THE ZERO DIVISOR GRAPH OF THE RING Z_{qp}

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ABSTRACT:

In this paper we construct a star zero divisor graph from the zero divisor graph of the ring Z_{qp}. The star zero divisor graph is obtained by removing some vertices from the zero divisor graph $\Gamma(Z_{qp})$, in different ways , but the best way to get star zero divisor graph $S\Gamma(Z_{qp})$ is by removing an odd number of zero divisors. Finally the crossing number, the girth and the diameter of this graph is also determined.

KEYWORDS: *Commutative rings, zero divisor graphs, star zero divisor graph, crossing number, girth of the graph.*

1-INTRODUCTION

Let R be a commutative ring with identity and let Z(R) be the set of zero- divisors of R. We associate a simple graph $\Gamma(R)$ to R with zero divisor vertices $Z^*(R)=Z(R)-\{0\}$ the set of non-zero zero divisors of R, and for distinct x, y $\in Z^*(R)$, the vertices x and y are adjacent if and only if xy = 0 as zero divisor elements. Note that $\Gamma(R)$ is empty if and only if R is an integral domain, and the zero divisor graph is always simple undirected connected graph.

The concept of a zero divisor graph was first introduced by Beck [5] in (1988), further studied by many authors like [1, 2,3,5,8,9] and [10]. In this work we consider the ring Z_{pq}, where p and q are primes, and we construct a zero divisor star graph which will be denoted by $S\Gamma(R)$, where R $\cong Z_{pq}$.

1-1: The zero divisor graph of the ring Z_n, n=3p, p is prim.

The zero divisor set $Z^*(Z_{3p}) = \{3, 6, 9, \ldots, 3(p-1), p, 2p\}$, which contains p+1 elements. This type of zero divisor graphs are special graphs since they contains only two types of vertices V_1 and V_2 such that $V(Z_{3p}) = V_1 \cup V_2$, where $V_1=\{p, 2p\}$ and $V_2=\{3, 6, 9, \ldots, 3(p-1)\}$. The vertices in V_1 (p and 2p) are of degree (p-1) and they are adjacent with all other vertices of the zero divisor graph while p and 2p are non adjacent together, since $p.2p = 2p^2$ not divide, 3p while the vertices in V_2 has the same degree 2 since they are adjacent with both p and 2p but not adjacent with each other. Then, we can called these types of graphs by double star zero divisor graph (star graph has one vertex adjacent with all other vertices but $\Gamma(Z_{3p})$ has two vertices adjacent with all other vertices). In the other hand the two vertices p and 2p which has the same properties in the graph, the centers of the zero divisor graph $\Gamma(Z_{3p})$, C_1=p and C_2 =2p as shown In figure (1) below in the graph $\Gamma(Z_{3p})$, p=7:

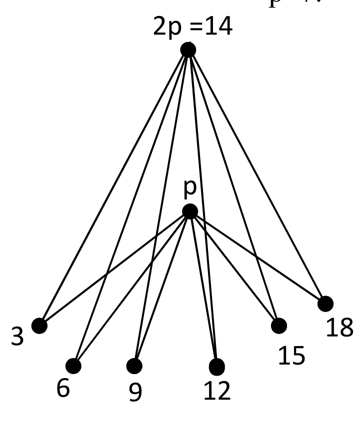

Fig. 1: The zero divisor graph (double star graph) $\Gamma(Z_{21})$

Next, we shall give the following example.

Example – 1: Let p= 11, then the zero divisor graph of the ring Z_{3p}, is $\Gamma(Z_{3p})=\Gamma(Z_{3.11})=\Gamma(Z_{33})$, and the zero divisor set is defined as $Z^*(Z_{33})$={3, 6, 9, 12, 15, 18, 21, 24, 27, 11, 22, 33, 44. As shown in figure(2) bellow, this set contains two types of vertices $V_1$and V_2, which contains the vertices p , 2p and the vertices u_i, i=1, 2,…, (p-1) respectively, where u_i represented the vertices 3, 6, …, 3(p-1).

Observe that, deg(p) =deg(2p) =p-1 =11-1 =10
deg(3)=deg(6)= deg(9) = … =deg(3(p-1)) = 2.
Now up to the adjacency these vertices are divided in to two partite sets:
V_1={3, 6, 9, 12, 15, 18, 21, 24, 27, 30}, they are 10 vertices of degree two, all adjacent with the centers C_1 and C_2.
V_2= {p,2p} ={11, 22}, they are two vertices of degree 10, for this we call it the double star graph.

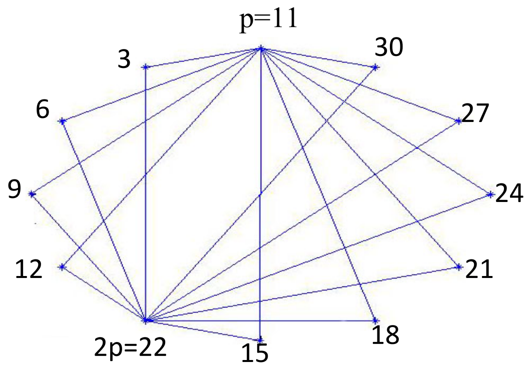

Fig. 2: The zero divisor graph of Z_{33}

1-2: Construction of star zero divisor graph $\Gamma(Z_{3p})$.

In this section we construct a star zero divisor graph from the zero divisor graph $\Gamma(Z_{3p})$. Since the elements in the zero divisor set are only two kinds of different degree and the vertices p, 2p have the same degree (p-1) and they are adjacent with all other vertices in type two, then we can easily construct a star graph from the zero divisor graph $\Gamma(Z_{3p})$.

Theorem 1-2-1: The zero divisor graph $\Gamma(Z_{3p})$ is star graph by removing one of its center vertices.

Proof: The zero divisor graph $\Gamma(Z_{3p})$ have two centers p and 2p with greatest degree in the graph since they are adjacent with all other vertices u_i, i from 1 to (P-1) and non adjacent together, since p.2p $=2p^2\neq 0$ as a zero divisor element. Then removing one center element p or 2p the all other vertices u_is will be end vertices after removing one incident edges between vertices u_i and center, means that we remover only one vertex from the zero divisor graph $\Gamma(Z_{3p})$ to get star zero divisor graph, $S_{1,m}$, but the total number of zero divisors in $Z^*((Z_{3p}))$ is equal p+1, implies that m = p-1 and the star zero divisor graph is $S_{1,p-1}$as shown in the figure (3).

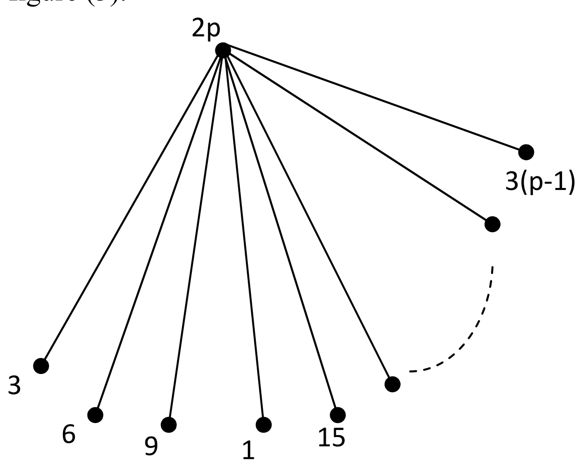

Fig. 3: The star zero divisor graph $S_{1,p-1}$.

Definition 1-2-2: [Akbari 2013] A **bipartite graph** (or bigraph) is a graph whose vertices can be divided into two disjoint sets V_1 and V_2 , that is, V_1 and V_2 are each independent sets (no any vertices are adjacent) such that every edge connects a vertex in V_1 to one in V_2. Vertex sets V_1 and V_2 are often denoted as partite sets.

Definition 1-2-3: [Gubta 2013] A **complete bipartite graph** is a graph whose vertices can be partitioned into two subsets V_1 and V_2 such that no edge has both endpoints in the same subset, and every possible edge that could connect vertices in different subsets is part of the graph. That is, it is a bipartite graph (V_1, V_2, E) such that for every two vertices $v_1 \in V_1$ and $v_2 \in V_2$, v_1v_2 is an edge in E. A complete bipartite graph with partitions of size $|V_1|$=m and $|V_2|$=n, is denoted $K_{m,n}$.

In the mathematical field of graph theory, a complete bipartite graph or biclique is a special kind of bipartite graph where every vertex of the first set is connected to every vertex of the second set.

Theorem 1-2-4 : The zero divisor graph $\Gamma(Z_{3p})$ is complete bipartite graph $K_{2,p-1}$.

Proof: According to the two centers of the zero divisor graph, C_1 and C_2 , where $C_1 . C_2 \neq 0$, means that are non adjacent together but C_1 and C_2 are adjacent with all other vertices u_i, i=1, 2, 3,…, (p-1) (u_i represented the second type of vertices 3, 6, 9, …, 3(p-1)) and they are also non adjacent together, $u_i. u_j \neq 0$ for all i, j and i, j = 1, 2, 3, …, (p-1). Then by the behavior of the zero divisor elements we can divide the vertex set $V(\Gamma(Z_{3p}))$ in to two disjoint part V_1 and V_2 such that $V = V_1 \cup V_2$, where V_1 = {p, 2p} and V_2= { u_i, i = 1, 2, …, (p-1)} . Up to the properties of the zero divisor graph $\Gamma(Z_{3p})$ the vertices in the set V_1(contains two elements) is adjacent with all the vertices u_i in the other set V_2 (contains (p-1) elements)but the vertices in each vertex set V_1 nor V_2 are adjacent together, implies that the zero divisor graph $\Gamma(Z_{3p})$ is complete bipartite graph $K_{2,(p-1)}$ (by definition of complete bipartite).

Example -2:

In the following figure of the zero divisor graph $\Gamma(Z_{33})$, p=11 is a complete bipartite graph $K_{2,10}$.

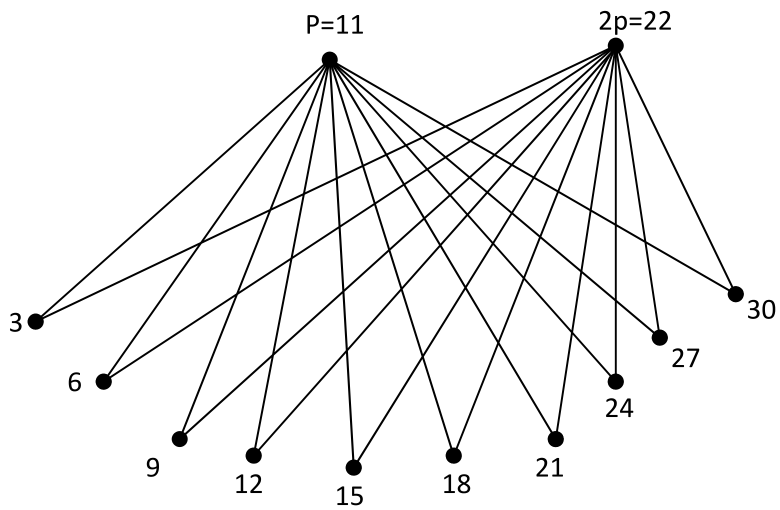

Fig. 4: The zero divisor graph $\Gamma(Z_{33}) = K_{2,10}$

With two partite sets V_1 ={11,22} and V_2={3, 6, 9, 12, 15, 18, 21, 24, 27, 30} , $V_1 \cup V_2$ =V =$Z^*(Z_{3p})$. Now in V_1 we have 11.22=242 $\neq$ 0, means they are non adjacent, so the vertices in V_2 are non adjacent since $u_i.u_j \neq 0$ for all i, j, u_i, $u_j \in V_2$, while each vertex of V_1 is adjacent with all the vertices in V_2 and vice versa.

Theorem 1-2-5 : The girth of $\Gamma(Z_{3p})$ is 4.

Proof: The zero divisor set of the zero divisor graph $\Gamma(Z_{3p})$ is Z*(Z3p) = { 3, 6, 9, …, 39P-1), P, 2P}. Let u_1, u_2, u_3, …, u_n represent the first part of vertices 3, 6, 9, …, 3(p-1) and v_1, v_2 represent the vertices p and 2p respectively. Then all the cycles in the graph $\Gamma(Z_{3p})$ are of length 4 , they are the smallest and they are of the form $v_1 \to u_i \to v_2 \to u_j \to v_1$ for $i \neq j$, i, j =1, 2,…, n. Then the girth is 4.

The chromatic number [Hararry 1969] of the zero divisor graph $\Gamma(Z_{3p})$ is defined by the next theorem.

Theorem 1-2-6 : The chromatic number of $\Gamma(Z_{3p})$ is equal 2.

Proof: Since no two adjacent vertices take the same color in coloring vertices in any graph, then p, 2p have any color and all other vertices u_i, i=1, 2, …, n which represent the vertices 3, 6, 9, …, 3(P-1) has another color. Therefore the total number of colors used in the coloration of this graph is 2, implies that the chromatic number is 2.

Example-3: The chromatic number of the zero divisor graph $\Gamma(Z_{3p})$ is two as shown in the figure (5) bellow:

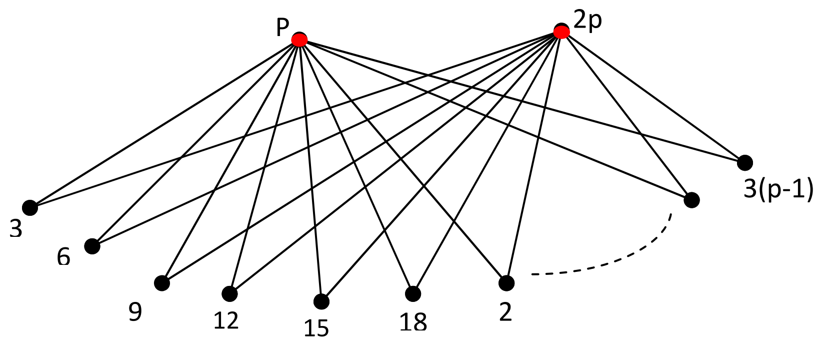

Fig. 5: The chromatic number of $\Gamma(Z_{3p})$ =2 $_{2,10}$

The vertices p and 2p have the red color and the other vertices have the black color since the adjacent vertices must have different color in the coloring vertices in any graph.

In the next we determine the crossing number of the graph $\Gamma(Z_{3p})$.

Lemma 1-2-7: The crossing number of the zero divisor graph $\Gamma(Z_{3p})$ is zero.

Proof: Clearly if we put the vertices p and 2p on the positive and negative y-axis and other vertices u_i on the two sides of x-axis, after connecting all the adjacent vertices, the graph $\Gamma(Z_{3p})$ is planar graph and there is no crossing edges in the zero divisor graph as shown in figure (6) bellow.

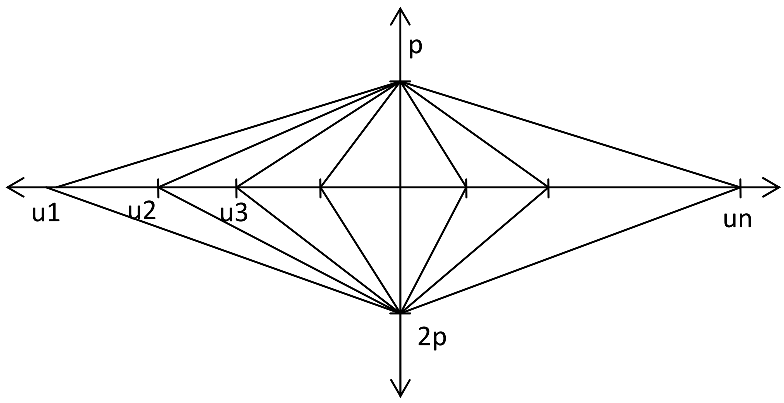

Fig. 6: The crossing number of $\Gamma(Z_{3p})$.

2-The zero divisor graph of Z_{5p}, p prime, p>5

The zero divisor graph of the ring Z_{5p} , contains four center vertices have the same property and same degree (p-1), since they are adjacent with all other vertices in the zero divisor graph $\Gamma(Z_{5p})$ except themselves, they have the greatest degree among all vertices.

Example -4: If p=7, then $Z_{5.7}$=Z_{35}, the zero divisor elements of Z_{35} are as follow: $Z^*(Z_{35})$ = { 5, 10, 15, 20, 25, 30, 7,14, 21, 28}, it contains two types of vertices 5n, n from 1 to 6 all are of degree 4,and the vertices {7, 14, 21, 28} are of degree 6.

If p=11, $Z_{5.11}$=Z_{55}, $Z^*(Z_{55})$ = {5, 10, 15, 20, 25, 30, 35, 40, 45, 50, 11, 22, 33, 44} are 14 vertices of two types:

V_1= { 5, 10, 15, 20, 25, 30, 35, 40, 45, 50} each of degree 4 and V_2 ={ 7, 14, 21, 28} each of degree 10 , each vertex in V_1 is adjacent with all vertices in V_2, so each vertex in V_2 is adjacent with all vertices of V_1 as shown in figure (7) bellow:

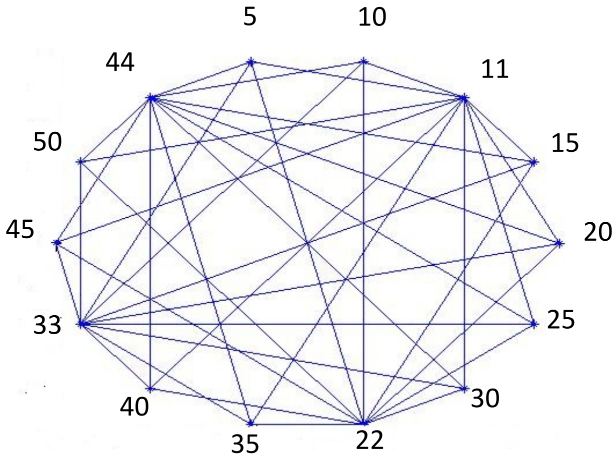

Fig. 7: The zero divisor graph $\Gamma(Z_{55})$

In general the set of non-zero zero divisor elements of the graph $\Gamma(Z_{5p})$ is :

$Z^*(Z_{5p})$ = { 5, 10, 15, 20, … , 5(p-1), p, 2p, 3p, 4p }, clearly this zero divisor graph contains two type of vertices V_1={ 5, 10, 15, …, 5(p-1)}, they are (p-1) vertices all of degree 4, and V_2={p, 2p, 3p, 4p} are four vertices of degree (p-1).

2-1: Construction of star zero divisor graph of $\Gamma(Z_{5p})$.

The zero divisor graph $\Gamma(Z_{5p})$ contains four vertices {p, 2p, 3p, 4p} f degree (p-1) , are the greatest degree vertices in the graph since they are adjacent with all vertices in the zero divisor set except the multiple of p, then we can call this zero divisor graph by 4- star graph, and called the set of these vertices center of $\Gamma(Z_{5p})$.

Definition 2-1-1: [Buckly and Harary 1990] The **distance** between two vertices u and v, denoted d(u,v), is the length of a shortest u−v path, also called a u−v **geodesic**. The distance function is a metric on the vertex set of a graph G. In particular, it satisfies the triangle inequality: d(a, b) ≤ d(a, c) + d(c, b) for all vertices a, b, c of G. This follows from the fact that, if you want to go from a to b, then one possibility is to go via vertex c.

The **diameter** of a connected graph G, denoted diam(G), is the maximum distance between two vertices. The **eccentricity** of a vertex is the maximum distance from it to any other vertex. The **radius**, denoted rad(G), is the minimum eccentricity among all vertices of G. Of course the diameter is the maximum eccentricity among all vertices.

Lemma 2-1-2: The zero divisor graph $\Gamma(Z_{5p})$ is complete bipartite graph $K_{4,(p-1)}$.

Proof: Follows from Theorem 1.2.4, where q=5, p>q, as shown in figure (8).

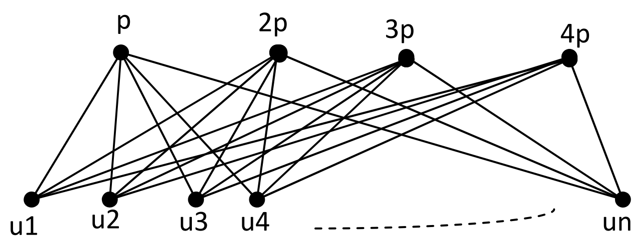

Fig. 8: zero divisor graph $\Gamma(Z_{5p}) = K_{4,(p-1)}$

Theorem 2-1-3: The zero divisor graph $\Gamma(Z_{5p})$ is star graph by removing the vertices 2p, 3p, and 4p.
Proof: Its obvious, the remind vertices are p with ui and the graph will by star graph $S_{1,n}$, where n= (p-1) as shown in figure (9).

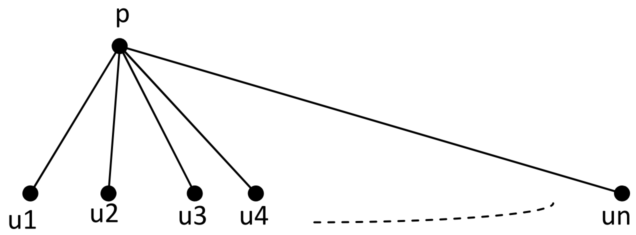

Fig. 9: The star zero divisor graph $K_{1,(p-1)}$

Clearly the chromatic number of $\Gamma(Z_{5p})$ is equal 2, and the girth of the zero divisor graph $\Gamma(Z_{5p})$ is four.

Theorem 2-1-4: The crossing number $Cr(Z_{5p})$ is equal $2\left\lfloor\frac{p-1}{2}\right\rfloor\left\lfloor\frac{p-2}{2}\right\rfloor$.

Proof: Since the vertices in the zero divisor set of the graph $\Gamma(Z_{5p})$ contains two different partite subset of vertices ui and v_j, i=1, 2, 3, 4, j=1, 2, 3, …, p-1. u_i represented the center vertices p, 2p, 3p, and 4p and v_j represented the other vertices 5, 10, 15, …, 5(p-1). The vertices in each partite set are non adjacent while the vertices of each set are adjacent with other vertices in the other set. Now to find the crossing number we arrange the vertices ui and v_j on the x and y-axes in the way that u_1, u_2 and u_3, u_4 on the +ve and –ve part of y-axes respectively, while we order the vertices v_j on both side of x-axes equally, i.e. $v_j/2$ in +ve and vj/2 in –ve side of x-axes. Then we get 4- equal quadratic part of xy-plane.

Now we complete the proof by mathematical induction, first for if p=7, clearly the zero divisor graph is $\Gamma(Z_{5p}) = \Gamma(Z_{5.7})\ \Gamma(Z_{35})$ and the set of the zero divisors is $Z^*(Z_{5p})$ ={ 7, 14, 21, 28, 5, 10, 15, 20, 25, 30} contains two partite sets of vertices ui= p, 2p, 3p, 4p =7, 14, 21, 28 and vj = 5, 10, 15, 20, 25, 30.

So we put u_1=p=7, u_2=2p=14 on the upper side (+ve) of y-axes and u_3=3p=21, u_4=4p=28 on the lower side (-ve) of y-axes, and the vertices vj are six vertices, clearly we must put three vertices on each side (+ve and –ve) of x-axes as shown in the figure(4) below. Clearly we get two equal upper and lower side in xy-plane, and the crossing number of the zero divisor graph $\Gamma(Z_{5p})$ is $Cr(\Gamma(Z_{5p}))$ = $\sum_{i=1}^{6} Cr(vj, ui)$, the summation is of over all the vertices ui and vj for i = 1, 2, 3, 4 and j = 1, 2, 3, 4, 5, 6, now to calculate the crossing number of the zero divisor graph $\Gamma(Z_{5p})$ we connect the vertices ui with vj as follow: first where we connect the six vertices vj with the vertex u1=p we get no crossing but when we connect vj with u2 = 2p in the upper side of xy- plane we get the crossing between vertices as the follow:

$Cr(vj, u_1) = 0$, $Cr(v_3,u_2) = Cr(v_4, u_2)=0$
$Cr(v_2,u_2) = Cr(v_5, u_2)=1$ and $Cr(v_1,u_2) = Cr(v_6, u_2)=2$

Then $Cr(\Gamma(Z_{5p}) = \sum_{i=1}^{6} Cr(vj,ui) = 2(1+2) = 2.3 = \left\lfloor\frac{7-1}{2}\right\rfloor\left\lfloor\frac{7-2}{2}\right\rfloor = \left\lfloor\frac{p-1}{2}\right\rfloor\left\lfloor\frac{p-2}{2}\right\rfloor$ in the upper side of xy-plane, but we have two equal sides in xy-plane, therefore $Cr(\Gamma(Z_{5p}) = 2\left\lfloor\frac{p-1}{2}\right\rfloor\left\lfloor\frac{p-2}{2}\right\rfloor$.

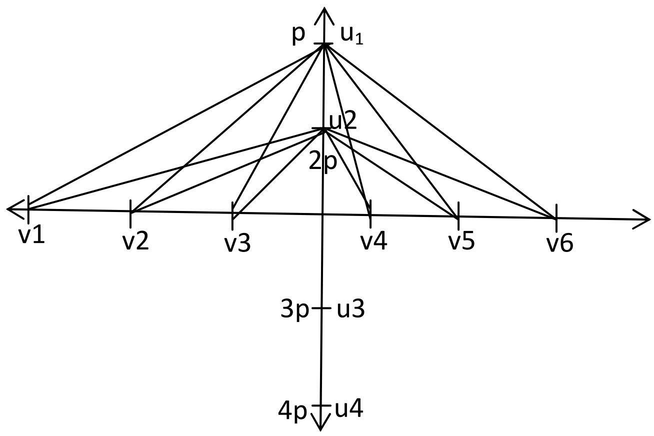

Fig. 10: the crossing number of $\Gamma(Z_{5p})$

In general for v_j, j=1, 2, ….,p-1, for any prime p the vertices of y-axes still four u_1, u_2, u_3, and u_4 (two in each side), while the vertices vj on the x-axes are increases by changing p and the number of crossing Cr(v_j, u_1) =Cr(v_j,u_4) =0 but the crossing number of vj with u_1 and u_2 is increases as 1, 2, 3, 4, …, for j.

Remark: The zero divisor graph $\Gamma(Z_{7p})$, p>7 is called 6- star graph since it has six centers p, 2p, 3p, 4p, 5p, and 6p, when they are adjacent with all other vertices in the zero divisor graph $\Gamma(Z_{7p})$. This zero divisor graph $\Gamma(Z_{7p})$ is also complete bi-partite graph $K_{6,(p-1)}$ as shown in the figure (11) bellow :

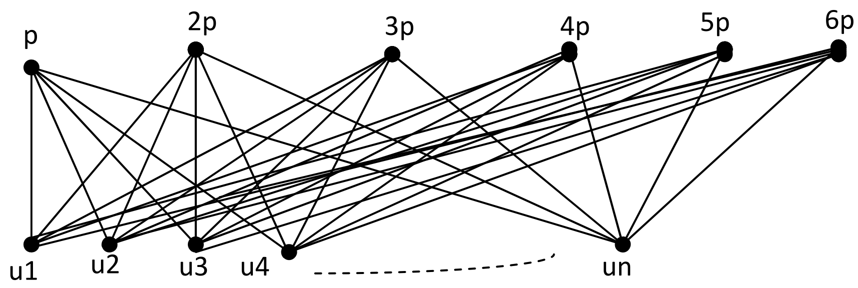

Fig. 11: zero divisor graph $\Gamma(Z_{7p}) = K_{6,(p-1)}$

Using the same proceed as for q =3, 5, and 7, the crossing number of $\Gamma(Z_{7p})$ is 6 $\left\lfloor \frac{p-1}{2} \right\rfloor \left\lfloor \frac{p-2}{2} \right\rfloor$ as shown in figure (12) bellow:

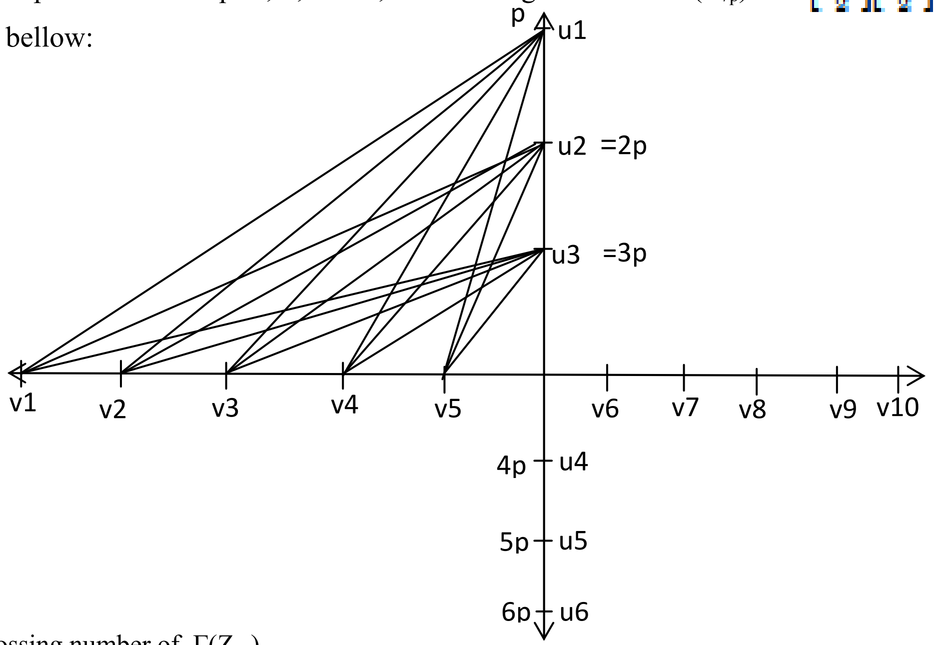

Fig. 12: the crossing number of $\Gamma(Z_{7p})$

Note that the crossing number of the zero divisor graph $\Gamma(Z_{11p})$ is 20 $\left\lfloor \frac{p-1}{2} \right\rfloor \left\lfloor \frac{p-2}{2} \right\rfloor$

3- The zero divisor graph of Z_{qp}, q, p are primes with p> q.

In general the zero divisor graph $\Gamma(Z_{qp})$ has the zero divisor set $Z^*(\Gamma(Z_{qp}))$ ={p, 2p, 3p, …, (q-1)p, q, 2q, 3q, …, (p-1)q} which has two partite sets V_1 ={p, 2p, 3p, …, (q-1)p} contains (q-1) vertices each of degree (p-1) and V_2 ={q, 2q, 3q, …, (p-1)q} contains (p-1) vertices of degree (q-1).

The vertices in V_1 and V_2 are non adjacent together but all the vertices in V_1 are adjacent with all the vertices of V_2.

Next we shall give the following results, by considering q>p in the zero divisor graph $\Gamma(Z_{qp})$.

<u>Theorem 3-1:</u> The zero divisor graph $\Gamma(Z_{qp})$ is complete bi-partite graph of order (p+q-2).

Proof: The zero divisor graph $\Gamma(Z_{qp})$ contains two type of vertices np, n from 1 to (q-1) and mq, m from 1 to (p-1) of different degree, then we can divided the vertex set of the graph in to distinct two partite sets V_1 and V_2 each contain the vertices np, mq respectively.

The vertices in V_1 are non adjacent together, $np.kp=nkp^2$ which does not divide qp (q is prime certainly not equal nk, for any n,k in Z+), also the vertices in V_2 are non adjacent together, mq.hq=mhq2 not divided qp (p is prime certainly not equal mh, for any m,h in Z+). Clearly the vertex set is consist of only two partite set and the vertices in both are non-adjacent together which Achieves bipartite condition in any graph, so the zero divisor graph Γ(Zqp) is bipartite graph.

Now since the vertices in V_1 are adjacent with all other vertices in V_2, implies that the graph is complete bipartite set as shown in figure (13) below.

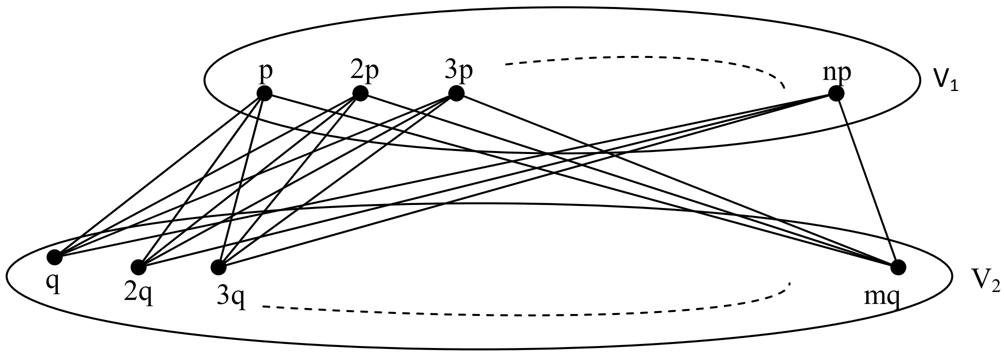

Fig. 13: The zero divisor graph $\Gamma(Z_{qp}) = K_{(p-1),(q-1)}$

Theorem 3-2: The zero divisor graph $\Gamma(Z_{qp})$ is star graph $S_{1,(p-1)}$, by removing the vertices 2p, 3p, …, (q-1)p.

Proof: Since the vertices in the zero divisor graph are np, mq, n from 1 to (q-1) and m from 1 to (p-1) , where deg(np) = (q-1) and deg(mq) =(p-1), since each vertex of the form np is adjacent with all the vertices mq and vise versa as shown in figure (10), therefore when we remove all the vertices np except p for example, then we get star graph $S_{1,(p-1)}$.

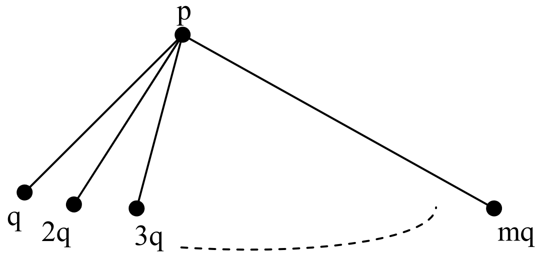

Fig. 14: The star zero divisor graph $S_{1,(p-1)}$

Theorem 3-3: The crossing number of the zero divisor graph $\Gamma(Z_{qp})$ is $(\frac{q-1}{2})(\frac{q-2}{2}) \left\lfloor\frac{p-1}{2}\right\rfloor\left\lfloor\frac{p-2}{2}\right\rfloor$.

Proof: The proof is by mathematical induction on p and q for p>q.

If q=5 and p >5, we have $Cr(Z_{qp}) = 2 \left\lfloor\frac{p-1}{2}\right\rfloor\left\lfloor\frac{p-2}{2}\right\rfloor$ as proved in Theorem(2-1-4).

If q=7 and p >7, then the crossing number $Cr(Z_{qp})= 6 \left\lfloor\frac{p-1}{2}\right\rfloor\left\lfloor\frac{p-2}{2}\right\rfloor$ as proved before.

If q=11, p >11, the crossing number Cr(Zqp) $=20 \left\lfloor\frac{p-1}{2}\right\rfloor\left\lfloor\frac{p-2}{2}\right\rfloor$

If q=13, p> 13, we get $Cr(Z_{qp}) =30 \left\lfloor\frac{p-1}{2}\right\rfloor\left\lfloor\frac{p-2}{2}\right\rfloor$, and so on we get the result for all p and all q.

While the chromatic number of the zero divisor graph $\Gamma(Z_{qp})$ is two, since the zero divisor graph $\Gamma(Z_{qp})$ is complete bi partite, then each partite set takes a different color from the other, then we use only two color in the zero divisor graph, implies that the chromatic number is two.

Theorem 3-4: The diameter of the zero divisor graph $\Gamma(Z_{qp})$ is equal or less than 2, i.e. $diam(\Gamma(Z_{qp})) \leq 2$.

Proof: Let Z* be the set of all zero divisors of the zero divisor graph $\Gamma(Z_{qp})$, then $Z^*=V(Z_{pq}) =V_1 \cup V_2$, where V_1 and V_2 are two partite set of the zero divisor graph $\Gamma(Z_{qp})$ containing the vertices {p,2p,…,np}, {q, 2q, …, mq} respectively.

If u, v ϵ Z*, then we have the following cases:

Case(1): If uv=0 then u is adjacent with v thus u ─ v is a path of length one and d(u,v) =1, so diam(u,v)=1.

Case(2): Suppose uv ≠ 0, implies that u and v are non adjacent, so they are in the same partite set say V_1, thus there exist zero divisor element w ϵ Z*/ {u, v}such that uw=0 and vw=0,(w must be in the different partite set with u and v), then u─ w ─ v is a path of length 2 in the graph which implies d(u, v) =2, and diam(u,v)=2.

Case (3): If uw=0, implies u and ware in different partite set(w ϵ V_2), then there exist an element z ϵ Z*/ {u, v, w} such that zw≠0, if uz =0 or vz=0, then u─ w ─ v ─ z is a path of length three, thus d(u,z) =3. In the similar way the same result hold if vw=0.

Case(4): Since the zero divisor graph Γ(Zqp) is complete bipartite graph, clearly there exist an element y ϵ Z*/ {u, v} in the same partite set with u and v such that zy =0 and wy=0 , then the length of the path between u and v is u ─ w ─ y ─ z ─ v is a path of length 4, and d(u,v)=4.

Thus d(u, v) ≤ 4 for all distinct vertices u,v ϵ Z*, hence the zero divisor graph $\Gamma(Z_{qp})$ is connected graph with $diam(\Gamma(Z_{qp})) \leq 4$.

Remark: In the case of the zero divisor graph Γ(Zqp) where q=p>2, then $\Gamma(Z_{qp}) = \Gamma(Z_{p2})$ is special graph called complete graph of order (p-1), that is the zero divisor graph $\Gamma(Z_{p2})$ has (p-1) vertices each of degree (p-2) as shown in the example bellow:

Example-5: Conceder p=q=11, then the zero divisor graph is $\Gamma(Z_{p2})= \Gamma(Z_{121})$

$Z^*(\Gamma(Z_{p2}))$={ 11, 22, 33, 44, 55, 66, 77, 88, 99, 110}, the order of this graph is (p-1) =10, means it contains 10 vertices and all the vertices has degree (p-2) = 9, and all the vertices are adjacent to each together, then the zero divisor graph is a complete graph of order 10 and denoted by K_{p-1} as shown in the figure (15) .

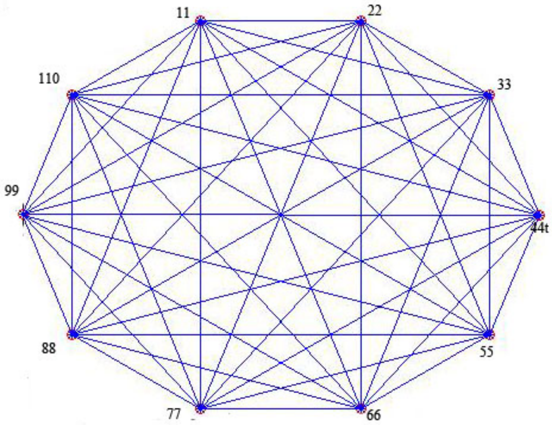

Fig. 15: The zero divisor graph $\Gamma(Z_{p2})$ is a complete graph

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پوخته:

لەم تویژینەوەیەدا باسی پیکهینانی گرافی ئەستیرە لە گرافی کولکەی سفری بۆ جۆغەزی Zqp کراوە. پیکهینانی گرافی ئەستیرە لە ریگای لابردنی چەند سەریکی هەلبژیردراو لە گرافەکە بە چەند ریگای جیاواز، باشترینیان بۆ بەدەست هینانی گرافی ئەستیرە $S\Gamma(Z_{qp})$ لابردنی سەرە تاکەکان وچەندچارەکانی. هەروەها ژمارەی یەکتربرەکانی گراف (یەکتربرینی لاکان) وپیوانەو تیرەی گرافی کولکەی سفریمان دۆزیەوە بۆ جۆزەغی$\Gamma(Z_{qp})$.

الخلاصة :

في هذا البحث تم دراسة تكوين بيان النجمة من البيان قاسم الصفرللحلقة Zpq، أن تكوين بيان النجمة من البيان قاسم الصفر تنتج عن طريق حذف بعض الرؤوس المختارة من البيان بطرق مختلفة لكن افضل طريقة للحصول على بيان نجمة قاسم الصفر$S\Gamma(Z_{qp})$ هي بحذف عنصر القاسم الصفر الفردية ومضاعفاتها. كما وجدنا درجة التقاطع، المقاس و القطر للبيان قاسم الصفرللبيان $\Gamma(Z_{qp})$.