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NUMERICAL SOLUTION OF CUBIC-QUINTIC NONLINEAR SCHRÖDINGER EQUATION

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ABSTRACT:

This paper is devoted to investigating and comparing the variational iteration method (VIM) and the residual power series method (RPSM) for solving the cubic-quintic nonlinear Schrödinger equation (CQNLSE) initially developed to elucidate the propagation of pulses in optical fibers. Next, we use the initial conditions to get the numerical solutions of the CQNLSE. We compared the known exact solutions with the approximate results obtained using both the VIM and RPSM. The exact solution and the results from RPSM are evaluated against those from VIM. The findings demonstrated that VIM outperformed RPSM in terms of accuracy, efficiency, and ease of implementation for solving the CQNLSE. In addition, the current results are shown graphically and in the table.

KEYWORDS: Numerical solution, Cubic-Quintic Nonlinear Schrödinger equation, Residual Power Series Method, Variational Iteration Method, Lagrange multiplier.

1. INTRODUCTION

The CQNLSE is a universal mathematical model describing various physical applications and approximating more complex systems, such as Bose-Einstein condensates (BECs), nonlinear optics, a range of interaction phenomena in plasmas, including plasma physics, condensed matter physics, and nuclear physics (Tang & Shukla, 2007,Seadawy *et al.*, 2022, Peleg & Chakraborty, 2023). Notably in nonlinear optics and BECs. In optics, it models the propagation of light beams through layered media, where the nonlinear response of the material changes with position or time. In BECs, it describes atomic interactions influenced by Feshbach resonances. Even in the absence of external potentials, the CQNLSE with nonlinearity management is essential for regulating soliton dynamics, particularly when nonlinearities fluctuate spatially or temporally. This enables the formation of stable soliton structures and multi-soliton bound states (Luo, 2022). The CQNLSE is used in modelling light propagation through various optical media, including non-Kerr crystals, chalcogenide glasses, organic materials, colloids, dye solutions, and ferroelectrics (Seadawy & Sayed, 2017). Studies on the cubic-quintic nonlinear Schrödinger equation also extend to optical fiber communications and nuclear hydrodynamics.

Researchers have extensively studied the CQNLSE using various solution methods. For example, Hafez used the $\exp(-\varphi(\xi))$ approach in his work to extract both singular and periodic solutions (Golam Hafez *et al.*, 2014). By assuming an ansatz solution, Serkin (Serkin *et al.*, 2001) discovered the novel stable bright and dark soliton management regimes for the CQNLSE model. Hao (Hao *et al.*, 2004) published solitary wave analytical solutions. He found the explicit spatial self-similar, bright, and dark soliton solutions of the CQNLSE using distributed coefficients and an external potential (He *et al.*, 2014). Caplan addressed the problems of solitary vortex existence, interactions, and stability in the two-dimensional CQNLSE using both analytical and numerical techniques (Caplan *et al.*, 2009). Numerous researchers have focused on applying various techniques to identify numerical analysis solutions in recent years. Among these are the Adomian and Adomian-Padé Techniques (Sabali, Manaa, & Easif, 2018), Variational Iteration Method (Easif *et al.*, 2015), Sumudu-Decomposition Method (Azzo & Manaa, 2022), Successive Approximation Method (Sabali *et al.*, 2021), Residual Power Series Method (Manaa *et al.*, 2021), and others.

Inokuti (1978) was the first to propose a generic Lagrange multiplier approach for solving quantum mechanical problems. This method allowed for the solution of nonlinear problems. In 2006 and 2007, Chinese mathematician Ji-Huan He, a professor at Donghua University, transformed the Lagrange multiplier method into an iterative technique known as the VIM (Al-Saif *et al.*, 2011). For a wide range of applications in physics, chemistry, biology, and engineering, homogeneous or inhomogeneous equations, including linear or nonlinear (NL), and systems of equations, the VIM approach provides a reliable and efficient

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process. Numerous writers have demonstrated (Faradilla *et al*., 2021) that this approach offers advantages over current numerical methods. This method offers quick converging successive approximations of the precise answer if one exists; if not, a few approximations may be used for numerical purposes (Shihab *et al*., 2023). The VIM does not need special treatments like the Adomian method (Odibat, 2010), perturbation methods, etc, it just employs the starting conditions that are specified.

The RPSM, a relatively new methodology based on the generalized Taylor series, is explained in depth (Alquran, 2014). The RPSM (Korpinar, 2019) is a helpful technique for determining the coefficient values in the solution of fuzzy differential equations using power series. The repeating algorithm is what makes up the RPSM. The RPSM offers several significant advantages for solving both linear and nonlinear differential equations (El-Ajou *et al*., 2015). In contrast to conventional approaches, RPSM can effectively and immediately tackle severely nonlinear situations since it does not need linearization, perturbation techniques, or discretization (Inc *et al*., 2016). With each term calculated using basic chains of linear equations, it offers an analytical solution in the form of a convergent power series, which is both computationally simple and extremely precise. Furthermore, RPSM may be implemented flexibly given a suitable starting estimate, and does not require reformulation when the solution order is increased. This makes it a practical and adaptable tool for a variety of issues, such as differential and integral equation systems (Moaddy *et al*., 2015).

Numerous writers from a variety of subjects have employed the RPSM approach, such as the Klein-Gordon Schrödinger equation (Manaa *et al*., 2021), fractional diffusion equations (Jumarie, 2011), fractional Burger-type equations (Zunino *et al*., 2008), Boussinesq–Burgers equations (Mahmood & Yousif, 2017). Systems of Fredholm integral equations (Komashynska *et al*., 2016), nonlinear fractional KdV–Burgers equation (He, 1999), time fractional nonlinear coupled Boussinesq–Burgers equations (Ubriaco, 2009), fuzzy differential equations (Mainardi, 2012), for linear and nonlinear Lane–Emden equations (Oldham & Spanier, 1974), higher order initial value problems (Miller & Ross, 1993), time-fractional Fokker–Planck equations (Cifani & Jakobsen, 2011).

The following sections make up this paper: Section 2 presents the CQNLSE's mathematical model. In section 3, the basic ideas of VIM and RPSM are provided. In section 4, the derivation of VIM and RPSM are illustrated for the CQNLSE, and in section 5, a numerical example demonstrates the approaches' accuracy and efficiency.

2. MATHEMATICAL MODEL

Cubic-Quintic Nonlinear Schrödinger Equation:

Consider the CQNLSE (Lai *et al*., 2006).

$$iu_t = \frac{\beta_1}{2}u_{xx} + i\frac{\beta_2}{6}u_{xxx} + \frac{\beta_3}{24}u_{xxxx} - \alpha_1|u|^2\,u - \alpha_2|u|^4\,u, \tag{1}$$

where $i = \sqrt{-1},\ \beta_1, \beta_2, \beta_3, \alpha_1$ and $\alpha_2 \in \mathcal{R}$.

Where $u(x,t)$is the slowly changing electric field envelope, x is the distance along the velocity dispersion direction, t is the time, with respect to the group velocity dispersion, β_1 is the second-order dispersion, β_2 is the 3rd-order dispersion, and β_3 is the fourth-order dispersion. For the cubic-quintic terms, the coefficients are α_1 and α_2.

Bright Soliton:

The bright soliton for the CQNLSE (1) is (Lai *et al*., 2006):

$$u(x,t) = sech\left(\frac{4\,t}{3} - x\right)e^{i\left(-\frac{t}{6}-x\right)}. \tag{2}$$

With the initial condition

$$u(x,0) = sech(x)\,e^{-ix}, \quad x \in I \tag{3}$$

The Method's Description:

In this section, we will display the main ideas of the VIM and RPSM.

Basic Idea of the Variational Iteration Method:

To demonstrate the fundamental idea of VIM, consider the following general NL partial differential equation (PDE), (Odibat, 2010):

$$L\big(u(x,t)\big) + N(u(x,t)) = g(u(x,t)), \tag{4}$$

with the initial condition:

$$u(x,0) = f(x), \tag{5}$$

When g is a known analytic function, N is the NL operator component, and $L = \frac{\partial^{\mathrm{r}}}{\partial t^{\mathrm{r}}}$, $\mathrm{r} \in \mathbb{N}$ is a linear operator part and is the term of the highest-order derivative.

We may create the following iteration formula using VIM:

$$u_{n+1}(x,t) = u_n(x,t) + \int_0^t \lambda(\tau)[Lu_n(x,\tau) + N\tilde{u}_n(x,\tau) - g(x,t)]d\tau\,, \qquad n = 0,1,2,\dots \tag{6}$$

A generic Lagrangian multiplier, denoted by λ (Inokuti, 1978), can be optimally determined from the stationary conditions of equation (6) concerning u_n using the variational theory (Anjum & He, 2019), the subscript n indicates the n-th approximation, and $\tilde{u}_n$is regarded as a confined variation, i.e. $\delta\tilde{u}_n = 0$.

The Lagrange multiplier can be identified as (He and Wu, 2007).

$$\lambda(t,\tau) = \frac{(-1)^{\mathrm{r}}}{(\mathrm{r}-1)!}(\tau - t)^{\mathrm{r}-1}\,,\ \mathrm{r} \ge 1. \tag{7}$$

By applying the Laplace transform to both sides of (4), we may more readily find the Lagrange multiplier, following the methodology of Tsai and Chen. This converts the linear portion with constant coefficients into an algebraic one (Wu, 2013).

Next, the approximate solution to equation (4) is thus provided by: $u(x,t) = \lim_{n\to\infty} u_n(x,t)$

The approximation is denoted by the subscript n, and $\tilde{u}_n$ is regarded as a constrained variant (Momani & Abuasad, 2006).

Basic Idea of the Residual Power Series:

3 METHOD

Considering the general form of an NLPDE:

$$L(u(x,t)) = N(u(x,t)) + g(u(x,t)), \tag{8}$$

with the initial condition equation (5),

where $L = \frac{\partial^k}{\partial t^k}$, $k \in \mathbb{N}$, is the highest order partial derivative with respect to time t. $N(x,t)$ is a nonlinear term and $g(x,t)$ a linear term. The standard RPSM defines the solution $u(x,t)$ as the power series of the form:

$$u(x,t) = \sum_{n=0}^{\infty} f_n(x)\, t^n. \tag{9}$$

Where $n = 0,1,2,\dots$, afterward, we may define $u_n(x,t)$ to denote the *n*th truncated series of $u(x,t)$, i.e.

$$u_m(x,t) = \sum_{n=0}^{m} f_n(x)\, t^n. \tag{10}$$

The zeroth residual power series approximation solution for $u(x,t)$ is

$$u_0 = u(x,0) = f(x) = f_0(x). \tag{11}$$

Where $f_0(x)$ is the initial condition. Now, if we replace equation (11) with equation (10), we obtain:

$$u_n(x,t) = u(x,0) + \sum_{n=1}^{m} f_n(x)\, t^n,\ for\ t \ge 0,\ x \in I,\ m = 1,2,\dots \tag{12}$$

to complete the coefficients $f_n(x)$, $n = 1,2,\dots,m$ of equation (12), the residual function for equation (8) is first defined as follows:

$$Resu(x,t) = L(u(x,t)) - N(u(x,t)) - g(u(x,t)), \tag{13}$$

and the m^{th} Residual function is of the form:

$$Resu_m(x,t) = L(u_m(x,t)) - N(u_m(x,t)) - g\big(u_m(x,t)\big),\ m = 1,2,3,\dots \tag{14}$$

We present a few RPSM findings from (Cifani & Jakobsen, 2011), which are crucial to RPSM:

- $Resu(x,t) = 0$,
- $\lim_{m\to\infty} Resu_m(x,t) = Resu(x,t),\ t \ge 0,\ \forall\, x \in I$,
- $\left.\frac{\partial^k Res\, u_m(x,t)}{\partial t^k}\right|_{t=0} = 0,\ k = 0,1,2,\dots,m.$ (15)

As a result, we may acquire all of the necessary coefficients $f_n(x)$ of the power series of equation (8).

Illustrative Application:

This section will apply the aforementioned techniques to the CQNLSE.

Derivation of VIM for Solving Cubic-Quintic Nonlinear Schrödinger Equation:

Consider the CQNLSE (1) with the initial condition equation (3) by means of VIM, the correcting function is provided as

$$u_{n+1}(x,t) = u_n(x,t) + \int_0^t \lambda(\tau)\left(\frac{\partial u_n(x,\tau)}{\partial \tau} + i\frac{\beta_1}{2}\frac{\partial^2 \tilde{u}_n(x,\tau)}{\partial x^2} - \frac{\beta_2}{6}\frac{\partial^3 \tilde{u}_n(x,\tau)}{\partial x^3} + i\frac{\beta_3}{24}\frac{\partial^4 \tilde{u}_n(x,\tau)}{\partial x^4} - \alpha_1 i\left(|\tilde{u}_n(x,\tau)|^2\, \tilde{u}_n(x,\tau)\right) - \alpha_2 i(|\tilde{u}_n(x,\tau)|^4\, \tilde{u}_n(x,\tau))\right) d\tau, \tag{16}$$

where

$|u|^2\ u = u^2\, \bar{u}$, and $|u|^4\, u = u^3\ \bar{u}^2$,

$\bar{u}$ is the conjugate of u.

In our equation, $\mathfrak{r} = 1$. Then by formula (7), $\lambda = -1$, substituting in equation (16), we get:

$$u_{n+1}(x,t) = u_n(x,t) - \int_0^t \left(\frac{\partial u_n(x,\tau)}{\partial \tau} + i\frac{\beta_1}{2}\frac{\partial^2 u_n(x,\tau)}{\partial x^2} - \frac{\beta_2}{6}\frac{\partial^3 u_n(x,\tau)}{\partial x^3} + i\frac{\beta_3}{24}\frac{\partial^4 u_n(x,\tau)}{\partial x^4} - \alpha_1 i\left((u_n(x,\tau)^2\, \overline{u_n(x,\tau)}\,\right) - \alpha_2 i\left(u_n(x,\tau)^3\, \overline{u_n(x,\tau)^2}\right)\right) d\tau, \quad n = 0,1,2,\ldots \tag{17}$$

Equation (3) provides us with the initial approximation $u(x,0)$. The following is how we may retrieve the additional components using the iteration formula (17):

$$u_0(x,t) = u(x,0) = f(x). \tag{18}$$

For $n = 0$;

$$u_1(x,t) = u_0(x,t) - \int_0^t \left(\frac{\partial u_0(x,\tau)}{\partial \tau} + i\frac{\beta_1}{2}\frac{\partial^2 u_0(x,\tau)}{\partial x^2} - \frac{\beta_2}{6}\frac{\partial^3 u_0(x,\tau)}{\partial x^3} + i\frac{\beta_3}{24}\frac{\partial^4 u_0(x,\tau)}{\partial x^4} - \alpha_1 i\left((u_0(x,\tau)^2\, \overline{u_0(x,\tau)}\,\right) - \alpha_2 i(u_0(x,\tau)^3\, \overline{u_0(x,\tau)^2})\right) d\tau, \tag{19}$$

For $n = 1$;

$$u_2(x,t) = u_1(x,t) - \int_0^t \left(\frac{\partial u_1(x,\tau)}{\partial \tau} + i\frac{\beta_1}{2}\frac{\partial^2 u_1(x,\tau)}{\partial x^2} - \frac{\beta_2}{6}\frac{\partial^3 u_1(x,\tau)}{\partial x^3} + i\frac{\beta_3}{24}\frac{\partial^4 u_1(x,\tau)}{\partial x^4} - \alpha_1 i\left((u_1(x,\tau)^2\, \overline{u_1(x,\tau)}\,\right) - \alpha_2 i\left(u_1(x,\tau)^3\, \overline{u_1(x,\tau)^2}\,\right)\right) d\tau. \tag{20}$$

And after similarly for $n = 3,4,\ldots$

Derivation of RPSM for Solving Cubic-Quintic Nonlinear Schrödinger Equation:

Consider the CQNLS equation (1), with the initial condition

$$u(x,0) = u_0(x,t) = f(x) = f_0(x). \tag{21}$$

Where $u_0(x,t)$ is the initial condition. Using equation (21) to apply RPSM to equation (1). Next, the following is the RPSM solution to equation (1) around the beginning point $t = 0$:

$$u(x,t) = \sum_{n=0}^{\infty} f_n(x)\, t^n,\ t \geq 0, x \in I. \tag{22}$$

Where $n = 0, 1, 2, \ldots$, then, we can define $u_n(\,x,t)$ to indicate the *n*th truncated series of $u(x,t)$, that's

$$u_m(x,t) = \sum_{n=0}^{m} f_n(x)\, t^n,\ \ t \geq 0, x \in I. \tag{23}$$

When we now replace equation (21) with equation (23), we obtain:

$$u_m(x,t) = f_0(x) + \sum_{n=1}^{m} f_n(x)\, t^n, t \geq 0, x \in I. \tag{24}$$

To calculate the value of the coefficients $f_n(x)$, of equation (24), $\forall\ n = 1, 2, 3, \ldots, m$. We defined the residual function for equation (1) as:

$$Resu(x,t) = iu_t - \frac{\beta_1}{2}u_{xx}(x,t) - i\frac{\beta_2}{6}u_{xxx}(x,t) - \frac{\beta_3}{24}u_{xxxx}(x,t) + \alpha_1\, |u(x,t)|^2\, u(x,t) + \alpha_2\, |u(x,t)|^4\, u(x,t), \tag{25}$$

Thus, the m^{th}residual functions, Res_m, are to the form

$$Res\, u_m(x,t) = i\,(u_m)_t(x,t) - \frac{\beta_1}{2}(u_m)_{xx}(x,t) - i\frac{\beta_2}{6}(u_m)_{xxx}(x,t) - \frac{\beta_3}{24}(u_m)_{xxxx}(x,t) + \alpha_1 |u_m(x,t)|^2 u_m(x,t) + \alpha_2 |u_m(x,t)|^4 u_m(x,t), \text{ for } m = 1,2,3,\ldots \quad (26)$$

to determine $f_1(x)$, we write $m = 1$ in equation (26). Then, we will have:

$$Res\, u_1(x,t) = i(u_1)_t(x,t) - \frac{\beta_1}{2}(u_1)_{xx}(x,t) - i\frac{\beta_2}{6}(u_1)_{xxx}(x,t) - \frac{\beta_3}{24}(u_1)_{xxxx}(x,t) + \alpha_1 |u_1(x,t)|^2 u_1(x,t) + \alpha_2 |u_1(x,t)|^4 u_1(x,t), \quad (27)$$

where

$$u_1(x,t) = f_0(x) + f_1(x)t. \quad (28)$$

We obtain the 1^{st}residual functions as follows by changing equation (28) into equation (27).

$$Res\, u_1(x,t) = \Big[i f_1(x) - \frac{\beta_1}{2}(f_0''(x) + f_1''(x)t) - i\frac{\beta_2}{6}(f_0'''(x) + f_1'''(x)t) - \frac{\beta_3}{24}(f_0''''(x) + f_1''''(x)t) + \alpha_1\big[\overline{(f_0(x)+f_1(x)t)}\,(f_0(x)+f_1(x)t)^2\big] + \alpha_2\big[\overline{(f_0(x)+f_1(x)\,t)^2}\,(f_0(x)+f_1(x)\,t)^3\big]\Big]\Big|_{t=0}, \quad (29)$$

$$Res\, u_1(x,t) = \Big[i f_1(x) - \frac{\beta_1}{2}(f_0''(x) + f_1''(x)\,t) - i\frac{\beta_2}{6}(f_0'''(x) + f_1'''(x)\,t) - \frac{\beta_3}{24}(f_0''''(x) + f_1''''(x)\,t) + \alpha_1\Big[{f_0}^2(x)\,\overline{f_0(x)} + {f_0}^2(x)\,\overline{f_1(x)}\,t + 2f_0(x)\,\overline{f_0(x)}\,f_1(x)\,t + 2\,f_0(x)\,\overline{f_1(x)}\,f_1(x)\,t^2 + \overline{f_0(x)}\,{f_1}^2(x)\,t^2 + \overline{f_1(x)}\,{f_1}^2(x)\,t^3\Big] + \alpha_2\Big[{f_0}^3(x)\,\overline{{f_0}^2(x)} + 2{f_0}^3(x)\,\overline{f_0(x)}\;\overline{f_1(x)}\,t + {f_0}^3(x)\,\overline{{f_1}^2(x)}\,t^2 + 3{f_0}^2(x)f_1(x)\,\overline{{f_0}^2(x)}\,t + 6\,{f_0}^2(x)\,f_1(x)\,\overline{f_0(x)}\;\overline{f_1(x)}\,t^2 + 3\,{f_0}^2(x)\,f_1(x)\,\overline{{f_1}^2(x)}\,t^2 + 3\,f_0(x)\,{f_1}^2(x)\,\overline{{f_0}^2(x)}\,t^2 + 6\,f_0(x)\,{f_1}^2(x)\overline{f_0(x)}\;\overline{f_1(x)}\,t^3 + 3\,f_0(x)\,{f_1}^2(x)\,\overline{{f_1}^2(x)}\,t^4 + {f_1}^3(x)\,\overline{{f_0}^2(x)}\,t^3 + 2\,{f_1}^3(x)\,\overline{f_0(x)}\;\overline{f_1(x)}\,t^4 + {f_1}^3(x)\,\overline{{f_1}^2(x)}\,t^5\Big]\Big]\Big|_{t=0}, \quad (30)$$

$$Res\, u_1(x,t) = i f_1(x) - \frac{\beta_1}{2} f_0''(x) - i\frac{\beta_2}{6} f_0'''(x) - \frac{\beta_3}{24} f_0''''(x) + \alpha_1\big[{f_0}^2(x)\,\overline{f_0(x)}\big] + \alpha_2\Big[{f_0}^3(x)\,\overline{{f_0}^2(x)}\Big], \quad (31)$$

According to $Res\, u_1(x,t) = 0$, we get

$$f_1(x) = -i\,\frac{\beta_1}{2} f_0''(x) + i\frac{\beta_2}{6} f_0'''(x) - i\frac{\beta_3}{24} f_0''''(x) + i\,\alpha_1\big[{f_0}^2(x)\,\overline{f_0(x)}\big] + i\,\alpha_2\Big[{f_0}^3(x)\,\overline{{f_0}^2(x)}\Big]. \quad (32)$$

So

$$u_1(x,t) = f_0(x) + \Big(-i\,\frac{\beta_1}{2} f_0''(x) + i\frac{\beta_2}{6} f_0'''(x) - i\frac{\beta_3}{24} f_0''''(x) + i\,\alpha_1\big[{f_0}^2(x)\,\overline{f_0(x)}\big] + i\,\alpha_2\Big[{f_0}^3(x)\,\overline{{f_0}^2(x)}\Big]\Big)\,t. \quad (33)$$

When $m = 2, 3, \ldots$ the same technique can be used to obtain a higher degree of an approximate solution.

Numerical Results:

Within this segment, we use the methods from the previous part to solve the CQNLS problem numerically. We then compare the results with exact solutions.

Solving Cubic-Quintic Nonlinear Schrödinger Equation with the use of VIM:

Consider the CQNLS equation (1), assume that

$\beta_1 = \beta_2 = \beta_3 = \alpha_1 = \alpha_2 = 1$, (Lai *et al.*, 2006).

With the initial condition

$$u(x,0) = sech\,(x)\,e^{-ix} \quad (34)$$

The exact solution is given by equation (2), and that $u_0(x,t) = u(x,0)$ and from equations (19) and (20).

$$u_1(x,t) = e^{-x\,i}\,sech(x) + \frac{t\,e^{-x\,i}e^{x}(78\,e^{2x}\,i + e^{4x}(8-i) - 8 - i)}{3\,(e^{2x}+1)^3}, \quad (35)$$

$$u_2(x,t) = \frac{cos(x) - sin(x)\,i}{cosh(x)} + \frac{t\,e^{x\,(1-i)}\,(78\,e^{2x}\,i + e^{4x}\,(8-i) - 8 - i)}{3\,(e^{2x}+1)^3} -$$

$$
t^2\left(\begin{array}{c}
610250760\,e^{15x}+e^{5x}\,(-1388745+15130800\,i)\\
+e^{7x}\,(46777500+67080960\,i)\\
+\,e^{9x}\,(203073885+168292080\,i)\\
+e^{11x}\,(421505370+252901440\,i)\\
+\,e^{13x}\,(569719575+202506480\,i)\\
+e^{3x}\,(-2029050+1477440\,i)\\
+e^{17x}\,(569719575-202506480\,i)\\
+e^{19x}\,(421505370-252901440\,i)\\
+e^{21x}\,(203073885-168292080\,i)\\
+e^{23x}\,(46777500-67080960\,i)\\
+e^{25x}\,(-1388745-15130800\,i)\\
+e^{27x}\,(-2029050-1477440\,i)\\
+e^{29x}\,(-25515+6480\,i)\\
+e^{x}\,(-25515-6480\,i)
\end{array}\right)\Bigg/\mu+t^3\left(\begin{array}{c}
e^{3x}\,(-17280+208440\,i)\\
+e^{5x}\,(1071360+4272480\,i)\\
+e^{7x}\,(20252160+25939440\,i)\\
+e^{9x}\,(98046720+136620000\,i)\\
+e^{11x}\,(211697280+492669000\,i)\\
+e^{13x}\,(204802560+1056818880\,i)\\
+1357788960\,e^{15x}\,i\\
+e^{17x}\,(-204802560+1056818880\,i)\\
+e^{19x}\,(-211697280+492669000\,i)\\
+e^{21x}\,(-98046720+136620000\,i)\\
+e^{23x}\,(-20252160+25939440\,i)\\
+e^{25x}\,(-1071360+4272480\,i)\\
+e^{27x}\,(17280+208440\,i)
\end{array}\right)\Bigg/\mu
$$

$$
-\,t^4\left(\begin{array}{c}
e^{3x}\,(-8775+70200\,i)\\
+e^{5x}\,(444420+3043440\,i)\\
+e^{7x}\,(17238690+13525920\,i)\\
+e^{9x}\,(103400820+187062480\,i)\\
+e^{11x}\,(851733495+671402520\,i)\\
+e^{13x}\,(2740719240+820925280\,i)\\
+3951202140\,e^{15x}\\
+e^{17x}\,(2740719240-820925280\,i)\\
+e^{19x}\,(851733495-671402520\,i)\\
+e^{21x}\,(103400820-187062480\,i)\\
+e^{23x}\,(17238690-13525920\,i)\\
+e^{25x}\,(444420-3043440\,i)\\
+e^{27x}\,(-8775-70200\,i)
\end{array}\right)\Bigg/\mu+\,t^5\left(\begin{array}{c}
e^{5x}\,(-149760+1502280\,i)\\
+e^{7x}\,(11741184-1330992\,i)\\
+e^{9x}\,(-17830656+157771944\,i)\\
+e^{11x}\,(987918336+193120704\,i)\\
+e^{13x}\,(2077042176+2242802448\,i)\\
+4420573920\,e^{15x}\,i\\
+e^{17x}\,(-2077042176+2242802448\,i)\\
+e^{19x}\,(-987918336+193120704\,i)\\
+e^{21x}\,(17830656+157771944\,i)\\
+e^{23x}\,(-11741184-1330992\,i)\\
+e^{25x}\,(149760+1502280\,i)
\end{array}\right)\Bigg/\mu
$$

$$
+\,t^6\left(\begin{array}{c}
e^{5x}\,(42250-338000\,i)\\
+e^{7x}\,(-3498300+1622400\,i)\\
+e^{9x}\,(23849410-63572080\,i)\\
+e^{11x}\,(-642117840+148711680\,i)\\
+e^{13x}\,(1829294900-2780480800\,i)\\
-27770394600\,e^{15x}\\
+e^{17x}\,(1829294900+2780480800\,i)\\
+e^{17x}\,(1829294900+2780480800\,i)\\
+e^{17x}\,(1829294900+2780480800\,i)\\
+e^{21x}\,(23849410+63572080\,i)\\
+e^{23x}\,(-3498300-1622400\,i)\\
+e^{25x}\,(42250+338000\,i)
\end{array}\right)\Bigg/\mu. \qquad (36)
$$

Where

$\mu = 14580\,e^{x\,i} + 218700\,e^{x\,(2+i)} + 1530900\,e^{x\,(4+i)} + 6633900\,e^{x\,(6+i)} + 19901700\,e^{x\,(8+i)} + 43783740\,e^{x\,(10+i)} + 72972900\,e^{x\,(12+i)} + 93822300\,e^{x\,(14+i)} + 93822300\,e^{x\,(16+i)} + 72972900\,e^{x\,(18+i)} + 43783740\,e^{x\,(20+i)} + 19901700\,e^{x\,(22+i)} + 6633900\,e^{x\,(24+i)} + 1530900\,e^{x\,(26+i)} + 218700\,e^{x\,(28+i)} + 14580\,e^{x\,(30+i)}$.

By the same way, we can find $u_3, u_4, \ldots$ and so on.

Solving Cubic-Quintic Nonlinear Schrödinger Equation with the use of RPSM:

Subject to the initial condition equation (3), equation (33), which is obtained by applying RPSM to the CQNLSE, is obtained.

For $m = 1$, we get

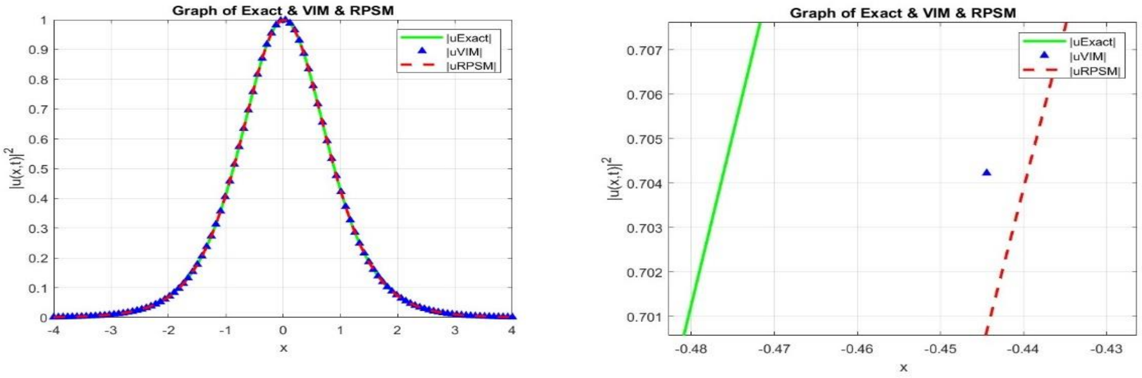
$$u_1(x,t) = e^{-x\,i}\,sech(x) + t\left(\frac{7\,i}{6}e^{-x\,i}sech(x) + \frac{4\,i}{3}e^{-x\,i}\,sinh(x)\,sech^2(x) - \frac{i}{3}e^{-x\,i}\,sech(x)\,tanh^2(x) + ie^{-x\,i}\,sech^3(x) - i\,e^{-x\,i}\,sech(x)\,tanh^4(x) + ie^{-x\,i}\,sech^5(x)\right), \qquad (37)$$

For $m = 2$, we get

$$u_2(x,t) = \frac{1}{72} e^{-x\,i} (63\, t^2\, sech(x) - 16i\, tanh(x)\, sech(x)\, t^2 + 1032\, sech^3(x)\, t^2 + 960\, i\, sech^2(x)\, tanh^2(x)\, t^2 - 3240\, sech^5(x)\, t^2 + 1680\, t^2 - 12\, i\, sech(x)\, t + 96\, tanh(x)\, sech(x) + 240\, i\, sech^3(x)\, t + 72\, sech\,(x)). \quad (38)$$

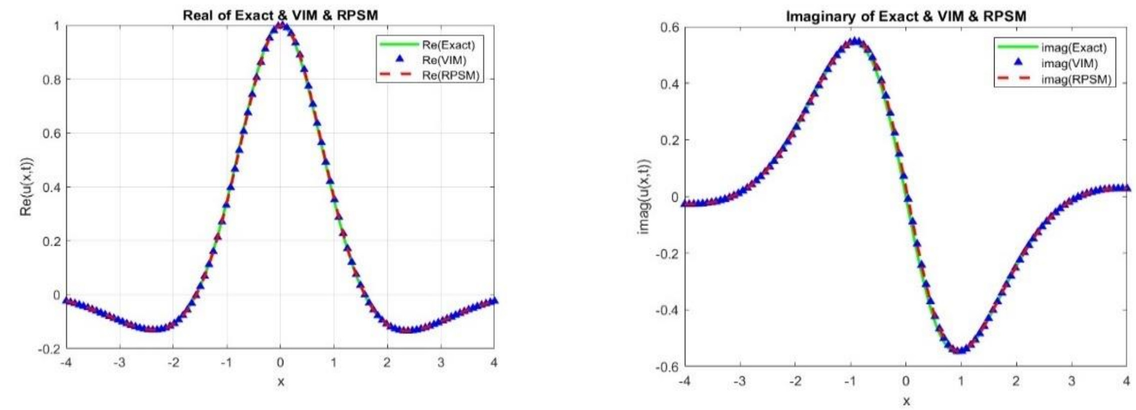
Table 1: Shows exact solution, VIM, RPSM, and absolute error between the exact solution and the approximate solutions by VIM and RPSM for $-4 \leq x \leq 4$ and $t = 0.01$.

x	Exact	VIM	RPSM	Absolute Error |Exact-VIM|	Absolute Error |Exact-RPSM|
-4	1.305687614 593779e-03	1.305694106 940856e-03	1.305694107 461191e-03	6.492347076 614280e-09	6.492867412 125070e-09
-3.2	6.450450147 789773e-03	6.450587452 890997e-03	6.450587464 300128e-03	1.373051012 245469e-07	1.373165103 455906e-07
-2.4	3.154516942 753750e-02	3.154815107 800896e-02	3.154815119 298163e-02	2.981650471 460540e-06	2.981765444 137752e-06
-1.6	1.468686431 534473e-01	1.469139522 828903e-01	1.469139360 725920e-01	4.530912944 300525e-05	4.529291914 459610e-05
-0.8	5.491884863 153710e-01	5.492309418 458221e-01	5.492296088 036173e-01	4.245553045 112427e-05	4.112248824 506004e-05
0	9.998222432 900585e-01	9.997116874 145271e-01	9.997115282 118056e-01	1.105558755 314373e-04	1.107150782 528876e-04
0.8	5.689860006 351928e-01	5.690434416 150851e-01	5.690447578 043778e-01	5.744097989 235364e-05	5.875716918 457563e-05
1.6	1.542683812 584001e-01	1.543150573 297396e-01	1.543150734 634280e-01	4.667607133 951313e-05	4.669220502 773186e-05
2.4	3.324429013 803103e-02	3.324730828 582197e-02	3.324730817 127070e-02	3.018147790 946613e-06	3.018033239 682305e-06
3.2	6.802609526 142632e-03	6.802740545 064941e-03	6.802740533 695097e-03	1.310189223 086011e-07	1.310075524 642990e-07
4	1.377165454 304141e-03	1.377170079 041128e-03	1.377170078 522558e-03	4.624736986 598432e-09	4.624218416 602460e-09
		Total		3.087168260 274371e-04	3.088590996 873099e-04



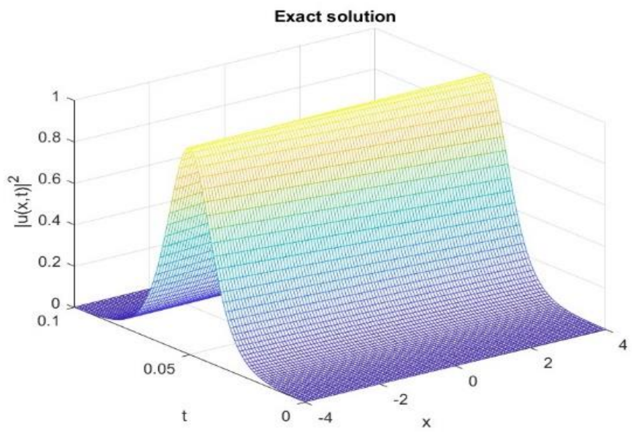
(a) The 2D graph of $|u(x,t)|^2$. (b) The zoomed 2D graph of $|u(x,t)|^2$.

Figure 1: Exact solution and VIM and RPSM at $-4 \le x \le 4$ and $t = 0.01$

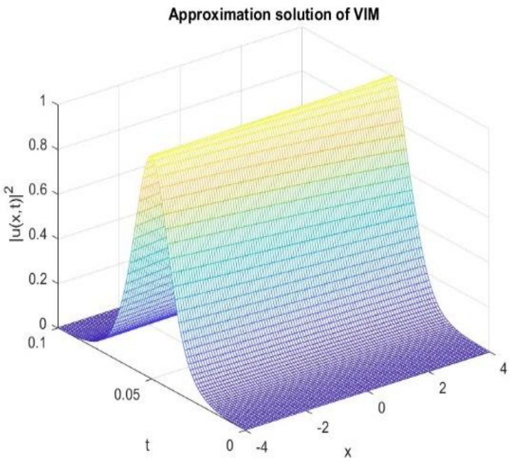


. (a) The 2D graph of $Re(u(x,t))$. (b) The 2D graph of $Im(u(x,t))$.

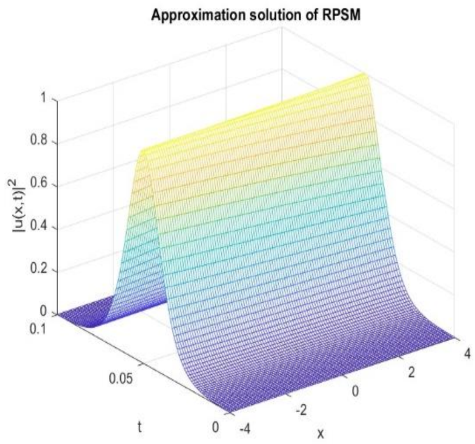
Figure 2: The curves of exact solution, VIM, and RPSM for real and imaginary, when $-4 \le x \le 4$ and $t = 0.01$.



(a) The 3D graph of $|u(x,t)|^2$.

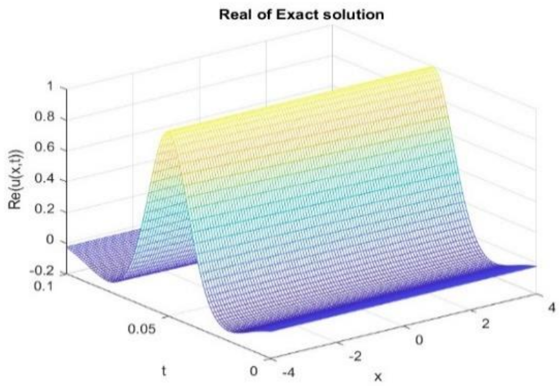


(b) The 3D graph of $|u(x,t)|^2$.

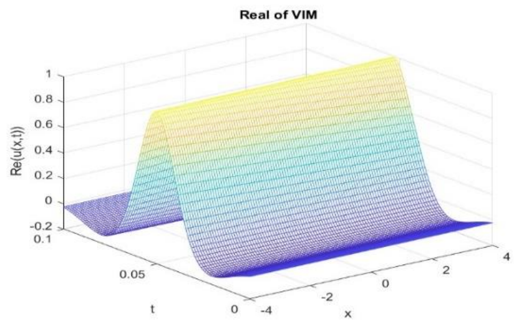


(c) The 3D graph of $|u(x,t)|^2$.

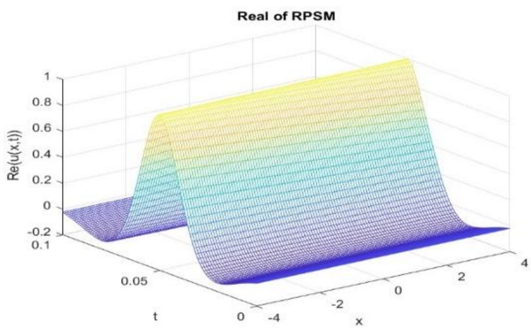
Figure 3: The surfaces of the exact solution, VIM, and RPSM CQNLSE, when $-4 \leq x \leq 4$ and $0 \leq t \leq 0.1$.



(a) The 3D graph of $Re\big(u(x,t)\big)$.

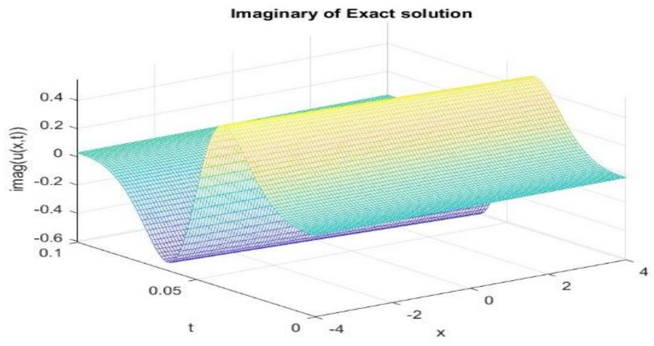


(b) The 3D graph of $Re\big(u(x,t)\big)$.

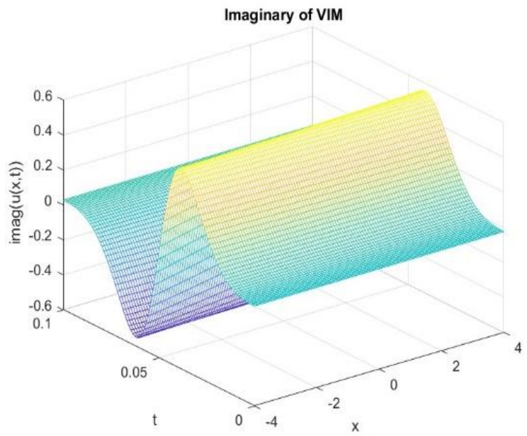


(c) The 3D graph of $Re\big(u(x,t)\big)$.

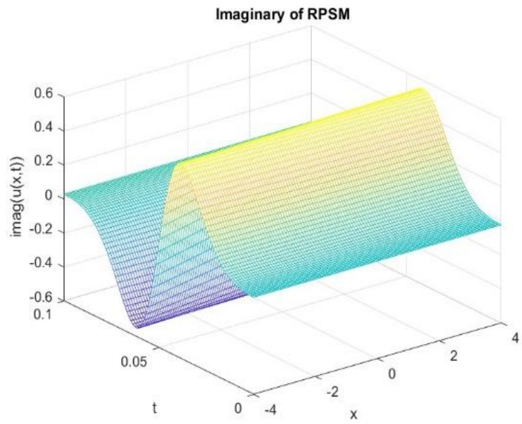
Figure 4: The 3D surfaces for the real part of $u(x,t)$, when $-4 \leq x \leq 4$ and $0 \leq t \leq 0.1$.



(a) The 3D graph of $Im\big(u(x,t)\big)$.



(b) The 3D graph of $Im\big(u(x,t)\big)$.



(c) The 3D graph of $Im\big(u(x,t)\big)$.

Figure 5: The 3D surfaces for the imaginary part of $u(x,t)$, when $-4 \leq x \leq 4$ and $0 \leq t \leq 0.1$.

CONCLUSION

The VIM and RPSM are both utilized in this paper to obtain approximate analytical solutions for the cubic-quintic nonlinear Schrödinger equation. We took an example of the CQNLS equation to compare between the 2nd order of VIM and RPSM with the exact solution. The results obtained by the two methods are compared with the exact solution of the equation. Moreover, we concluded that VIM is powerful, reliable, and elegant, and it yields solutions in a rapidly converging sequence compared to RPSM. It was also found that VIM is significantly more accurate and efficient, matching the exact solution more closely than RPSM. The solutions that have been obtained with real and imaginary parts are plotted.

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Author Contributions:

K. H. O. was responsible for writing the main script and conducting the primary research. F. H. E. supervised, provided guidance, and reviewed the manuscript. Both authors contributed to the final version of the paper.

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The authors declare that they have no conflict of interest.

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