{b} = m(ę\hat{a})\hat{b}$. And $0 = (1 - ę)m\hat{a}\hat{b} = m((1 - ę)\hat{a})\hat{b}$. By hypothesis, we get $0 = m\hat{b}Ϭ(ę\hat{a})$ and $0 = m\hat{b}Ϭ(1 - ę)\hat{a}$,

$0 = m\hat{b}Ϭ(ę)Ϭ(\hat{a})$ and $0 = m\hat{b}Ϭ(1 - ę)Ϭ(\hat{a})$,

$0 = m\hat{b}ęϬ(\hat{a})$ and $0 = m\hat{b}(1 - ę)Ϭ(\hat{a})$,

$0 = m\hat{b}ęϬ(\hat{a}) + m\hat{b}Ϭ(\hat{a}) - m\hat{b}ęϬ(\hat{a})$,

$0 = m\hat{b}Ϭ(\hat{a})$.

$\widehat{\mathcal{M}}^{\mathfrak{R}}$ is a R-Ϭ-$\mathcal{N}^{\mathcal{L}}\mathcal{S}$-module. ■

According to (Lee & Zhou, 2004), the module $\widehat{\mathcal{M}}^{\mathfrak{R}}$ is said to be Ϭ-reduced, if for each $m \in \widehat{\mathcal{M}}^{\mathfrak{R}}$ and each $\hat{r} \in \mathfrak{R}$, with $m\hat{r} = 0$, then $m\mathfrak{R} \cap \hat{r}\widehat{\mathcal{M}} = 0$.

Lemma 5.8 ([(Raphael, 1975), Lemma 1.2]). Let $\widehat{\mathcal{M}}^{\mathfrak{R}}$ be an $\mathfrak{R}$-module. Then the following statements are equivalent:

1. $\widehat{\mathcal{M}}^{\mathfrak{R}}$ is Ϭ-reduced;
2. The following statements are true: For each $m \in \widehat{\mathcal{M}}^{\mathfrak{R}}$ and $\hat{r} \in \mathfrak{R}$,

a. $m\hat{r} = 0 \to m\mathfrak{R}\hat{r} = m\mathfrak{R}Ϭ(\hat{r}) = 0$;

b. $m\hat{r}Ϭ(\hat{r}) = 0 \to m\hat{r} = 0$;

c. $m\hat{r}^2 = 0 \to m\hat{r} = 0$.

If the module $\widehat{\mathcal{M}}^{\mathfrak{R}}$ is 1-red-module, it is referred to as reduced. Hence, a ring $\mathfrak{R}$ is a red-ring if and only if $\mathfrak{R}$ is is 1-red-module as an $\mathfrak{R}$-module $\widehat{\mathcal{M}}^{\mathfrak{R}}$.

Proposition 5.9 Every Ϭ-reduced module is a R-Ϭ-$\mathcal{N}^{\mathcal{L}}\mathcal{S}$-module.

Proof. Consider $m \in \widehat{\mathcal{M}}^{\mathfrak{R}}$ and $ŭ, ρ́ \in \mathcal{N}^{\mathcal{L}}(\mathfrak{R})$ with $mŭρ́ = 0$, we prove $mρ́Ϭ(ŭ) = 0$. We apply conditions of Ϭ-reduced module in the process. Now $0 = mŭρ́ = mŭϬ(ρ́) = 0$. Then,

$mϬ(ρ́)ŭϬ(ρ́)ŭ = m(Ϭ(ρ́)ŭ)Ϭ(Ϭ(ρ́)ŭ) = mϬ(ρ́)ŭ = mϬ(ρ́)Ϭ(Ϭ(ŭ)) = mϬ(ρ́Ϭ(ŭ)) = mρ́Ϭ(ŭ)$. Hence $\widehat{\mathcal{M}}^{\mathfrak{R}}$ is a R-Ϭ-$\mathcal{N}^{\mathcal{L}}\mathcal{S}$-module. ■

The following illustration shows that, in general, Proposition 5.9's converse is not true.

Example 5.10 Consider $\bar{\mathcal{Z}}_4$ denote the ring of integer modulo 4. Let the ring $\mathfrak{R} = \left\{\begin{pmatrix} ŭ & ρ́ \\ 0 & ŭ \end{pmatrix}; ŭ, ρ́ \in \bar{\mathcal{Z}}_4\right\}$ and the $\mathfrak{R}$-module $\widehat{\mathcal{M}}^{\mathfrak{R}} = \left\{\begin{pmatrix} 0 & q \\ ɥ & ƅ \end{pmatrix}; q, ɥ, ƅ \in \bar{\mathcal{Z}}_4\right\}$ and a homomorphism $Ϭ: \mathfrak{R} \to \mathfrak{R}$ is defined by $Ϭ\left(\begin{pmatrix} ŭ & ρ́ \\ 0 & ŭ \end{pmatrix}\right) = \begin{pmatrix} ŭ & -ρ́ \\ 0 & ŭ \end{pmatrix}$. $\widehat{\mathcal{M}}^{\mathfrak{R}}$ is R-Ϭ-$\mathcal{N}^{\mathcal{L}}\mathcal{S}$-module but not Ϭ-reduced.

For, if $m = \begin{pmatrix} 0 & 0 \\ 2 & 1 \end{pmatrix} \in \widehat{\mathcal{M}}^{\mathfrak{R}}$ and $\hat{r} = \begin{pmatrix} 2 & 3 \\ 0 & 2 \end{pmatrix} \in \mathfrak{R}$. Then $m\hat{r} = 0$ but $\begin{pmatrix} 0 & 0 \\ 0 & 2 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 2 & 1 \end{pmatrix}\begin{pmatrix} 2 & 0 \\ 0 & 2 \end{pmatrix} = \begin{pmatrix} 0 & 0 \\ 2 & 0 \end{pmatrix}\begin{pmatrix} 2 & 3 \\ 0 & 2 \end{pmatrix} \in m\mathfrak{R} \cap \widehat{\mathcal{M}}\hat{r} \neq 0$. Hence $\widehat{\mathcal{M}}^{\mathfrak{R}}$ is not Ϭ-reduced.

Proposition 5.11 For a ring $\mathfrak{R}$ and $\mathfrak{R}$-module $\widehat{\mathcal{M}}^{\mathfrak{R}}$. Then the following conditions are equivalent,

i. $\widehat{\mathcal{M}}^{\mathfrak{R}}$ is R-Ϭ-$\mathcal{N}^{\mathcal{L}}\mathcal{S}$-module.

ii. Each submodule of $\widehat{\mathcal{M}}^{\mathfrak{R}}$ is R-Ϭ-$\mathcal{N}^{\mathcal{L}}\mathcal{S}$-module.

iii. Each finitely generated submodule of $\widehat{\mathcal{M}}^{\mathfrak{R}}$ is Ϭ-$\mathcal{N}^{\mathcal{L}}\mathcal{S}$-module.

iv. Each cyclic submodule of $\widehat{\mathcal{M}}^{\mathfrak{R}}$ is R-Ϭ-$\mathcal{N}^{\mathcal{L}}\mathcal{S}$-module.

Proof. It is a direct result of definitions and Proposition 3.6.

Theorem 5.12 Every flat module over an R-Ϭ-$\mathcal{N}^{\mathcal{L}}\mathcal{S}$-ring is an R-Ϭ-$\mathcal{N}^{\mathcal{L}}\mathcal{S}$-module.

Proof. Assume $\widehat{\mathcal{M}}^{\mathfrak{R}}$ be a flat module over the R-Ϭ-$\mathcal{N}^{\mathcal{L}}\mathcal{S}$-ring $\mathfrak{R}$ and $0 \to Ƙ \to Ƒ \to \widehat{\mathcal{M}}^{\mathfrak{R}} \to 0$ a short exact sequence with Ƒ free $\mathfrak{R}$-module. By [(Lee & Zhou, 2004), Theorem 2.3] is a R-Ϭ-$\mathcal{N}^{\mathcal{L}}\mathcal{S}$-module and we write $\widehat{\mathcal{M}}^{\mathfrak{R}} = Ƒ/Ƙ$ and any element $\bar{y} = y + Ƙ \in \widehat{\mathcal{M}}^{\mathfrak{R}}$ for $y \in Ƒ$. Let $\bar{y}\hat{a}\hat{b} = 0$ where $\bar{y} \in \widehat{\mathcal{M}}^{\mathfrak{R}}$ and $\hat{a}, \hat{b} \in \mathcal{N}^{\mathcal{L}}(\mathfrak{R})$. Since $\widehat{\mathcal{M}}^{\mathfrak{R}}$ is flat there exists a homomorphism $\hat{\star}: Ƒ \to Ƙ$ such that $\hat{\star}(y\hat{a}\hat{b}) = y\hat{a}\hat{b}$ Now set $u = \hat{\star}(y) - y \in Ƒ$. Then $u\hat{a}\hat{b} = 0$. Since Ƒ is R-Ϭ-$\mathcal{N}^{\mathcal{L}}\mathcal{S}$-module, $u\hat{b}Ϭ(\hat{a}) = 0$. Then $\hat{\star}\left(y\hat{b}Ϭ(\hat{a})\right) = y\hat{b}Ϭ(\hat{a})$. Since $\hat{\star}(y) \in Ƙ$, we have $y\hat{b}Ϭ(\hat{a}) \in Ƙ$. Therefore $\bar{y}\hat{b}Ϭ(\hat{a}) = 0$. Therefore $\widehat{\mathcal{M}}^{\mathfrak{R}}$ is R-Ϭ-$\mathcal{N}^{\mathcal{L}}\mathcal{S}$-module. ■

Proposition 5.13 Assume $\mathfrak{R}, ɠ$ are rings and $\vartheta: \mathfrak{R} \to ɠ$ be a ring *endo*. If $\widehat{\mathcal{M}}^{ɠ}$ is a right $\mathfrak{R}$-module, then $\widehat{\mathcal{M}}^{\mathfrak{R}}$is a right $\mathfrak{R}$-module via $mr = m\vartheta(r)$ for all $r \in \mathfrak{R}$ and $m \in \widehat{\mathcal{M}}^{\mathfrak{R}}$. Moreover, $\widehat{\mathcal{M}}^{\mathfrak{R}}$ is R-Ϭ-$\mathcal{N}^{\mathcal{L}}\mathcal{S}$-module, if and only if $\widehat{\mathcal{M}}^{ɠ}$ is R-Ϭ-$\mathcal{N}^{\mathcal{L}}\mathcal{S}$-module.

Proof. Let $\widehat{\mathcal{M}}^{ɠ}$ be an R-Ϭ-$\mathcal{N}^{\mathcal{L}}\mathcal{S}$-module. Consider $\hat{a}, \hat{b} \in \mathcal{N}^{\mathcal{L}}(\mathfrak{R})$ and $m \in \widehat{\mathcal{M}}^{\mathfrak{R}}$ Such that $m\hat{a}\hat{b} = 0$ Then $m\vartheta(\hat{a}\hat{b}) = m\vartheta(\hat{a})\vartheta(\hat{b}) = 0$. Since $\widehat{\mathcal{M}}^{ɠ}$ is R-Ϭ-$\mathcal{N}^{\mathcal{L}}\mathcal{S}$-module, we have,

$$m\vartheta(\hat{b})Ϭ(\vartheta(\hat{a})) = 0,$$
$$m\vartheta(\hat{b})\vartheta(\hat{a}) = 0,$$
$$m\vartheta(\hat{b}Ϭ(\hat{a})) = 0.$$

Hence $\widehat{\mathcal{M}}^{\mathfrak{R}}$ is a Ϭ-$\mathcal{N}^{\mathcal{L}}\mathcal{S}$-module.

Conversely. Assume that ϑ is onto and $\widehat{\mathcal{M}}^{\mathfrak{R}}$ is a R-Ϭ-$\mathcal{N}^{\mathcal{L}}\mathcal{S}$-module. Let $ŭ, ρ́ \in \mathcal{N}^{\mathcal{L}}(ɠ)$ and $m \in \widehat{\mathcal{M}}^{ɠ}$ such that $mŭρ́ = 0$. Since ϑ is onto, there exists $\hat{a}, \hat{b} \in \mathcal{N}^{\mathcal{L}}(\mathfrak{R})$ such that $ŭ = \vartheta(\hat{a})$ and $ρ́ = \vartheta(\hat{b})$. Then $0 = m\vartheta(\hat{a})\vartheta(\hat{b}) = m\vartheta(\hat{a}\hat{b}) = m\hat{a}\hat{b}$. Since $\widehat{\mathcal{M}}^{\mathfrak{R}}$ is right Ϭ-$\mathcal{N}^{\mathcal{L}}\mathcal{S}$-module, we have $0 = m\hat{b}Ϭ(\hat{a})$. Hence $0 = m\vartheta(\hat{b}Ϭ(\hat{a})) = 0 = m\vartheta(\hat{b})Ϭ(\vartheta(\hat{a})) = mρ́Ϭ(ŭ)$. Thus $\widehat{\mathcal{M}}^{ɠ}$ is R-Ϭ-$\mathcal{N}^{\mathcal{L}}\mathcal{S}$-module. ■

Now we study the $\mathcal{N}^{\mathcal{L}}$-symmetric property on some module extensions and module localizations like $\widehat{\mathcal{M}}[ɴ], \widehat{\mathcal{M}}[ɴ, ɴ^{-1}], \widehat{\mathcal{M}}[ɴ, ɴ^{-1}; Ϭ]$.

The following concepts were introduced by Lee and Zhou. For a module $\widehat{\mathcal{M}}$, We examine $\widehat{\mathcal{M}}[ɴ] = \{\sum_{i=0}^{s} m_i ɴ^i : s \geq 0, m_i \in \widehat{\mathcal{M}}\}$, $\widehat{\mathcal{M}}[ɴ]$ is an Abelian group under clearly addition operation. Additionally, the next, scalar product operation turns $\widehat{\mathcal{M}}[ɴ]$ into a right $\mathfrak{R}[ɴ]$-module:

For $m(ɴ) = \sum_{\sigma=0}^{s} m_\sigma ɴ^\sigma \in \widehat{\mathcal{M}}[ɴ]$ and $f(ɴ) = \sum_{\sigma=0}^{t} a_\sigma ɴ^\sigma \in \mathfrak{R}[ɴ]$,

$$m(ɴ)f(ɴ) = \sum_{d=0}^{3+\epsilon}\left(\sum_{\sigma+\dot{\sigma}=d} m_\sigma a_\sigma\right)x^d.$$

$\widehat{\mathcal{M}}[и]$ becomes a right module over $\mathfrak{R}[и]$ as a result of these operations. In the same way, the Laurent polynomial extension $\widehat{\mathcal{M}}[и, и^{-1}]$ becomes a right module over $\mathfrak{R}[и, и^{-1}]$ with a similar scalar product. Zhou and Lee (Lee & Zhou, 2004) also introduced notations for $\widehat{\mathcal{M}}$ module as,

$\widehat{\mathcal{M}}[и; б] = \left\{\sum_{\sigma=0}^{p} m_\sigma и^\sigma \mid p \geq 0, m_\sigma \in \widehat{\mathcal{M}}\right\}$. Each of the above is abelian group underneath the addition condition. Furthermore, $\widehat{\mathcal{M}}[и; б]$ is a module for $\mathfrak{R}[и; б]$ under the product operation as:

$$m(и) = \sum_{\sigma=0}^{\mu} m_\sigma и^\sigma \in \widehat{\mathcal{M}}[и; б],$$

$$f(и) = \sum_{\hat{o}=0}^{\sigma} f_{\hat{o}} и^{\hat{o}} \in \mathfrak{R}[и; б]$$

$$m(и)f(и) = \sum_{d=0}^{\mu+\sigma} \left(\sum_{\sigma+v=d} m_\sigma \, \alpha^\sigma(f_\sigma) \right) и^d$$

In the same way, the skew Laurent polynomial module $\widehat{\mathcal{M}}[и, и^{-1}; б]$ transforms into a module on $\mathfrak{R}[и, и^{-1}; б]$.

Again, from (Lee & Zhou, 2004), module $\widehat{\mathcal{M}}$ is known as б-Armendariz if the below conditions holds: (i) For $m \in \widehat{\mathcal{M}}$ and $a \in \mathfrak{R}, ma = 0$ for the case if $mб(a) = 0$ (ii) any $m(и) = \sum_{\sigma=0}^{t} m_\sigma и^\sigma \in \widehat{\mathcal{M}}[и; б]$ and $f(и) = \sum_{\hat{o}=0}^{n} a_{\hat{o}} и^{\hat{o}} \in \mathfrak{R}[и; б], m(и)f(и) = 0$ imply $m_\sigma б^\sigma(a_{\hat{o}}) = 0$ for all σ and $\hat{o}$. And then, Anderson and Camillo (Anderson & Camillo, 1999), extended the concept of Armendariz ring to Armendariz module, as follows: A $\mathfrak{R}$-module $\widehat{\mathcal{M}}^{\mathfrak{R}}$ is Armendariz when, if $m(и) = \sum_{\sigma=0}^{\mu} m_\sigma и^\sigma \in \widehat{\mathcal{M}}[и]$ and $g(и) = \sum_{\hat{o}=0}^{\sigma} a_{\hat{o}} и^{\hat{o}} \in \mathfrak{R}[и]$, such that $m(и)g(и) = 0$ implies $m_\sigma \, a_v = 0$ for all σ and $\hat{o}$. The Armendariz property is applicable for any finite product of polynomials. Clearly, $\mathfrak{R}$ is an Armendariz ring if and only if ${}_{\mathfrak{R}}\mathfrak{R}$ is an Armendariz $\mathfrak{R}$-module.

Theorem 5.14 Consider $\widehat{\mathcal{M}}^{\mathfrak{R}}$ is a б-Armendariz module. Then, the statements that follow are equivalent:

1. $\widehat{\mathcal{M}}^{\mathfrak{R}}$ is R-б-$\mathcal{N}^{\mathcal{L}}\mathcal{S}$-module;
2. $\widehat{\mathcal{M}}[и; б]^{\mathfrak{R}[и;б]}$ is R-б-$\mathcal{N}^{\mathcal{L}}\mathcal{S}$-module;
3. $\widehat{\mathcal{M}}[и, и^{-1}; б]^{\mathfrak{R}[и,и^{-1};б]}$ is R-б-$\mathcal{N}^{\mathcal{L}}\mathcal{S}$-module.

Proof. It suffices to demonstrate that $1 \Rightarrow 3$. Let $m(и) = \sum_{\sigma=0}^{\infty} m_\sigma и^\sigma \in \widehat{\mathcal{M}}[и, и^{-1}; б]^{\mathfrak{R}[и,и^{-1};б]}$ and $\mathfrak{A}(и) = \sum_{\hat{o}=0}^{\infty} a_{\hat{o}} и^{\hat{o}}, \mathfrak{B}(и) = \sum_{q=0}^{\infty} m_q и^q \in \mathcal{N}^{\mathcal{L}}(\mathfrak{R}[и, и^{-1}; б])$. Then we obtain $a_{\hat{o}}, b_q \in \mathcal{N}^{\mathcal{L}}(\mathfrak{R})$. Let $m(и)\mathfrak{A}(и)\,\mathfrak{B}(и) = 0$ this implies $m_\sigma a_{\hat{o}} b_q = 0$ for all $\sigma, \hat{o}, q$. Thus, by hypothesis $m_\sigma b_q a_{\hat{o}} = 0$. Therefore $m(и)\mathfrak{B}(и)\,\mathfrak{A}(и) = 0$, and so $\widehat{\mathcal{M}}[и, и^{-1}; б]^{\mathfrak{R}[и,и^{-1};б]}$ is a R-б-$\mathcal{N}^{\mathcal{L}}\mathcal{S}$-module. ■

Corollary 5.15 Consider $\widehat{\mathcal{M}}^{\mathfrak{R}}$ be an Armendariz module. Then the following are equivalent:

1. $\widehat{\mathcal{M}}^{\mathfrak{R}}$ is R-б-$\mathcal{N}^{\mathcal{L}}\mathcal{S}$-module;
2. $\widehat{\mathcal{M}}[и]^{\mathfrak{R}[и]}$ is R-б-$\mathcal{N}^{\mathcal{L}}\mathcal{S}$-module;
3. $\widehat{\mathcal{M}}[и, и^{-1}]^{\mathfrak{R}[и,и^{-1}]}$ is R-б-$\mathcal{N}^{\mathcal{L}}\mathcal{S}$-module.

Proposition 5.16 Consider б is an *endo* of a ring $\mathfrak{R}$ and $\widehat{\mathcal{M}}^{\mathfrak{R}}$is б-reduced module. Then $\widehat{\mathcal{M}}^{\mathfrak{R}}$is R-б-$\mathcal{N}^{\mathcal{L}}\mathcal{S}$-module over $\mathfrak{R}$ if and only if $\widehat{\mathcal{M}}^{\mathfrak{R}}[и]/\widehat{\mathcal{M}}^{\mathfrak{R}}[и](и^n)$ is R-б-$\mathcal{N}^{\mathcal{L}}\mathcal{S}$-module over $\frac{\mathfrak{R}[и]}{<и^n>}$ for integer $n \geq 2$.

Proof. Let $\widehat{\mathcal{M}}^{\mathfrak{R}}$ is right б-$\mathcal{N}^{\mathcal{L}}\mathcal{S}$-module with $pqh = 0$, where $\bar{и} = и + <и^n>$. Note that $a_\sigma b_{\hat{o}} c_q \bar{и}^{\,i+j+k} = 0$, for each $\sigma, \hat{o}$ and q with $\sigma + \hat{o} + q \geq n$. Therefore, it is sufficient to display the cases $\sigma + \hat{o} + q \leq n - 1$. Since $pqh = 0$, The following equations are available to us:

(1) $m_0 s_0 t_0 = 0,$

(2) $m_0 s_0 t_1 + m_0 s_1 t_0 + m_1 s_0 t_0 = 0,$

(3) $m_0 s_0 t_2 + m_0 s_1 t_1 + m_0 s_2 t_0 + m_1 s_0 t_1 + m_1 s_1 t_0 + m_2 s_0 t_0 = 0,$

⋮

$(n-2)$ $m_0 s_0 t_{n-2} + m_0 s_1 t_{n-3} + \cdots + m_{n-3} s_1 t_0 + m_{n-2} s_0 t_0 = 0,$

$(n-1)$ $m_0 s_0 t_{n-1} + m_0 s_1 t_{n-2} + \cdots + m_{n-2} s_0 t_1 + m_{n-2} s_1 t_0 + m_{n-1} s_0 t_0 = 0.$

Since $\widehat{\mathcal{M}}^{\mathfrak{R}}$ is б-reduced for any $m \in \widehat{\mathcal{M}}^{\mathfrak{R}}, a \in \mathfrak{R}, ma^2 = 0 \to ma = 0$, and each б-reduced module is semi-commutative**.** These facts are used as follows:

Eq(1) and Eq(2) $\times s_0 t_0$ gives $m_1 (s_0 t_0)^2 = 0$, and so $m_1 s_0 t_0 = 0$ and $m_0 s_0 t_1 + m_0 s_1 t_0 = 0$, multiplying by $s_1 t_0$ gives $0 = m_0 s_1 ({t_0}^2) = m_0 s_1 t_0$, so we have, $m_0 s_0 t_1 = 0, m_0 s_1 t_0 = 0$ and $m_1 s_0 t_0 = 0$. From Eq(1),(2) and (3) $\times s_0 t_0$, we get $m_2 s_0 t_0 = 0$ and,

$m_0 s_0 t_2 + m_0 s_1 t_1 + m_0 s_2 t_0 + m_1 s_0 t_1 + m_1 s_1 t_0 = 0$, in a similar way. If we multiply the right side of Eq(3) by $s_1 t_0, s_0 t_1, s_2 t_0$ and $s_1 t_1$ respectively, then we obtain $m_1 s_1 t_0 = 0, m_1 s_0 t_1 = 0, m_0 s_2 t_0 = 0, m_0 s_1 t_1 = 0,$ and $m_0 s_0 t_2 = 0$ in turn Inductively we assume that $m_\sigma s_v t_k = 0$ where $\sigma + \hat{o} + q = 0, 1, \ldots, (n-2)$. We apply the above method to Eq. $(n-1)$. First, the induction hypotheses and Eq. $(n-1)$ $\times s_0 t_0$ give $m_{n-1} s_0 t_0 = 0$ and,

$$(n-1)\;\; m_0 s_0 t_{n-1} + m_0 s_1 t_{n-2} + \cdots + m_{n-2} s_0 t_1 + m_{n-2} s_1 t_0 + m_{n-1} s_0 t_0 = 0.$$

If we multiply Eq. $(n-1)$ on the right side by $s_1 t_0, s_0 t_1, \ldots,$ and $s_1 t_{n-2}$ respectively, then we obtain $m_{n-2} s_1 t_0 = 0, m_{n-2} s_0 t_1 = 0, \ldots, m_0 s_1 t_{n-2} = 0$ and so $m_0 s_0 t_{n-1} = 0$. In turn. This shows that $m_\sigma s_{\hat{o}} t_q = 0$ for all $\sigma, \hat{o}$ and q with $\sigma + \hat{o} + q = n - 1$. Consequently, $m_\sigma s_{\hat{o}} t_q = 0$ for all $\sigma, \hat{o}$ and q with $\sigma + \hat{o} \leq n - 1$, and thus $m_\sigma t_q б^\sigma(s_{\hat{o}}) = 0, \forall \sigma \in Z^+$ by [(Kwak, 2007), Theorem 2.5(1)]. This yields $ph\bar{б}(q) = 0$, and therefore $\widehat{\mathcal{M}}^{\mathfrak{R}}[и]/\widehat{\mathcal{M}}^{\mathfrak{R}}[и](и^n)$ is R-б-$\mathcal{N}^{\mathcal{L}}\mathcal{S}$-module. ■

If $ur = 0$ implies $r = 0$ for $r \in \mathfrak{R}$, then an element u of a ring $\mathfrak{R}$ is right regular. Regular indicates that it is both left and right regular (and so not a zero divisor), while left regular is defined similarly. Assume that $\widehat{\mathcal{M}}$ is a subset of $\mathfrak{R}$ that is multiplicatively closed and made up of central regular elements. Let б be an automorphism of $\mathfrak{R}$ and consider $б(m) = m, \forall m \in \widehat{\mathcal{M}}$. Then $б(m^{-1}) = m^{-1}$ in $\widehat{\mathcal{M}}^{-1}\mathfrak{R}$ and the induced map $б: \widehat{\mathcal{M}}^{-1}\mathfrak{R} \to \widehat{\mathcal{M}}^{-1}\mathfrak{R}$ defined by $б(u^{-1}a) = u^{-1}б(a)$ is also an automorphism.

Proposition 5.17 Consider a ring $\mathfrak{R}$ and a subset Ω of $\mathfrak{R}$ that is multiplicatively closed and consists of central regular elements. Then

(1) $\mathfrak{R}$ is a R-б-$\mathcal{N}^{\mathcal{L}}\mathcal{S}$-ring if and only if is $\Omega^{-1}\mathfrak{R}$ is a R-б-$\mathcal{N}^{\mathcal{L}}\mathcal{S}$-ring.

(2) A module $\widehat{\mathcal{M}}^{\mathfrak{R}}$is R-б-$\mathcal{N}^{\mathcal{L}}\mathcal{S}$-module if and only if $\Omega^{-1}\widehat{\mathcal{M}}^{\Omega^{-1}\mathfrak{R}}$ is a R-б-$\mathcal{N}^{\mathcal{L}}\mathcal{S}$-module.

Proof.(1) Assume $ʒ\mathfrak{т}\kappa = 0$ with $ʒ = \hat{u}^{-1}\hat{a}, \mathfrak{т} = \hat{v}^{-1}\hat{b}, \kappa = \hat{w}^{-1}\hat{c}$, $\hat{u}, \hat{v}, \hat{w} \in \Omega$ and $\hat{a} \in R^\circ$, $\hat{b}, \hat{c} \in \mathcal{N}^{\mathcal{L}}(\mathfrak{R})$. Since Ω is included in the centre of $\mathfrak{R}$,

we have $0 = ʒ\mathfrak{т}\kappa = \hat{u}^{-1}\hat{a}\hat{v}^{-1}\hat{b}\hat{w}^{-1}\hat{c} = (\hat{u}^{-1}\hat{v}^{-1}\hat{w}^{-1})\hat{a}\hat{b}\hat{c} = (\hat{u}\hat{v}\hat{w})^{-1}\hat{a}\hat{b}\hat{c}$ and so $s\hat{a}\hat{b}\hat{c} = 0$ for some $s \in \Omega$. But $\mathfrak{R}$ is R-б-$\mathcal{N}^{\mathcal{L}}\mathcal{S}$-ring by the condition, so $s\hat{a}\hat{c}б(\hat{b}) = 0$ and $sʒ\kappa б(\mathfrak{т}) = s(\hat{u}^{-1}\hat{a})(\hat{w}^{-1}\hat{c})б\left((\hat{v}^{-1}\hat{b})\right) = s(\hat{u}\hat{w}\hat{v})^{-1}\hat{a}\hat{c}б(\hat{b}) = 0.$

Hence $\Omega^{-1}\mathfrak{R}$ is R-б-$\mathcal{N}^{\mathcal{L}}\mathcal{S}$-ring.

(2) Since a submodule of a R-б-$\mathcal{N}^{\mathcal{L}}\mathcal{S}$-module is likewise a R-б-$\mathcal{N}^{\mathcal{L}}\mathcal{S}$-module, it is sufficient to verify the required condition. Assume that $\widehat{\mathcal{M}}^{\mathfrak{R}}$is a R-б-$\mathcal{N}^{\mathcal{L}}\mathcal{S}$-module and $(q^{-1}\mathfrak{m})(\mu^{-1}\breve{u})(\sigma^{-1}\acute{\rho}) = 0$ for $q^{-1}\mathfrak{m} \in \Omega^{-1}\widehat{\mathcal{M}}^{\Omega^{-1}\mathfrak{R}}$ and $\mu^{-1}\breve{u}, \sigma^{-1}\acute{\rho} \in \mathcal{N}^{\mathcal{L}}(\Omega^{-1}\mathfrak{R})$ where $\mathfrak{m} \in \widehat{\mathcal{M}}^{\mathfrak{R}}$, $\breve{u}, \acute{\rho} \in \mathcal{N}^{\mathcal{L}}(\mathfrak{R})$. Since Ω is included in the centre of $\mathfrak{R}$, we have $0 = (q^{-1}\mathfrak{m})(\mu^{-1}\breve{u})(\sigma^{-1}\acute{\rho}) = (\hat{s}\hat{t}\hat{r})^{-1}\mathfrak{m}\breve{u}\acute{\rho}$ and so $0 = \mathfrak{m}\breve{u}\acute{\rho}$. By

assumption $ɱ\acute{\rho}ϭ(\breve{\upsilon}) = O$. Therefore $(ɋ^{-1}ɱ)(\sigma^{-1}\acute{\rho})ϭ(\mu^{-1}\breve{\upsilon}) = (ɋ^{-1}ɱ)(\sigma^{-1}\acute{\rho})\left(\mu^{-1}ϭ(\breve{\upsilon})\right) = O$. Hence $\Omega^{-1}\widehat{\mathcal{M}}^{\Omega^{-1}\widehat{\Re}}$ is a R-ϭ-$\mathcal{N}^{\mathcal{L}}\mathcal{S}$-module. ∎

Corollary 5.18 (1) For a ring $\widehat{\Re}$, $\widehat{\Re}[ʞ]$ is R-ϭ-$\mathcal{N}^{\mathcal{L}}\mathcal{S}$-ring if and only if is $\widehat{\Re}[ʞ; ʞ^{-1}]$ a R-ϭ-$\mathcal{N}^{\mathcal{L}}\mathcal{S}$-ring**.**

(2) For a $\widehat{\Re}$-module $\widehat{\mathcal{M}}^{\widehat{\Re}}$, $\widehat{\mathcal{M}}[ʞ]^{\widehat{\Re}[x]}$ is R-ϭ-$\mathcal{N}^{\mathcal{L}}\mathcal{S}$-module if and only if $\widehat{\mathcal{M}}[x, x^{-1}]^{\widehat{\Re}[x,x^{-1}]}$ is a R-ϭ-$\mathcal{N}^{\mathcal{L}}\mathcal{S}$-module.

Proof (1). Consider $\Omega = \{1, ʞ, ʞ^2, \cdots\}$. Then clearly Ω is a multiplicatively closed subset of $\widehat{\Re}[ʞ]$. Since $\widehat{\Re}[ʞ; ʞ^{-1}] = \Omega^{-1}\widehat{\Re}[ʞ]$, it follows that $\widehat{\Re}[ʞ; ʞ^{-1}]$ is right ϭ-$\mathcal{N}^{\mathcal{L}}\mathcal{S}$-ring by proposition 5.17(1).

(2) It is evident from proposition 5.17(2). if $\Omega = \{1, ʞ, ʞ^2, \ldots\}$. Then Ω is a multiplicatively closed subset of $\widehat{\Re}[ʞ]$ consisting of regular central element of $\widehat{\Re}[ʞ]$. Since $\Omega^{-1}\widehat{\mathcal{M}}[ʞ]^{\widehat{\Re}[ʞ]} = \widehat{\mathcal{M}}[ʞ, ʞ^{-1}]^{\widehat{\Re}[ʞ,ʞ^{-1}]}$ and $\Omega^{-1}\widehat{\Re}[ʞ] = \widehat{\Re}[ʞ; ʞ^{-1}]$. ∎

$\bar{Q}(\widehat{\Re})$ is *a classical right quotient* for $\widehat{\Re}$ if every regular element of $\widehat{\Re}$ is *invertible* in $\bar{Q}$ and every element of $\bar{Q}$ can be written in the form ab^{-1} with $a, b \in \widehat{\Re}$ and b regular.

A right *Ore* ring is a ring $\widehat{\Re}$ where, for any $a, b \in \widehat{\Re}$ with b being regular, $\exists a_1, b_1 \in \widehat{\Re}$ with b_1 also regular, such that $ab_1 = ba_1$. It is well known that $\widehat{\Re}$ is a right *ore ring* if and only if its classical right quotient ring $\bar{Q}(\widehat{\Re})$ exists. Now, suppose $\widehat{\Re}$ is a ring with the classical right quotient ring $\bar{Q}(\widehat{\Re})$. Then any automorphism ϭ of $\widehat{\Re}$ extends to $\bar{Q}(\widehat{\Re})$ by defining its action on fractions as $ϭ(ab^{-1}) = ϭ(a)(ϭ(b))^{-1}$ for all $a, b \in \widehat{\Re}$, provided that $ϭ(b)$ remains regular whenever b is a regular element in $\widehat{\Re}$.

Theorem 5.19 Consider $\widehat{\Re}$ is *Ore ring* with an *endo* ϭ of $\widehat{\Re}$ and $\bar{Q}(\widehat{\Re})$ *is* the classical right quotient ring $\mathcal{NJ}$ ring of $\widehat{\Re}$. Then

(1) $\widehat{\Re}$ is a R-ϭ-$\mathcal{N}^{\mathcal{L}}\mathcal{S}$-ring if and only if $\bar{Q}(\widehat{\Re})$ is a R-ϭ-$\mathcal{N}^{\mathcal{L}}\mathcal{S}$-ring.

(2) $\widehat{\mathcal{M}}^{\widehat{\Re}}$ is a R-ϭ-$\mathcal{N}^{\mathcal{L}}\mathcal{S}$-module if and only if $\bar{Q}(\widehat{\mathcal{M}})$ is a R-ϭ-$\mathcal{N}^{\mathcal{L}}\mathcal{S}$-module.

Proof. (1) Consider $\widehat{\Re}$ is a R-ϭ-$\mathcal{N}^{\mathcal{L}}\mathcal{S}$-ring. Assume $A = a\mu^{-1} \in \bar{Q}(\widehat{\Re})$ and $B = bv^{-1}, C = c\omega^{-1} \in \mathcal{N}^{\mathcal{L}}(\bar{Q}(\widehat{\Re}))$with $ABC = a\mu^{-1}\, bv^{-1}c\omega^{-1}$ where $a, \mu \in \widehat{\Re}$ and $b, v, c, \omega \in \mathcal{N}^{\mathcal{L}}(\widehat{\Re})$ with μ, v, ω regular. Let $\bar{Q}(\widehat{\Re})$be an $\mathcal{NJ}$ ring Then $\widehat{\Re}$ is $\mathcal{NJ}$ and so $b, c \in \mathcal{N}^{\mathcal{L}}(\widehat{\Re})$. $\exists\, c_1, b_1 \in \widehat{\Re}$ with b_1 regular such that $bc_1 = cb_1$ and $c_1b_1^{-1} = b^{-1}c$. Now $\exists\, \mu_1, b_1 \in \widehat{\Re}$ with μ_1 regular such that $b\mu_1 = \mu b_1$, $\mu^{-1}b = b_1\mu_1^{-1}$. Hence $ABC = a\mu^{-1}bv^{-1}c\omega^{-1} = ab_1\mu_1^{-1}v^{-1}c\omega^{-1} = O$. Let I and J be the ideals in $\bar{Q}(\widehat{\Re})$, generated by B and C within $\mathcal{N}^{\mathcal{L}}(\bar{Q}(\widehat{\Re}))$, respectively. Then each of I and J are $\mathcal{N}^{\mathcal{L}}$ with $b = Bv \in I, c = C\omega \in J$, Since $\widehat{\Re}$ is right Ore, for $c, v \in \mathcal{N}^{\mathcal{L}}(\widehat{\Re})$ $\exists\, c_1, v_1 \in \mathcal{N}^{\mathcal{L}}(\widehat{\Re})$ with v_1 regular such that $cv_1 = vc_1$, $v^{-1}c = c_1v_1^{-1}$ .Here note that $c_1 \in \mathcal{N}^{\mathcal{L}}(\widehat{\Re})$. Indeed, $vc_1 = cv_1 \in J$ and so $c_1 = v^{-1}(vc_1) \in J$. So $ABC = ab_1\mu_1^{-1}c_1v_1^{-1}\,\omega^{-1} = O$.

Similarly, also there exists $c_2 \in \mathcal{N}^{\mathcal{L}}(\widehat{\Re})$ and $\mu_2 \in \widehat{\Re}$ with μ_2 regular such that $c_1\mu_2 = \mu_1c_2, \mu_1^{-1}c_1 = c_2\mu_2^{-1}$, Thus, we obtain that $ABC = ab_1c_2\,\mu_2^{-1}v_1^{-1}\omega^{-1} = O$ and hence $ab_1c_2 = O$. This implies $O = ab_1c_2\mu = a\mu b_1c_2 = ab\mu_1c_2 = abc_2\mu_1$, and $O = abc_2 = abc_2\mu_1 = ab\mu_1c_2 = abc_1\mu_1$. So we have $O = abc_1 = abc_1v = abvc_1 = abcv$. It follows that $acϭ(b) = O$, since $\widehat{\Re}$ is a R-ϭ-$\mathcal{N}^{\mathcal{L}}\mathcal{S}$-ring.

Similar, there exists $c_3, b_2, \omega_2, b_4 \in \mathcal{N}^{\mathcal{L}}(\widehat{\Re})$ and $\mu_3, \mu_4 \in \widehat{\Re}$ with μ_3, ω_2, μ_4 regular such that $c\mu_3 = \mu c_3$, $\mu^{-1}c = c_3\mu_3^{-1}$, $b\omega_2 = \omega b_2, \omega^{-1}b = b_2\omega_2{}^{-1}, b_2\mu_4 = \mu_3b_4$, and, $ACϭ(B) = ac_3\mu_3^{-1}\omega^{-1}ϭ(b)ϭ(v)^{-1} = ac_3\mu_3^{-1}b_2\omega_2^{-1}ϭ(v)^{-1} = ac_3b_4\mu_4^{-1}\omega_2^{-1}ϭ(v)^{-1}$. Form $acϭ(b) = O$. We have $O = acϭ(b)\omega_2 = ac\omega b_2 = acb_2\omega$, and hence $O = acb_2 = acb_2\mu_4 = ac\mu_3b_4 = acb_4\mu_3$. It follows that, $O = acb_4 = acb_4\mu_3 = ac\mu_3b_4 = a\mu c_3b_4 = ac_3b_4\mu$, and hence $ac_3b_4 = O$. Now we have $ACϭ(B) = O$, therefore $\bar{Q}(\widehat{\Re})$ is a R-ϭ-$\mathcal{N}^{\mathcal{L}}\mathcal{S}$-ring.

(2) Assume that $\widehat{\mathcal{M}}^{\widehat{\Re}}$ is a R-ϭ-$\mathcal{N}^{\mathcal{L}}\mathcal{S}$-module. Let $A = a\mu^{-1} \in \bar{Q}(\widehat{\mathcal{M}})$ and $B = bv^{-1}, C = c\omega^{-1} \in \mathcal{N}^{\mathcal{L}}(\bar{Q}(\widehat{\Re}))$ with $ABC = a\mu^{-1}\, bv^{-1}c\omega^{-1}$ where $a, \mu \in \widehat{\mathcal{M}}^{\widehat{\Re}}$ and $b, v, c, \omega \in \mathcal{N}^{\mathcal{L}}(\widehat{\Re})$ with μ, v, ω regular. Let $\bar{Q}(\widehat{\Re})$ be an $\mathcal{NJ}$ ring Then $\widehat{\Re}$ is $\mathcal{NJ}$ and so $b, c \in \mathcal{N}^{\mathcal{L}}(\widehat{\Re})$. then $\exists\, c_1, b_1 \in \widehat{\Re}$ with b_1 regular such that $bc_1 = cb_1$ and $c_1b_1^{-1} = b^{-1}c$. Now $\exists\, \mu_1 \in \widehat{\mathcal{M}}^{\widehat{\Re}}, b_1 \in \widehat{\Re}$ with μ_1 regular such that $b\mu_1 = \mu b_1$, $\mu^{-1}b = b_1\mu_1^{-1}$. Hence $ABC = a\mu^{-1}bv^{-1}c\omega^{-1} = ab_1\mu_1^{-1}v^{-1}c\omega^{-1} = O$. Let I and J be the ideals in $\bar{Q}(\widehat{\Re})$, generated by B and C within $\mathcal{N}^{\mathcal{L}}(\bar{Q}(\widehat{\Re}))$, respectively. Then each of I and J are $\mathcal{N}^{\mathcal{L}}$ with $b = Bv \in I$ and $c = C\omega \in J$, Since $\widehat{\Re}$ is right Ore, for $c, v \in \mathcal{N}^{\mathcal{L}}(\widehat{\Re})$ $\exists\, c_1, v_1 \in \mathcal{N}^{\mathcal{L}}(\widehat{\Re})$ with v_1 regular such that $cv_1 = vc_1$, $v^{-1}c = c_1v_1^{-1}$ .Here note that $c_1 \in \mathcal{N}^{\mathcal{L}}(\widehat{\Re})$. Indeed, $vc_1 = cv_1 \in J$ and so $c_1 = v^{-1}(vc_1) \in J$. So $ABC = ab_1\mu_1^{-1}c_1v_1^{-1}\,\omega^{-1} = O$.

Similarly, also $\exists\ c_2 \in \mathcal{N}^{\mathcal{L}}(\widehat{\Re})$ and $\mu_2 \in \widehat{\mathcal{M}}^{\widehat{\Re}}$ with μ_2 regular such that $c_1\mu_2 = \mu_1c_2, \mu_1^{-1}c_1 = c_2\mu_2^{-1}$, Thus, we obtain that $ABC = ab_1c_2\,\mu_2^{-1}v_1^{-1}\omega^{-1} = O$ and hence $ab_1c_2 = O$. This implies $O = ab_1c_2\mu = a\mu b_1c_2 = ab\mu_1c_2 = abc_2\mu_1$, and $O = abc_2 = abc_2\mu_1 = ab\mu_1c_2 = abc_1\mu_1$. So we have $O = abc_1 = abc_1v = abvc_1 = abcv$. It follows that $acϭ(b) = O$, since $\widehat{\mathcal{M}}^{\widehat{\Re}}$ is a R-ϭ-$\mathcal{N}^{\mathcal{L}}\mathcal{S}$-module.

Similar, $\exists c_3, b_2, \omega_2, b_4 \in \mathcal{N}^{\mathcal{L}}(\widehat{\Re})$ and $\mu_3, \mu_4 \in \widehat{\mathcal{M}}^{\widehat{\Re}}$ with μ_3, ω_2, μ_4 regular such that $c\mu_3 = \mu c_3$, $\mu^{-1}c = c_3\mu_3^{-1}$, $b\omega_2 = \omega b_2, \omega^{-1}b = b_2\omega_2{}^{-1}, b_2\mu_4 = \mu_3b_4$, and, $ACϭ(B) = ac_3\mu_3^{-1}\omega^{-1}ϭ(b)ϭ(v)^{-1} = ac_3\mu_3^{-1}b_2\omega_2^{-1}ϭ(v)^{-1} = ac_3b_4\mu_4^{-1}\omega_2^{-1}ϭ(v)^{-1}$. Form $acϭ(b) = O$. We have $O = acϭ(b)\omega_2 = ac\omega b_2 = acb_2\omega$, and hence $O = acb_2 = acb_2\mu_4 = ac\mu_3b_4 = acb_4\mu_3$. It follows that, $O = acb_4 = acb_4\mu_3 = ac\mu_3b_4 = a\mu c_3b_4 = ac_3b_4\mu$, and hence $ac_3b_4 = O$. Now we have $ACϭ(B) = O$, therefore $\bar{Q}(\widehat{\mathcal{M}})$ is a R-ϭ-$\mathcal{N}^{\mathcal{L}}\mathcal{S}$-module. ∎

CONCLUSION

This article introduced the concept right ϭ -$\mathcal{N}^{\mathcal{L}}$ symmetric rings and then extends it to right ϭ-$\mathcal{N}^{\mathcal{L}}$ symmetric modules, which serve as generalizations of both ϭ-symmetric rings and ϭ-symmetric modules. Several results were founded as the characterization of ϭ-$\mathcal{N}^{\mathcal{L}}$-symmetric rings in section 2, also for ϭ-$\mathcal{N}^{\mathcal{L}}$-symmetric modules in section 5. In addition to that we investigated the concept of an ϭ-$\mathcal{N}^{\mathcal{L}}$-symmetric rings on some of ring extensions and localizations in section 3 and 4, also for ϭ-$\mathcal{N}^{\mathcal{L}}$-symmetric modules in section. As a proposal for a future work, the following questions are presented;

1. Are all right ϭ-$\mathcal{N}^{\mathcal{L}}$-symmetric rings and ϭ-$\mathcal{N}^{\mathcal{L}}$-symmetric modules necessarily non-commutative?
2. Is there a relationship between ϭ-$\mathcal{N}^{\mathcal{L}}$-symmetric module and ϭ-semi-commutative?
3.Are there a class of modules which are $\mathcal{N}^{\mathcal{L}}$-symmetric over their endomorphism?

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Author Contributions:

Ibrahim A. Mustafa conceptualized the study and drafted the initial version of the manuscript. Chenar A. Ahmed contributed to the development of the theoretical framework, reviewed the manuscript critically, and assisted in its final revision. Both authors read and approved the final version of the manuscript.

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