PERIODIC SOLUTION FOR NONLINEAR SYSTEM OF DIFFERENTIAL EQUATIONS DEPENDING ON THE GAMMA DISTRIBUTION

Raad **N**. **B**utris
Department of Mathematics, Faculty of Science, University of Zakho, Kurdistan Region –Iraq.
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Abstract

In this paper we study the periodic solution of nonlinear system of differential equations depending on the gamma distribution by using the numerical analytic method to investigate periodic solutions of ordinary differential equation which given by Samoilenko A. M. .These investigations lead us to improving and extending this method. Also we expand the results gained by Samoilenko A. M. to change the periodic system of nonlinear differential equations to periodic system of nonlinear differential equations depending the on gamma distribution.

Keyword: *Periodic Solution, Differential Equations, Gamma Distribution, Numerical Analytic.*

1.Introduction.

They are many subjects in physics and technology using mathematical methods that depends on the linear and nonlinear differential equations, and it become clear that the existence of periodic solutions and its algorithm structure form important problems, to present time where many of studies and researches [1,4,5,6] dedicates for treatment the autonomous and non-autonomous periodic systems and specially with differential equations.

Numerical-analytic method [1,2,3,5] owing to the great possibilities of exploiting computers are becoming versatile means of the finding and approximate construction of periodic solutions of differential equations. Samoilenko [4] assumes the numerical-analytic method to study the periodic solutions for ordinary differential equations and its algorithm structure and this method include uniformly sequences of periodic functions and the results of that study is using of the periodic solutions on wide range in the difference of new processes industry and technology as in [3,5,6].

Samoilenko[4] has been used the numerical-analytic method to studythe periodic solution for nonlinear system of differential equation which has the form:

$$\frac{dx}{dt} = f(t,x) \quad \cdots (1.1)$$

where

$x \in D$, $all\ real\ t\ \ and\ \ D\ is\ the\ closure\ of\ bounded\ domain\ and\ connected\ in\ R^n$.

In this study we have employed the numerical-analytic method of Samoilenko .[4] to investigate the existence and approximation of periodic solution for nonlinear system of differential equations which depends on th gamma distribution. The study of such differential equations leads to improving and extending Samoilenko method [4].

Thus, the differential equations which depends on the gamma distribution that we have introduced in this study, becomes more general and detailed than those introduced by Samoilenko .[4]. The study is considered a theoretical one, however, the results that we have got, many several applications in the physical field as well as mathematical problems.

Consider the following system of differential equations which has the form

$$\frac{dx}{dt} = f(t, \gamma(t,\alpha), x) \ \cdots (1.2)$$

where

$x \in D$, D *is the closure of bounded domain and connected in* R^n. The vector function $f(t, \gamma(t,\alpha), x)$ is defined on the domain:

$$(t, \gamma(t,\alpha), x) \in R^1 \times [0,T] \times D = (-\infty, \infty) \times [0,T] \times D \qquad \cdots (1.3)$$

Continuous for all variables, periodic in t of period T and satisfies the inequalities:

$$|f(t, \gamma(t,\alpha), x)| \le MM_\alpha \quad , \qquad M_\alpha > 0 \qquad \cdots (1.4)$$

$$|f(t, \gamma(t,\alpha), x_1) - f(t, \gamma(t,\alpha), x_2)| \le M_\alpha (K|x_1 - x_2|) \qquad \cdots (1.5)$$

for all $t \in R^1$ and $x, x_1, x_2 \in D$, where $M = (M_1, M_2, \cdots, M_n)$is a positive constant vectors and the gamma distribution is defined as

$$\left.\begin{array}{l} \gamma(t,\alpha) = \dfrac{t^{\alpha-1}e^{-t}}{\Gamma(\alpha)} \quad , \ \alpha > 0 \, , \\ \text{where} \qquad T < \left(\dfrac{\Gamma(\alpha+1)}{(\alpha+1)}\right)^{\frac{1}{\alpha}} . \end{array}\right\} \qquad \cdots (1.6)$$

We define the non-empty sets as follows:

$$D_{\gamma f} = D - MM_\alpha \frac{T}{2} \ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots\ldots . (1.7)$$

Furthermore, we suppose that the greatest eigen value λ_{max} of the matrix $\Lambda = M_\alpha \, K \frac{T}{2}$ does not exceed unity, i.e.

$$\lambda_{max}(\Lambda) < 1 \qquad \cdots (1.8)$$

Lemma 1.1[4]. Let $f(t)$ be a continuous vector function defined on the interval $[0,T]$, then

$$\left| \int_0^t \left(f(s) - \frac{1}{T}\int_0^T f(s)ds\right)ds \right| \le \alpha(t) \max_{t\in[0,T]} |f(t)|$$

where $\alpha(t) = 2t(1 - \frac{t}{T})$. (For the proof see [4]) .

By using Lemma 1.1 , we can state and prove the following Lemma.

Lemma 1.2. Suppose that the function $\gamma(t,\alpha)$ of gamma distribution is continuous on the interval $[0,T]$. Then

$$\left| \int_0^t \left(\gamma(s,\alpha) - \frac{1}{T}\int_0^T \gamma(s,\alpha)ds\right)ds \right| \le M_\alpha \alpha(t)$$

is hold for all values of α . where $M_\alpha = \max_{t\in[0,T]} |\gamma(t,\alpha)|$.

Proof. Taking

$$\left| \int_0^t \left(\gamma(s,\alpha) - \frac{1}{T}\int_0^T \gamma(s,\alpha)ds\right)ds \right| \le \left(1 - \frac{t}{T}\right)\int_t^T |\lambda(s,\alpha)|ds + \frac{t}{T}\int_t^T |\lambda(s,\alpha)|\,ds$$

$$= \left(1 - \frac{t}{T}\right)\int_0^t \frac{T^{\alpha-1}e^{-t}}{\Gamma(\alpha)}ds + \frac{t}{T}\int_t^T \frac{T^{\alpha-1}e^{-t}}{\Gamma(\alpha)}ds$$

$$\le \frac{T^{\alpha-1}e^{-t}}{\Gamma(\alpha)}\left[\left(1 - \frac{t}{T}\right)t + \frac{t}{T}(T - t)\right]$$

$$= \alpha(t) \ M_\alpha$$

so that

$$\left|\int_0^t (\gamma(s,\alpha) - \frac{1}{T}\int_0^T \gamma(s,\alpha)ds)ds\right| \leq \alpha(t)M_\alpha \cdots (1.10)$$

for all $t \in [0,T]$ and $\alpha(t) \leq \frac{T}{2}$.

2. Approximate solution.

The investigation of periodic approximate solution of (1.2) is formulated by the following theorem.

Theorem 2.1. If the system (1.2) satisfy the inequalities (1.4),(1.5) and conditions (1.6),(1.7) has a periodic solution $x = x(t,\gamma(t,\alpha),x_0)$, then the sequence of functions

$$x_{m+1}(t,\gamma(t,\alpha),x_0) = x_0 + \int_0^t [f(s,\gamma(s,\alpha),x_m(s,\gamma(s,\alpha),x_0) - \frac{1}{T}\int_0^T (f(s,\gamma(s,\alpha),x_m(s,\gamma(s,\alpha),x_0)ds]ds \ldots. (2.1)$$

with

$x_0(t,\gamma(t,\alpha),x_0) = x_0$, $\quad m = 0,1,2,\cdots$

is periodic in t of period T, and uniformly convergent to the limit function $x^0(t,\gamma(t,\alpha),x_0)$ as $m \to \infty$ in the domain

$(t,\gamma(t,\alpha),x_0) \in R^1 \times [0,T] \times D_{\gamma f} \cdots (2.2)$

periodic in t of period T and satisfying the system of integral equations

$$x(t,\gamma(t,\alpha),x_0) = x_0 + \int_0^t [f(s,\gamma(s,\alpha),x(s,\gamma(s,\alpha),x_0) - \frac{1}{T}\int_0^T (f(s,\gamma(s,\alpha),x(s,\gamma(s,\alpha),x_0)ds]ds \cdots (2.3)$$

which is a unique solution of the problem (1.2) provided that:

$$|x^0(t,\gamma(t,\alpha),x_0) - x_0| \leq M\, M_\alpha\, \alpha(t) \ldots \ldots \ldots \ldots \ldots \ldots \ldots \ldots \cdots (2.4)$$

and

$$|x^0(t,\gamma(t,\alpha),x_0) - x_m(t,\gamma(t,\alpha),x_0)| \leq \Lambda^m (E - \Lambda)^{-1} M\, M_\alpha\, \alpha(t) \cdots (2.5)$$

for all $m \geq 1$ and $t \in R^1$, where E is the identity matrix.

Proof:

By Lemma 1.1, 1.2 and using (2.1), when $m = 0$, we get:

$$|x_1(t,\gamma(t,\alpha),x_0) - x_0| \leq (1 - \frac{t}{T})\int_0^t |f(s,\gamma(s,\alpha),x_0, \int_s^{s+T} g(\tau,\gamma(\tau,\alpha),x_0)\, d\tau)|ds$$

$$+\frac{t}{T}\int_{t}^{T}|f(s,\gamma(s,\alpha),x_0,\int_{s}^{s+T}g(\tau,\gamma(\tau,\alpha),x_0)\,d\tau)|ds$$
$$\leq (1-\frac{t}{T})\int_{0}^{t} M\,M_\alpha\,ds+\frac{t}{T}\int_{t}^{T} M\,M_\alpha\,ds$$
$$= M\,M_\alpha\,[(1-\frac{t}{T})t+\frac{t}{T}(T-t)]$$
$$= 2t(1-\frac{t}{T})\;M\,M_\alpha$$
$$= \alpha(t)\,M\,M_\alpha$$

So that

$|x_1(t,\gamma(t,\alpha),x_0)-x_0| \leq M\,M_\alpha\alpha(t) \cdots (2.6)$

i.e. $x_1(t,\gamma(t,\alpha),x_0) \in D$, for all $t \in R^1, x_0 \in D_{\gamma f}$.

Thus by mathematical induction, we find that:

$$|x_m(t,\gamma(t,\alpha),x_0)-x_0| \leq M\,M_\alpha\alpha(t) \cdots (2.8)$$

for all $t \in R^1$ and $x_0 \in D_{\gamma f}$.

i.e. $x_m(t,\gamma(t,\alpha),x_0) \in D$, for all $t \in R^1$ and $x_0 \in D_{\gamma f}$.

We claim that the sequence of functions (2.1) is uniformly convergent on the domain (2.2).

By using the Lemmas 1.1,1.2 and putting $m=1$ in (2.1), we have:

$$|x_2(t,\gamma(t,\alpha),x_0)-x_1(t,\gamma(t,\alpha),x_0)| \leq K[(1-\frac{t}{T})\int_{0}^{t}|\gamma(s,\alpha)||x_1(s,\gamma(s,\alpha),x_0)-x_0|ds$$
$$+\frac{t}{T}\int_{t}^{T}|\gamma(s,\alpha)|\,|x_1(s,\gamma(s,\alpha),x_0)-x_0|ds]$$
$$\leq M_\alpha\;M\;M_\alpha K\frac{T}{2}\alpha(t)$$
$$|x_2(t,\gamma(t,\alpha),x_0)-x_1(t,\gamma(t,\alpha),x_0)| \leq M\,{M_\alpha}^2 K\frac{T}{2}\alpha(t) \cdots (2.9)$$

Suppose that the following inequality is true

$$|x_m(t,\gamma(t,\alpha),x_0)-x_{m-1}(t,\gamma(t,\alpha),x_0)| \leq M\,{M_\alpha}^m[K\frac{T}{2}]^{m-1}\alpha(t) \cdots (2.10)$$

for all $m \geq 1$.

Now, we shall prove the following:

$$|x_{m+1}(t,\gamma(t,\alpha),x_0)-x_m(t,\gamma(t,\alpha),x_0)|$$
$$\leq K[\left(1-\frac{t}{T}\right)\int_{0}^{t} M_\alpha\;|x_m(s,\gamma(s,\alpha),x_0)-x_{m-1}(s,\gamma(t,\alpha),x_0)|ds$$
$$+\frac{t}{T}\int_{t}^{T} M_\alpha\,|x_m(s,\gamma(s,\alpha),x_0)-x_{m-1}(s,\gamma(t,\alpha),x_0)|ds] \leq$$
$$\leq \left(1-\frac{t}{T}\right)\int_{0}^{t} M_\alpha\,M\,{M_\alpha}^m\left[K\frac{T}{2}\right]^{m-1}\;\alpha(s)ds$$

$$+\frac{t}{T}\int_{t}^{T} M_\alpha\, M\, {M_\alpha}^{m}\left[K\frac{T}{2}\right]^{m-1}\alpha(s)ds$$

$$= M\, M_\alpha\left[M_\alpha K\frac{T}{2}\right]^{m}\alpha(t)$$

and hence

$$|x_{m+1}(t,\gamma(t,\alpha),x_0)-x_m(t,\gamma(t,\alpha),x_0)|\le M M_\alpha\left[M_\alpha K\frac{T}{2}\right]^{m}\alpha(t)\cdots(2.11)$$

for all $m\ge 0$.

From (2.11) we conclude that for any$k\ge 1$, we have the inequality

$$|x_{m+k}(t,\gamma(t,\alpha),x_0)-x_m(t,\gamma(t,\alpha),x_0)|\le\sum_{i=0}^{k-1}\Lambda^{m+i}\, M\, M_\alpha\ \alpha(t)$$

such that

$$|x_{m+k}(t,\gamma(t,\alpha),x_0)-x_m(t,\gamma(t,\alpha),x_0)|$$

$$\le\textstyle\sum_{i=0}^{\infty}|x_{m+1+i}(t,\gamma(t,\alpha),x_0)-x_{m+i}(t,\gamma(t,\alpha),x_0)|$$

$$\le\sum_{i=0}^{\infty}M M_\alpha\alpha(t)\,\Lambda^{m+1+i}$$

$$\le M\, M_\alpha\,\alpha(t)\,\Lambda^{m}\sum_{i=0}^{\infty}\Lambda^{i+1}$$

$$\le M\, M_\alpha\,\alpha(t)\,\Lambda^{m}\,(\mathrm{E}-\Lambda)^{-1}$$

so that

$$|x_{m+k}(t,\gamma(t,\alpha),x_0)-x_m(t,\gamma(t,\alpha),x_0)|\le\ \Lambda^{m}\,(\mathrm{E}-\Lambda)^{-1}M\, M_\alpha\,\alpha(t)\qquad\cdots(2.12)$$

for all $k\ge 1$.

From (2.12) and the condition (1.9), we find that:

$$\lim_{m\to\infty}\Lambda^{m}=0\qquad\cdots(2.13)$$

Relations (2.12) and (2.13) insures the uniform convergence of the sequence of functions (2.1) on the domain (2.2).

Let

$$\lim_{m\to\infty}x_m(t,\gamma(t,\alpha),x_0)=x^0(t,\gamma(t,\alpha),x_0)\cdots(2.14)$$

Since the sequence of functions (2.2) is periodic in t of period T, then the limiting function $x^0(t,\gamma(t,\alpha),x_0)$ is also periodic in t of period T.

Moreover, by Lemmas 1.1,1.2 and inequality (2.12) the inequalities (2.4) and (2.5) are holds.

Finally, we have to show that $x(t,\gamma(t,\alpha),x_0)$ is a unique solution of the system (1.1).Assume that $r(t,\gamma(t,\alpha),x_0)$ is another solution of the system (1.1), i.e.

$$r(t,\gamma(t,\alpha),x_0)=x_0+\int_0^t[f(s,\gamma(s,\alpha),r(s,\gamma(s,\alpha),x_0)$$
$$-\frac{1}{T}\int_0^T(f(s,\gamma(s,\alpha),r(s,\gamma(s,\alpha),x_0)ds]ds,\qquad\ldots(2.15)$$

Now, we prove that $x(t,\gamma(t,\alpha),x_0)=r(t,\gamma(t,\alpha),x_0)$ for all $x_0\in D_{\gamma f}$ and to do this, we need to derive the following inequality:

$$|r(t,\gamma(t,\alpha),x_0)-x(t,\gamma(t,\alpha),x_0)|\le\ \Lambda^{m}\,(\mathrm{E}-\Lambda)^{-1}M^{*}\, M_\alpha\,\alpha(t)\qquad\cdots(2.16)$$

where $M^* = max_{x_0 \in D_{\gamma f}} \left| f\big(t, r(t, \gamma(t,\alpha), x_0), x(t, \gamma(t,\alpha), x_0)\big) \right|$.
Suppose that (2.15) is true for $m = k$, i.e.
$|r(t,\gamma(t,\alpha),x_0) - x(t,\gamma(t,\alpha),x_0)| \le \Lambda^k\ (\mathrm{E}-\Lambda)^{-1} M^*\ M_\alpha\ \alpha(t)$
then

$$|r(t,\gamma(t,\alpha),x_0) - x(t,\gamma(t,\alpha),x_0)|$$
$$\le K[\left(1-\frac{t}{T}\right)\int_0^t M_\alpha\ |r(s,\gamma(s,\alpha),x_0) - x(s,\gamma(s,\alpha),x_0)|ds$$
$$+\frac{t}{T}\int_t^T M_\alpha\ |r(s,\gamma(s,\alpha),x_0) - x(s,\gamma(s,\alpha),x_0)|ds]$$
$$\le K[\left(1-\frac{t}{T}\right)\int_0^t M_\alpha\ \Lambda^k (E-\Lambda)^{-1} M^* M_\alpha \alpha(s) ds$$
$$+\frac{t}{T}\int_t^T M_\alpha\ \Lambda^k (E-\Lambda)^{-1} M^* M_\alpha \alpha(s) ds]$$
$$= \Lambda^{k+1}(E-\Lambda)^{-1} M^* M_\alpha \alpha(t).$$

By induction, inequality (2.16) is true for $m = 0,1,2,\cdots$.
Thus from (2.14) and (2.16), we have:

$$\lim_{m\to\infty} |r(t,\gamma(t,\alpha),x_0) - x_m(t,\gamma(t,\alpha),x_0)| = 0$$

and hence

$$\lim_{m\to\infty} x_m(t,\gamma(t,\alpha),x_0) = r(t,\gamma(t,\alpha),x_0)$$

By using the relation (2.14), we get:

$$x(t,\gamma(t,\alpha),x_0) = r(t,\gamma(t,\alpha),x_0)$$

i.e. $x(t,\gamma(t,\alpha),x_0)$ is a unique solution of (1.1) on the domain (1.2). ■

3. Existence of solution.

The problem of existence of a periodic solution of period T of the system (1.1) is uniquely connected with the existence of zeros of the function $\Delta(0,\gamma(0,\alpha),x_0)$ which has the form:

$$\Delta(0,\gamma(0,\alpha),x_0) = \frac{1}{T}\int_0^T f(t,\gamma(t,\alpha), x^0(t,\gamma(t,\alpha),x_0)dt \ldots\ldots\ldots\ldots \cdots (3.1)$$

where $x^0(t,\gamma(t,\alpha),x_0)$ is the limiting function of the sequence of functions (2.1).
The function (3.1) can be determined only approximately, say by computing the following functions:

$$\Delta_{\mathrm{m}}(0,\gamma(0,\alpha),x_0) = \frac{1}{T}\int_0^T f(t,\gamma(t,\alpha), x_m(t,\gamma(t,\alpha),x_0)dt \ \ldots\ldots\ldots \cdots (3.2)$$
$$m = 0,1,2,\cdots.$$

Now, we prove the following theorem.

Theorem3.1. If all assumptions and conditions of theorem 2.1 are hold, then the inequality:

$$|\Delta(0,\gamma(0,\alpha),x_0) - \Delta_{\mathrm{m}}(0,\gamma(0,\alpha),x_0)| \le \Lambda^{m+1}(E-\Lambda)^{-1}\ M\ M_\alpha \cdots (3.3)$$

will be satisfied for all $m \ge 0, x_0 \in D_{\gamma f}$.

Proof. From relations (3.1) and (3.2), the estimate

$$|\Delta(0,\gamma(0,\alpha),x_0) - \Delta_{\mathrm{m}}(0,\gamma(0,\alpha),x_0)|$$
$$\le \frac{K}{T}\int_0^T |\gamma(t,\alpha)|\ |x^0(t,\gamma(t,\alpha),x_0) - x_m(t,\gamma(t,\alpha),x_0)|dt$$

$$\leq \frac{K}{T}\int_0^T M_\alpha \Lambda^m (E-\Lambda)^{-1} M M_\alpha \alpha(t)dt$$
$$= \Lambda^{m+1}(E-\Lambda)^{-1} M M_\alpha$$

Thus the inequality (3.2) is hold for all $m \geq 0$.

Next, we prove the following theorem taking into account that the inequality (3.3) will be satisfied for all $m \geq 0$.

Theorem 3.2. If the system (1.1) satisfies the following condition

(i) The sequence of functions (3.2) has an isolated singular point $x_0 = x^0$, $\Delta_{\rm m}(0,\gamma(0,\alpha),x^0) \equiv 0$, for all $t \in R^1$.

(ii) The index of this point is nonzero;

(iii) There exists a closed convex domain D_γ^* belonging to domain $D_{\gamma f}$ and possessing a unique singular point x^0 such that on its boundary $\Gamma_{D_\gamma^*}$ the following inequality hold

$$inf_{x_0\in\Gamma_{D_\gamma^*}} \|\Delta_{\rm m}(t,\gamma(t,\alpha),x_0)\| \geq \|\Lambda^m(E-\Lambda)^{-1} M M_\alpha\| \cdots \quad (3.4)$$

Where $x_0 \in \Gamma_{D_\gamma^*}$ for all $m \geq 0$. Then the system (1.1) has a periodic solution $x = x(t,\gamma(t,\alpha),x_0)$ for which $x(0,\gamma(0,\alpha),x_0)$ belongs to the domain D_γ^* .

Proof. By using the inequality (3.1) the proof is similar to that of theorem 7.1[4].

Remark 3.1.[4]. When $R^n = R^1$, i.e. when $x_0$is a scalar, the existence of solution can be strengthens by giving up the requirement that the singular point shout be isolated, thus we have

Theorem3.3. Let the system of nonlinear differential equations (1.1) are defined on the interval $[a,b]$. Suppose that for $m \geq 0$, the function $\Delta_{\rm m}(0,\gamma(0,\alpha),x_0)$ defined according to formula (3.2) satisfies the inequalities:

$$\left.\begin{aligned} \min_{a+h\leq x_0\leq b-h} \|\Delta_{\rm m}(t,\gamma(t,\alpha),x_0)\| \leq -\sigma_m \quad ; \\ \max_{a+h\leq x_0\leq b-h} \|\Delta_{\rm m}(t,\gamma(t,\alpha),x_0)\| \geq \sigma_m \quad . \end{aligned}\right\} \cdots (3.5)$$

Then the system (1.1) has a periodic solution $x = x(t,\gamma(t,\alpha)x_0)$ for which $x_0 \in [a+h, b-h]$, where $h = \|M M_\alpha\| \frac{T}{2}$ and $\sigma_m = \|\Lambda^{m+1}(E-\Lambda)^{-1} M M_\alpha\|$.

Proof. Let x_1 and x_2 be any two points on the interval $[a,b]$ such that:

$$\left.\begin{aligned} \Delta_{\rm m}(0,\gamma(0,\alpha),x_1) = \min_{a+h\leq x_0\leq b-h} \Delta_{\rm m}(0,\gamma(0,\alpha),x_0) \quad ; \\ \Delta_{\rm m}(0,\gamma(0,\alpha),x_2) = \max_{a+h\leq x_0\leq b-h} \Delta_{\rm m}(0,\gamma(0,\alpha),x_0) \quad . \end{aligned}\right\} \cdots (3.6)$$

From the inequalities (3.3) and (3.5), we have

$$\left.\begin{aligned} \Delta(0,\gamma(0,\alpha),x_1) = \Delta_{\rm m}(0,\gamma(0,\alpha),x_1) + [\Delta(0,\gamma(0,\alpha),x_1) - \Delta_{\rm m}(0,\gamma(0,\alpha),x_1)] \\ \Delta(0,\gamma(0,\alpha),x_2) = \Delta_{\rm m}(0,\gamma(0,\alpha),x_2) + [\Delta(0,\gamma(0,\alpha),x_2) - \Delta_{\rm m}(0,\gamma(0,\alpha),x_2)] \end{aligned}\right\}$$
$$\cdots (3.7)$$

It follows from the inequalities (3.7) and the continuity of the function $\Delta(0,\gamma(0,\alpha),x_0)$, that there exist an isolated singular point $x^0, x^0 \in [x_1,x_2]$, such that $\Delta(0,\gamma(0,\alpha),x_0) \equiv 0$, this means that the system (1.1) has a periodic solution $x = x(t,\gamma(t,\alpha),x_0)$ for which $x_0 \in [a+h, b-h]$. ■.

Theorem3.4. If the function $\Delta(0,\gamma(0,\alpha),x_0)$ is defined by

$\Delta: D_{\gamma f} \to R^n$,

$$\Delta(0,\gamma(0,\alpha),x_0) = \frac{1}{T}\int_0^T f(t,\gamma(t,\alpha),x^0(t,\gamma(t,\alpha),x_0)dt \ldots\ldots \cdots (3.7)$$

where$x^0(t,\gamma(t,\alpha),x_0)$ is a limit of the sequence of functions (2.1). Then the following inequalities are holds

$|\Delta(0,\gamma(0,\alpha),x_0)| \leq \mathrm{M}\,\mathrm{M}_\alpha \cdots (3.8)$
and

$$|\Delta(0,\gamma(0,\alpha),x_0{}^1)-\Delta(0,\gamma(0,\alpha),x_0{}^2)| \leq \frac{2}{T}\Lambda(E-\Lambda)^{-1}M_\alpha \cdots (3.9)$$

for all $x_0, x_0{}^1,\ x_0{}^2 \in D_{\gamma f}$.

Proof. From the properties of the function $x^0(t,\gamma(t,\alpha),x_0)$ as in theorem 2.1; it follows that the function $\Delta(0,\gamma(0,\alpha),x_0)$ is continuous and bounded by $M\ M_\alpha$.

By using (3.7), we get:

$$|\Delta(0,\gamma(0,\alpha),x_0{}^1)-\Delta(0,\gamma(0,\alpha),x_0{}^2)| = \left|\frac{1}{T}\int_0^T f(t,\gamma(t,\alpha),x^0(t,\gamma(t,\alpha),x_0{}^1)dt\right.$$

$$-\frac{1}{T}\int_0^T [f(t,\gamma(t,\alpha),x^0(t,\gamma(t,\alpha),x_0{}^2)ds]ds$$

$$\leq \frac{K}{T}\int_0^T M_\alpha\ |x^0(t,\gamma(t,\alpha),x_0{}^1)-x^0(t,\gamma(t,\alpha),x_0{}^2)|dt$$

$$\leq M_\alpha K\frac{T}{2}.\frac{2}{T}|x^0(t,\gamma(t,\alpha),x_0{}^1)-x^0(t,\gamma(t,\alpha),x_0{}^2)|dt$$

$$=\frac{2}{T}\Lambda|x^0(t,\gamma(t,\alpha),x_0{}^1)-x^0(t,\gamma(t,\alpha),x_0{}^2)|$$

and hence

$$|\Delta(0,\gamma(0,\alpha),x_0{}^1)-\Delta(0,\gamma(0,\alpha),x_0{}^2)|$$

$$\leq \frac{2}{T}\ \Lambda\ |x^0(t,\gamma(t,\alpha),x_0{}^1)-x^0(t,\gamma(t,\alpha),x_0{}^2)|M_\alpha \cdots (3.10)$$

where $x_0{}^1(t,\gamma(t,\alpha),x_0)$ and $x_0{}^2(t,\gamma(t,\alpha),x_0)$ are solutions of the integral equation:

$$x(t,\gamma(t,\alpha),x_0{}^k) = x_0{}^k + \int_0^t [f(s,\gamma(s,\alpha),x(s,\gamma(s,\alpha),x_0{}^k)$$

$$-\frac{1}{T}\int_0^T (f(s,\gamma(s,\alpha),x(s,\gamma(s,\alpha),x_0{}^k)ds]ds \cdots (3.11)$$

with

$x_0{}^k(t,\gamma(t,\alpha),x_0) = x_0{}^k\ ,\qquad k=1,2.$

From (3.11), we have:

$$|x^0(t,\gamma(t,\alpha),x_0{}^1)-x^0(t,\gamma(t,\alpha),x_0{}^2)| \leq |x_0{}^1-x_0{}^2|$$

$$+K[\left(1-\frac{t}{T}\right)\int_0^t M_\alpha\ |x^0(s,\gamma(s,\alpha),x_0{}^1)-x^0(s,\gamma(s,\alpha),x_0{}^2)|ds$$

$$+\frac{t}{T}\int_t^T M_\alpha\,|x^0(s,\gamma(s,\alpha),x_0{}^1)-x^0(s,\gamma(s,\alpha),x_0{}^2)|ds]$$

$$\leq |x_0{}^1-x_0{}^2| + M_\alpha K\frac{T}{2}|x^0(t,\gamma(t,\alpha),x_0{}^1)-x^0(t,\gamma(t,\alpha),x_0{}^2)|\alpha(t)$$

$$\leq |x_0{}^1-x_0{}^2| + \Lambda|x^0(t,\gamma(t,\alpha),x_0{}^1)-x^0(t,\gamma(t,\alpha),x_0{}^2)|$$

thus

$|x^0(t,\gamma(t,\alpha),x_0{}^1)-x^0(t,\gamma(t,\alpha),x_0{}^2)| \leq (E-\Lambda)^{-1}|x_0{}^1-x_0{}^2| \cdots (3.12)$

using the inequality (3.12) in (3.10), we get (3.9).

Remark 3.2.[2]. The theorem 3.4 ensure the stability solution of the system (1.1) when there is a slight change on the point x_0 accompanied with noticeable change in the function $\Delta(0, \gamma(0, \alpha), x_0)$.

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الملخص

يتضمن البحث دراسة الحل الدوري لنظم المعادلات التناضلية اللأخطية المعتمدة على توزيع كاما وذللك بأستخدام الطريقة التحليلية _العددية لدراسة الحلول الدورية للمعادلات التفاضلية المعطاه في المرجع Samoilenko. هذه الدراسة تقودنا الى تحسين وتوسيع هذه الطريقة وكذلك تم توسيع نتائج Samoilenko وذلك بتحويل المعادلات التفاضلية اللاخطية الدورية الى معادلات تفاضلية لأخطية دورية معتمدة على توزيع كاما