NUMERICAL SOLUTION OF KAWAHARA EQUATION USING NEURAL NETWORK

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Abstract

An artificial neural network technique is proposed in this research to solve the well-known partial differential equations of the types: Kawahara and modified Kawahara equations. The mathematical model of the equation was developed with the help of artificial neural networks. The construction requires imposing certain constrains on the values of the input, bias and output weights, and on the attribution of certain roles of each aforementioned parameters. The results obtained from the proposed technique were very accurate, simple and convenient. Moreover, the comparison between the approximated solutions and the exact one has done. This comparison found them in a good agreement with each other due to of superior properties of the Neural Network.

***Keyword**: Kawahara Equation, Modified Kawahara Equation, Artificial Neural Network.*

1. INTRODUCTION

In the past several decades, the investigation of solutions for nonlinear equations has played an important role in the study of nonlinear physical phenomena.

The Kawahara equation was first proposed by Kawahara in 1972, as a model equation describing solitary wave propagation in media Kawahara T.1972. The Kawahara and Generalized Kawahara equation occurs in the theory of magneto-acoustic waves in plasma D. T. Pham, E. Koc, A. Ghanbarzadeh and S. Otri 2006. and in the theory of shallow water waves with surface tension. However, recently some researchers have used artificial neural network (ANN) to solve differential equationsin numerical calculation to overcome such limitations A. Junaid, M. A. Z. Raja, and I. M. Qureshi2009.H. A . Jalab, Rabha W. Ibrahim, Shayma A. Murad, Amera I. Melhum, and S. B.Hadid 2012. Many researchers have spent a great deal of effort to compute the solution of the Kawahara equation using various numerical methods. Recently, In V. G. Gupta and S. Gupta2010, the homotopy perturbation method (HPM) is employed to obtain approximate analytical solutions of the Kawahara equation and generalized Kawahara equation. M. Kurulay2012 applied the homotopy analysis to solve the modified Kawahara equation. Exp-function method used to study the exact solution for the Kawahara equation in N. A. Kudryashov2010.

Sh. Bahzadi 2011,solved the Kawahara equation by using the Adomain's decomposition method, modified Adomian's decomposition method, variational iteration method, modified variational iteration method, homotopy perturbation method, modified homotopy perturbation method and homotopy analysis method.

2. THE MODEL PROBLEMS

The analysis presented in this paper is based on the Kawahara equation (Sirendaoreji , 2004) :

$$u_t + uu_x + u_{xxx} - u_{xxxxx} = 0 \qquad (1)$$

with the initial condition: $u(x,0) = f(x)$

The exact solution of the above equation is:

$$u(x,t) = \frac{-72}{169} + \frac{105}{169}\, sech^4 \frac{1}{2\sqrt{13}}\left(x + \frac{36}{169}t\right)$$

For example

while we consider modified Kawahara equation **(** Sirendaoreji , 2004) , is:

$$u_t + u^2u_x + pu_{xxx} + qu_{xxxxx} = 0 \qquad (2)$$

where p,q are nonzero real constants, with the initial condition $u(x,0) = f(x)$

The exact solution is given for modified Kawahara equation by **(** Sirendaoreji , 2004):

$$u(x,t) = \frac{3p}{\sqrt{-10q}} sech^2(K(x-ct))$$

With $c = \frac{25q-4p^2}{25q}$ and $K = \frac{1}{2}\sqrt{\frac{-p}{5q}}$

In this paper we apply an artificial neural network to estimate the solutions of the Kawahara equation (1) and the for modified Kawahara equation (2).

3. ARTIFICIAL NEURAL NETWORK (ANN)

Neural networks are computational models of the biological brain. Like the brain, a neural network comprises a large number of interconnected neurons. Each neuron is capable of performing only simple computation M. Ghalambaz, A.R. Noghrehabadi, M.A. Behrang, E. Assareh, A. Ghanbarzadeh and N. Hedayat 2011, D. T. Pham, E. Koc, A. Ghanbarzadeh and S. Otri2006. Anyhow, the architecture of an artificial neuron is simpler than a biological neuron. ANNs are constructed in layer connected to one or more hidden layers where the factual processing is performance through weighted connections. Each neuron in the hidden layer joins to all neurons in the output layer. The results of the processing are acquired from the output layer. Learning in ANNs is achieved through particular training algorithms which are expanded in accordance with the learning laws, assumed to simulate the learning mechanisms of biological system A. Malek and R. S. Beidokhti 2006. However, as an assembly of neurons, a neural network can learn to perform complex tasks including pattern recognition, system identification, trend prediction, function approximation, and process control D. T. Pham, E. Koc, A. Ghanbarzadeh and S. Otri 2006. Multi-layer Perceptron (MLPs) are perhaps the most common type of feed forward networks D. T. Pham and X. Liu1995, A. S. Yilmaz and Z. Ozer2009 .

In this work, a standard back-propagation neural network (NN) is used to estimate the exact solution for the given fractional equation. The network consists of three layers; the first layer consists of neurons that are responsible for input data vectors into the neural network. The second layer is a hidden layer. This layer allows neural network to perform the error reduction, which is necessary to successfully achieve the desired output. The final layer is the output layer which is

Determined by the size of the set of desired outputs, which represent the estimated exact solution. Each possible output is represented by a separate neuron. There is one output from neural network. The neural network structure is shown in Fig. 1.

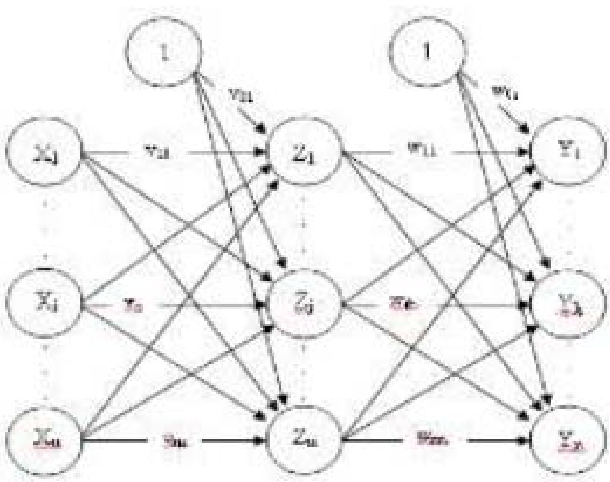

Fig.1 Neural Network structure

Each neuron j (Fig. 2) in the hidden layer sums up its input signals x_i after weighting them with the strengths of the respective connections w_{ij} from the input layer and adding the bias b_i to them, and computes its output η_i as a function g of the sum

$$\eta_i = g(\sum w_{ij} x_i + b_i) \quad (3)$$

where η_i is each neuron output and g can be a simple threshold function or a sigmoid.

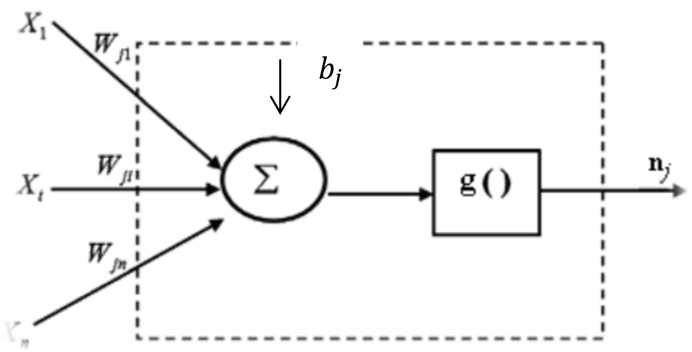

Fig.2 Details of a neuron

The general training algorithm for BPN is as follows

Step0. Initialize weights.

(Set to small random values)

Step1. While stopping, condition is false(output meet the goal), do Step 2-9.

Step2. For each training pair, do Steps 3-8.

Feedforward:

Step3. Each input unit $(X_i,\ i = 1 \dots n)$ receive input signal X_i and broadcasts this signal to all units in the layer above (the hidden units).

Step4. Each hidden unit $(Z_j,\ j = 1 \dots p)$ sums it

$z\_in_j = v_{oi} + \sum_{i=1}^{n} x_i\, v_{ij}$ (4)

Applies its activation function to compute its output signal.

$Z_i = f(z\_in_j)$ (5)

and send this signal to all units in the layer above (output units)

Step5. Each output unit $(Y_k,\ k = 1 \dots m)$ sums its weighted input signals

$y\_in_k = w_{0k} + \sum_{j=1}^{p} z_j\, w_{jk}$ (6)

And applies its activation function to compute its output signal

$y_k = f(y\_in_k)$ (7)

Backpropagation of error:

Step6. Each output unit $(Y_k,\ k = 1 \dots m)$ receives a target pattern corresponding to the input training pattern, computes its error information term.

$\delta_k = (t_k - y_k) f'(y\_ in_k)$ (8)

Calculates its weight correction term (used to update w_{jk} later).

$\Delta w_{ik} = \alpha \delta_k z_j$ (9)

Calculates its bias correction term (used to update w_{0k} later).

$\Delta w_{0k} = \alpha \delta_k$ (10)

and sends δ_k to units in the layer below.

Step7. Each hidden unit $(Z_j,\ j = 1 \dots p)$ sums its delta inputs (from units in the layer above).

$\delta\_in_j = w_{0k} + \sum_{k=1}^{p} \delta_k\, w_{jk}$ (11)

Multiplies by the derivative of its activation function to calculate its error information term

$\delta_j = \delta\_in_j\, f'(z\_in_j)$ (12)

Calculate its weight correction term (used to update v_{ij} later).

$\Delta v_{ij} = \alpha \delta_j x_i$ (13)

and calculate its bias correction term (used to update v_{0j} later)

$\Delta v_{0j} = \alpha \delta_j$(14)

Update weights and biases:

Step8. Each output unit $(Y_k,\ k = 1 \dots m)$ update its bias and weights

$(j = 0 \dots p)$:

$w_{jk}(new) = w_{jk}(old) + \Delta w_{jk}$ (15)

Each hidden unit $(Z_j,\ j = 1 \dots p)$ updates its bias and weights$(i = 0 \dots n)$:

$$v_{ij}(new) = v_{ij}(old) + \Delta v_{ij} \qquad (16)$$

Step9. Test stopping condition. [13].

4. Artificial Neural Network Methodology for Kawahara Equation

Neural networks generally provide improved performance with the normalized data. The use of original data as input to neural network may cause a convergence problem. All the data sets were therefore, transformed into values between -1 and 1 through dividing the difference of actual and minimum values by the difference of maximum and minimum values subtracted by 1. At the end of each algorithm, outputs were renormalized into the original data format for achieving the desired result.

Training goal for the networks was set to 10^{-5}. Finding appropriate architecture needs trial and error method.

Networks were trained for a fixed number of epochs. By this way, we found that two neurons for two hidden layer produce good result. Comparison the result of ANN and exact solution is shown in Fig. 3.

1. Create an architecture consists of two input nodes in the input layer, two hidden nodes in two hidden layers, one output node in the output layer. Assign the nodes to each layer.
2. Initialize the weights and bias to random values.
3. Initialize the network parameters.
4. Train the network with initialized parameters, and with sigmoid activation function.
5. Repeat the process until the maximum epochs are reached or the desired output is identified or the minimum gradient is reached.

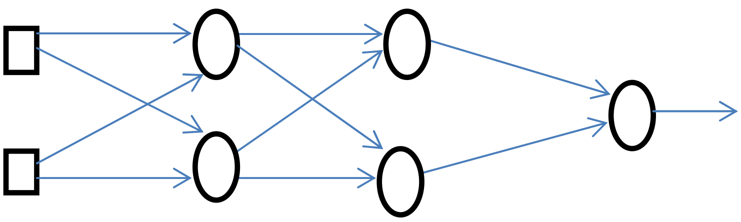

Fig. 3 Neural Network structure for kawahara equation

EXAMPLE 1. To show the procedure, we will examine Kawahara equation

$$u_t + uu_x + u_{xxx} - u_{xxxxx} = 0$$

With initial condition

$$u(x,0) = \frac{-72}{169} + \frac{105}{169} sech^4\left(\frac{1}{2\sqrt{13}} x\right)$$

The exact solution is

$$u(x,t) = \frac{-72}{169} + \frac{105}{169}\, sech^4 \frac{1}{2\sqrt{13}}\left(x + \frac{36}{169} t\right)$$

for computing work: Comparison between the exact solution and the artificial neural network solution will be given in the following (Table 1):

Table 1: Comparison of results for the solutions of example 1 where $x \in (0,2)$, $t \in (0,1)$

Kawahara Equation			
T	x	ANN Solution	Exact Solution
0.1	0.1	0.194818435344505	0.194915325641
0.1	0.2	0.194134854223350	0.194097364623
0.1	0.3	0.192782703802976	0.192805204738
0.1	0.4	0.191039486048669	0.191042568078
0.1	0.5	0.188833513582170	0.188812796301
0.1	0.6	0.186115595708242	0.186121568496
0.1	0.7	0.182957732770161	0.182977236589
0.1	0.8	0.179393463430124	0.179389100063
0.1	0.9	0.175383881161043	0.175364895242
0.1	1	0.170903532930394	0.170914456985
0.1	1.1	0.166008717725511	0.166051894564
0.1	1.2	0.160830163626410	0.160786354712
0.1	1.3	0.155484418862169	0.155513423562
0.1	1.4	0.149951486630059	0.149108256974
0.1	1.5	0.144126956820884	0.142723459831
0.1	1.6	0.138580203948702	0.1359952886166
0.2	0.6	0.185481153647747	0.1854893572
0.2	0.7	0.182265474605305	0.182249967
0.2	0.8	0.178640125774316	0.1785673955
0.2	0.9	0.174550793023224	0.174451295
0.2	1	0.169973265199086	0.1699123841

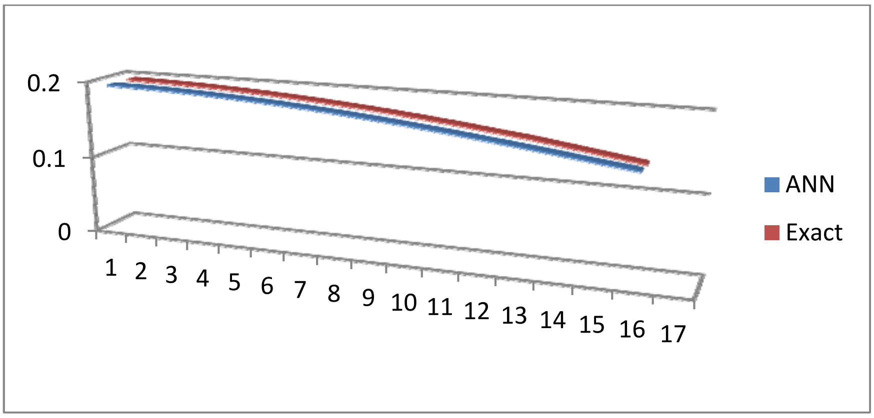

Fig3. Neural Network estimated solution for Example (1)

EXAMPLE 2. We have taken $p = 0.001$ *and* $q = -1$, then the modified Kawahara equation is

$$u_t + u^2 u_x + 0.001 u_{xxx} - u_{xxxxx} = 0$$

with initial condition $u(x,0) = \frac{0.003}{\sqrt{10}} sech^2(0.00707106x)$

The exact solution of modified Kawahara equation as follow:

$$u(x,t) = \frac{0.003}{\sqrt{10}} sech^2(0.00707106(x - 1.00000016t))$$

for computational work, , will be given in Table 2

Table 2. Comparison of results for the solution of example2 $= 0.001$ *and* $q = -1$, $x \in R$, $t \in (0,1)$

Modified Kawahara Equation			
t	x	ANN Solution	Exact Solution
0.02	-5	0.000947424133242253	0.0009474889415215798
0.04	-2.5	0.000948316063770784	0.0009483773375906261
0.06	0.0	0.000948590405265281	0.0009486831272874861
0.08	2.5	0.000948390995627245	0.0009484055588301033
0.10	5	0.000947705508736422	0.0009475453146340136
0.12	7.5	0.000946215002182919	0.0009461045077496208
0.14	10	0.000944010475646170	0.0009440866708438895
0.16	12.5	0.000941399960100077	0.0009414967378249370
0.18	15	0.000938351483701271	0.0009383410182732446
0.20	17.5	0.000934608800307114	0.0009346271649062464
0.22	20	0.000930050731366098	0.0009303641343629944
0.06	-1.25	0.000948730474574495	0.0009486031388265163
0.07	0	0.000948664477167285	0.0009486831272874861
0.08	1.25	0.000948514966415799	0.0009486172537859130
0.09	2.5	0.000948270673895690	0.0009484055588301033
0.10	3.75	0.000947888953488847	0.0009480481725713649
0.11	5	0.000947320814390300	0.0009475453146340136
0.12	6.25	0.000946548126945595	0.0009468972938286883

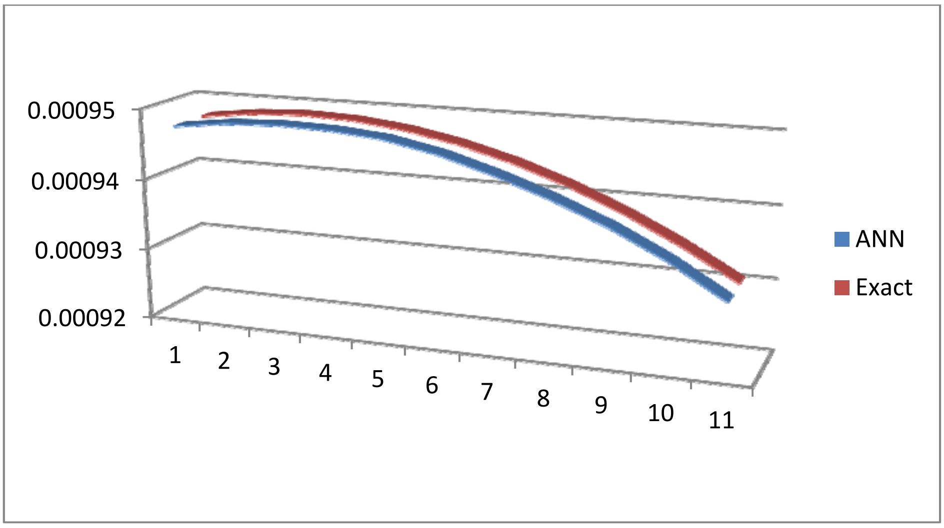

Fig4. Neural Network estimated solution for Example (2)

5. CONCLUSION

This research indicated that the artificial neural network (ANN) technique can play an important role to estimate the solutions of the well-known equations: Kawahara and modified Kawahara equation. Therefore, the research has shown that the estimated solution by using ANN technique is approximately to the exact solution. Although the obtained results have acceptable accuracy, the proposed technique can be improved by increasing the number of training data, and can be minimized the error and reduced the differences. Further, the parallel processing property of neural network had reduced the computational time which makes this method better than the conventional methods.

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الملخص

تناولنا في هذا البحث , دراسة الحل لبعض المعادلات التفاضلية الجزئية من نوع معادلة كاهوارا ومعادلة كاهوارا المحسنة باستخدام الشبكات العصبية. النتائج التي تم الحصول عليها من هذه الطريقة جدا متقاربة من الحل الحقيقي .

كورتي

د ڤێ ڤەكولينێدا هاتە خواندن شيكار كردن هندە ك هاوكێشەيێن جياكاريێن بشك ژلاى هاوكيشا كەهوارە وهاوكيشە كەهوارە ياجيتر ب رێكا تورين نيورين. ئانجمت كفتينا بن دەست گالك نيزيكن اش شيكاركردن درست.