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Original Article

CONVEXITY WITH SOFT APPROXIMATE FUNCTION UNDER FUZZY SOFT EXPERT SET ENVIRONMENT

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ABSTRACT

Soft set theory is considered the best tool to deal with problems involving uncertainties by controlling all the complexities of optimization theory, fuzzy set theory, and probability theory. By utilising this theory, the majority of soft set-like models have been developed to address issues in decision-making and medical diagnosis with a single expert. This is a challenge for researchers who employ questionnaires. This issue is solved by soft expert sets, which enable the decision-maker to access the opinions of all experts in a single model. In this study, some types of convexity, like strictly (strongly) convexity, (λ, ψ)-convexity with different senses over fuzzy soft expert set, have been described with the numerical examples. Convex hull and convex cone on fuzzy soft expert set have been developed, and some important results have been proved by applying the algebraic approach.

Keywords: Convexity, Fuzzy set, Fuzzy logic, Soft computing, Convex hull.

1. INTRODUCTION

There are many complications in different branches of science such as medicine, engineering, and economics that involve certain uncertainties. To deal with these uncertainties, different theories, such as probability and fuzzy set theory, remain unsatisfactory. The main reason for this insufficiency is the lack of a parameterization tool in these theories. In this situation, Molodtsov (1999) took the initiative and introduced the idea of the soft set (SS), which proved very helpful as a parameterization tool for dealing with uncertainties in many problems. The flexibility and effectiveness of this model lie in the fact that there is no such impediment to its application.

In order to further develop the algebraic structures in uncertainty modeling, Sezgin and Atagün (2023) presented a unique operation in soft set theory known as the complementary soft binary piecewise plus (+) operation. In order to increase the adaptability and expressive capacity of soft set frameworks in mathematical modeling and decision-making under uncertainty, this study is part of a larger research trend that defines new soft set operations. By developing and redefining a number of important operations on soft sets, Ali *et al.* (2009) produced one of the most significant foundational contributions to soft set theory. They demonstrated that some classical features, such as distributive behaviors and De Morgan's laws, do not always apply under the definitions of soft set operations proposed in previous research. The authors suggested new and enhanced operational definitions, such as restricted union, restricted intersection, restricted difference, and extended intersection of soft sets, to address these problems.

Zadeh (1965) extended the crisp set by introducing a new set called the fuzzy set. He made changes to the crisp set by assigning a membership degree to each element. Maji *et al.* (2001) integrated the work of Zadeh with the soft set and introduced a new concept called fuzzy soft set theory. By examining the basic functions and characteristics of fuzzy soft sets (FSS), Borah *et al.* (2012) advanced the field of fuzzy soft set theory. Instead of using Maji's original complement notion, the authors based their discussion on an alternative definition of the complement operation introduced by Neog and Sut. They concentrated on examining and assessing existing operations within the fuzzy soft set framework. By investigating crucial operations, including union, intersection, complement, and related properties, all of which are important for expanding fuzzy soft set models in uncertain decision-making environments, their study sought to enhance the algebraic structure of fuzzy soft sets.

In order to improve the mathematical flexibility of fuzzy soft sets, Das *et al.* (2022) introduced a number of novel operational structures that significantly advanced fuzzy soft set theory. The authors examined the basic characteristics of the new operations—Einstein sum, Einstein product, algebraic sum, algebraic product, bounded sum, and bounded product—in the context of fuzzy soft sets. These procedures enhance the ability of fuzzy soft models to handle complex decision-making situations and provide more tools for managing uncertain information. By presenting and examining a structured framework in which the set of parameters is furnished with an order relation, Aslam *et al.* (2019) expanded the theory of fuzzy soft sets. The qualities (parameters) in many real-world decision-making issues are not independent but rather adhere to a certain hierarchy or priority order, which serves as the inspiration for this work. This ordering information is not adequately captured by conventional fuzzy soft set models. Çelik and Yamak (2013) applied this theory to medical diagnosis by using fuzzy arithmetic operations. Xiao *et al.* (2009) developed a forecasting approach for this theory. Gogoi *et al.* (2014) applied this structure to day-to-day problems, especially decision problems. Tanay and Kandemir (2011) introduced topological structures into this model for the first time. Kong *et al.* (2018) introduced some new parameter reduction methods in this model.

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By presenting a fuzzy soft set-based method for resolving decision-making issues in ambiguous and imprecise settings, Roy and Maji (2007) significantly advanced fuzzy soft set theory. By adding fuzzy membership values to the traditional soft set structure, the authors were able to capture gradual and ambiguous information. They used fuzzy soft set parameters in their work to offer a comparison table-based method for object detection and decision-making. The suggested method offered a mathematical framework for managing multi-observer data with various characteristics and viewpoints. Roy and Maji's (2007) work laid a crucial foundation for later fuzzy soft set research, especially in the fields of hybrid soft computing frameworks, decision analysis, and uncertainty modeling. Their methodology showed that, by simultaneously handling ambiguity and parameterization, fuzzy soft sets may successfully combine the benefits of fuzzy sets and soft sets.

By examining the relationship between these two uncertainty-handling frameworks, Liu *et al.* (2019) offered a significant theoretical study examining the formal link between soft set theory and fuzzy set theory. Every fuzzy set defined on a universe can be represented as a soft set, according to the authors, and a soft set can also be understood as a fuzzy set under certain circumstances. This paper offered a fresh viewpoint on the relationship between membership-based uncertainty (fuzzy sets) and parameterized uncertainty (soft sets). By presenting a formal framework that combines the benefits of fuzzy sets and soft sets for processing ambiguous and imprecise information, Çağman *et al.* (2011) significantly advanced fuzzy soft set theory. The authors established a mathematical basis for conveying uncertainty through both parameterization and membership degrees by defining fuzzy soft sets and examining their basic characteristics. By adding fuzzy membership values to classical soft set theory, their study enhanced the capacity of soft models to explain intricate real-world issues involving vagueness.

One of the first and fundamental studies on the algebraic structure and operations of fuzzy sets was conducted by Mizumoto and Tanaka (1981), who made a substantial contribution to the mathematical advancement of fuzzy set theory. The basic operations of fuzzy sets, such as union, intersection, complement, and different t-norm- and t-conorm-based operations, were systematically evaluated in their study. Also, the behavior of these operations under various mathematical formulations was investigated.

It is impossible to account for the opinions of different decision-makers (experts) in a soft set-like model. Soft set-like models have been constructed for a single expert opinion. Alkhazaleh and Salleh (2011) addressed this limitation by introducing the soft expert set (SES). Alkhazaleh and Salleh (2014) embedded this structure into fuzzy set theory to form the fuzzy soft expert set (FSES). Hassan *et al.* (2017) applied this model to the prediction of coronary artery disease.

The soft expert set paradigm was critically reexamined by Khan *et al.* (2025) with an emphasis on its theoretical coherence and usefulness in multi-criteria decision-making (MCDM) scenarios. The authors pointed out basic discrepancies in the initial SES definition put forth by Alkhazaleh's work, emphasizing in particular how many common SES operations, such as union, intersection, and subset relations, do not completely correspond with the underlying mathematical structure of SESs.

In order to simulate complicated and multi-layered uncertainty in decision-making situations, Qayyum *et al.* (2016) presented the idea of cubic soft expert sets (CSESs), a substantial generalization of both soft expert sets and cubic sets. Integrating three distinct uncertainty representations—fuzzy sets, interval-valued fuzzy sets, and expert parameterization—into a single framework is the fundamental concept of cubic soft expert sets. Compared to traditional soft or fuzzy soft sets, this hybrid structure enables the model to capture more realistic and detailed information.

In order to create a more adaptable framework for managing uncertainty in complicated decision-making contexts, Alkhazaleh and Salleh (2014) presented the idea of the fuzzy soft expert set (FSES), a major extension of both soft expert sets and fuzzy soft sets. This study's main goal was to represent both qualitative (expert-based) and quantitative (fuzzy) uncertainty at the same time by combining expert views and fuzzy membership grades into a single mathematical framework. Al-Qudah and Hassan (2017) introduced the concept of bipolar fuzzy soft expert sets in this theory with some examples. Adam and Hassan (2016) introduced a new structure by combining this model with multi-Q fuzzy sets to form multi-Q fuzzy soft expert sets. They also generalized certain operators to ensure their applicability to decision-making problems.

2. RESEARCH GAP AND MOTIVATION

Convexity has various applications in operations research and mathematical economics, including optimization, pattern identification and classification, image processing and decomposition, discrete event simulation, duality problems, and many other related areas. Deli (2019) provided a mathematical tool to solve such difficulties in soft set surroundings. Soft expert set has been theorized by combining soft set with classical expert set. Soft expert set, an extension of soft set, is more advantageous as it provides the estimations of professionals in a single model. In order to incorporate the literature on classical convexity into such a situation, a need for the development of a novel model is observed that may tackle convexity-related problems with the management of uncertainties. One may wonder why convexity is studied. Convexity is a highly useful tool for understanding and solving optimization problems; thus, the solution is quite straightforward. In the past, linear programs were primarily concerned with optimization. Initially, it was thought that problems could be broadly divided into two groups: linear and nonlinear optimization problems. Subsequently, researchers discovered that the appropriate distinction between convex and non-convex problems was more suitable because some nonlinear problems were perceived as challenging by others. Deli presented the notion of convexity in relation to fuzzy soft set and soft set. He used operations such as union, intersection, and complement to demonstrate some significant findings.

Later on, Majeed (2016) worked on soft set and presented convex hull and cone using these structures. Salih and Sabir (2020) took an interest in soft set and introduced new characteristics of convexity, such as strict (strong) convexity (concavity). Rahman *et al.* (2021a) introduced some new features of convexity, such as (m, n)-convexity and (m, n)-convexity in the first (second) sense. Rahman *et al.* (2020) modified certain features of convexity using different structures, such as soft set and fuzzy soft set, and hypersoft set. Rahman *et al.* (2021c) embedded this idea in a refined fuzzy environment. Ihsan *et al.* (2021) embedded the concept of convexity in the fuzzy soft expert set environment.

Even though the idea of convexity in fuzzy soft expert sets offers a helpful geometric framework for managing uncertainty in expert-based decision-making, the traditional definition of convexity is still too limited to adequately represent the richness and diversity of real-world decision settings. Expert opinions are frequently not evenly distributed in real-world multi-criteria decision-making problems, and the relationships between options may show different degrees of curvature, strength, and interaction that traditional convexity alone is unable to effectively explain. Within the context of fuzzy soft expert sets, more adaptable and broader types of convexity are needed to overcome these constraints. More precise control over the degree and type of convex behavior is made possible by concepts like strict (strong) convexity and (λ, ψ)-convexity. This allows for the modeling of sharper preference structures and parameter-dependent convex interactions among expert judgments. The broader mathematical structure offered by these modified convexity types more accurately captures the graded and subjective character of expert knowledge under fuzziness. Moreover, the structural behavior of solution spaces in optimization and choice problems is not adequately captured by traditional convexity. Convex hull and convex cone growth in fuzzy soft expert sets thus becomes crucial. The convex cone adds scalability and directional expansion, which are crucial in optimization-based decision frameworks, while the convex hull offers the smallest convex fuzzy soft expert set containing a given collection of choices. Inspired by these constraints, this study introduces and analyzes several forms of convexity, as well as convex hull and convex cone constructions, in an effort to expand the theory of fuzzy soft expert sets. These enhancements improve

fuzzy soft expert sets' applicability in challenging decision-making scenarios while also enhancing their algebraic structure. The effectiveness and interpretability of the presented notions are further supported by illustrative numerical examples in the proposed framework.

The work is structured as follows: (λ,ψ)-convexity (concavity) and related results are presented in Section 3. Sections 4, 5, and 6 present the notions of certain types of convexity with related theorems. Section 7 discusses the concept of convex hull and cone. Section 8 presents the comparative analysis and discussion. Finally, Section 9 presents the conclusion and future directions.

3. PRELIMINARIES

This section contains some valuable definitions and important terms from literature, representing the main study.

Definition 3.1. (Zadeh, 1965)
A fuzzy set X over $\mathcal{Z}$ (the universe of discourse) is defined as a collection of pairs $X=\{\langle z,\zeta_x(z)\rangle : z\in\mathcal{Z}\}$, where $\zeta_x:\mathcal{Z}\to[0,1]$ represents the satisfaction (membership) function of X.

Definition 3.2. (Molodtsov, 1999)
Let $P(\mathcal{Z})$ denote the power set of $\mathcal{Z}$, and let F be a collection of parameters defining $\mathcal{Z}$. A soft set over $\mathcal{Z}$ is an ordered pair (h_M,F) defined by the mapping $h_M:F\to P(\mathcal{Z})$.

Definition 3.3. (Çelik & Yamak, 2013)
A pair (Δ,F) represents a fuzzy soft set, where Δ is a mapping given by $\Delta:F\to I^{\mathcal{Z}}$, and $I^{\mathcal{Z}}$ represents the set of all fuzzy subsets of $\mathcal{Z}$.

Definition 3.4. (Alkhazaleh & Salleh, 2011)
Let a triplet be defined by $\zeta = F\times\beth\times\Xi$, with $S\subseteq\zeta$, where the symbol $\beth$ represents a set of experts, while Ξ represents the set of conclusions (opinions). A pair (Π,S) is known as a soft expert set over $\mathcal{Z}$, where Π is a mapping given by $\Pi:S\to P(\mathcal{Z})$. The opinion set is described as $\Xi=\{0=\text{disagree},1=\text{agree}\}$.

Definition 3.5. (Khan *et al.*, 2025)
Let $\overline{h}$ be a mapping given by $\overline{h}:S\to I^{\mathcal{Z}}$. Then, a pair $(\overline{h},S)$ can be considered a fuzzy soft expert set over $\mathcal{Z}$, where the notation $I^{\mathcal{Z}}$ in the mapping represents the collection of all fuzzy subsets of $\mathcal{Z}$.

Definition 3.6. (Khan *et al.*, 2025)
The union and intersection operations between fuzzy soft expert sets $(\Re_1,\Lambda)$ and $(\Re_2,\Upsilon)$ are characterized as follows:
For the union operation, we have $(\mho,\rtimes)$, where $\rtimes=\Lambda\cup\Upsilon$, and for all $\dashv\in\rtimes$,

$$\mho(\dashv)=\begin{cases}\Re_1(\dashv) & ;\ \dashv\in\Lambda\setminus\Upsilon,\\ \Re_2(\dashv) & ;\ \dashv\in\Upsilon\setminus\Lambda,\\ s(\Re_1(\dashv),\Re_2(\dashv)) & ;\ \dashv\in\Lambda\cap\Upsilon\end{cases}$$

while s represents a s-norm.
For intersection operation, we have $(\mho,\rtimes)$ with $\rtimes=\Lambda\cap\Upsilon$, and for all $\dashv\in\rtimes$,

$$\mho(\dashv)=\begin{cases}\Re_1(\dashv) & ;\ \dashv\in\Lambda\setminus\Upsilon,\\ \Re_2(\dashv) & ;\ \dashv\in\Upsilon\setminus\Lambda,\\ t(\Re_1(\dashv),\Re_2(\dashv)) & ;\ \dashv\in\Lambda\cap\Upsilon\end{cases}$$

where t represents a t-norm.

Definition 3.7. (Ihsan *et al.*, 2021)
The fuzzy soft expert set over S is called a convex fuzzy soft expert set if

$$h_{F_{ses}}(\lambda\mu_1+(1-\lambda)\mu_2)\ \supseteq\ h_{F_{ses}}(\mu_1)\cap h_{F_{ses}}(\mu_2)$$

for each $\mu_1,\mu_2\in S$ and $\lambda\in\Gamma^{\bullet}=[0,1]$.

Example 3.8. Suppose that a BATA company produces new types of products and wants to take the opinions of some experts about these products. Let $\mathcal{Z}=\{\text{œ}_1,\text{œ}_2,\text{œ}_3,\text{œ}_4\}$ be a set of products, and $F=\{p_1,p_2,p_3\}$ be a set of decision parameters, where $p_i\,(i=1,2,3)$ denotes the parameters price, brand, and weight. Let $\beth=\{\overline{h}_1,\overline{h}_2,\overline{h}_3\}$ be an experts set and $S=F\times\beth\times\Xi$. We get 18 triplets. Suppose that

$$\begin{aligned}
\daleth_1&=\daleth(p_1,\overline{h}_1,1)=\left\{\tfrac{\text{œ}_1}{0.1},\tfrac{\text{œ}_2}{0.7},\tfrac{\text{œ}_3}{0.4},\tfrac{\text{œ}_4}{0.1}\right\},\\
\daleth_2&=\daleth(p_1,\overline{h}_2,1)=\left\{\tfrac{\text{œ}_1}{0.3},\tfrac{\text{œ}_2}{0.8},\tfrac{\text{œ}_3}{0.1},\tfrac{\text{œ}_4}{0.2}\right\},\\
\daleth_3&=\daleth(p_1,\overline{h}_3,1)=\left\{\tfrac{\text{œ}_1}{0.6},\tfrac{\text{œ}_2}{0.4},\tfrac{\text{œ}_3}{0.5},\tfrac{\text{œ}_4}{0.3}\right\},\\
\daleth_4&=\daleth(p_2,\overline{h}_1,1)=\left\{\tfrac{\text{œ}_1}{0.8},\tfrac{\text{œ}_2}{0.4},\tfrac{\text{œ}_3}{0.6},\tfrac{\text{œ}_4}{0.3}\right\},\\
\daleth_5&=\daleth(p_2,\overline{h}_2,1)=\left\{\tfrac{\text{œ}_1}{0.4},\tfrac{\text{œ}_2}{0.8},\tfrac{\text{œ}_3}{0.3},\tfrac{\text{œ}_4}{0.2}\right\},\\
\daleth_6&=\daleth(p_2,\overline{h}_3,1)=\left\{\tfrac{\text{œ}_1}{0.5},\tfrac{\text{œ}_2}{0.3},\tfrac{\text{œ}_3}{0.6},\tfrac{\text{œ}_4}{0.8}\right\},\\
\daleth_7&=\daleth(p_3,\overline{h}_1,1)=\left\{\tfrac{\text{œ}_1}{0.2},\tfrac{\text{œ}_2}{0.9},\tfrac{\text{œ}_3}{0.4},\tfrac{\text{œ}_4}{0.5}\right\},\\
\daleth_8&=\daleth(p_3,\overline{h}_2,1)=\left\{\tfrac{\text{œ}_1}{0.4},\tfrac{\text{œ}_2}{0.6},\tfrac{\text{œ}_3}{0.7},\tfrac{\text{œ}_4}{0.9}\right\},\\
\daleth_9&=\daleth(p_3,\overline{h}_3,1)=\left\{\tfrac{\text{œ}_1}{0.7},\tfrac{\text{œ}_2}{0.3},\tfrac{\text{œ}_3}{0.5},\tfrac{\text{œ}_4}{0.2}\right\},\\
\daleth_{10}&=\daleth(p_1,\overline{h}_1,0)=\left\{\tfrac{\text{œ}_1}{0.3},\tfrac{\text{œ}_2}{0.2},\tfrac{\text{œ}_3}{0.4},\tfrac{\text{œ}_4}{0.1}\right\},\\
\daleth_{11}&=\daleth(p_1,\overline{h}_2,0)=\left\{\tfrac{\text{œ}_1}{0.1},\tfrac{\text{œ}_2}{0.9},\tfrac{\text{œ}_3}{0.6},\tfrac{\text{œ}_4}{0.2}\right\},\\
\daleth_{12}&=\daleth(p_1,\overline{h}_3,0)=\left\{\tfrac{\text{œ}_1}{0.2},\tfrac{\text{œ}_2}{0.1},\tfrac{\text{œ}_3}{0.3},\tfrac{\text{œ}_4}{0.5}\right\},\\
\daleth_{13}&=\daleth(p_2,\overline{h}_1,0)=\left\{\tfrac{\text{œ}_1}{0.7},\tfrac{\text{œ}_2}{0.3},\tfrac{\text{œ}_3}{0.5},\tfrac{\text{œ}_4}{0.7}\right\},\\
\daleth_{14}&=\daleth(p_2,\overline{h}_2,0)=\left\{\tfrac{\text{œ}_1}{0.7},\tfrac{\text{œ}_2}{0.2},\tfrac{\text{œ}_3}{0.9},\tfrac{\text{œ}_4}{0.4}\right\},\\
\daleth_{15}&=\daleth(p_2,\overline{h}_3,0)=\left\{\tfrac{\text{œ}_1}{0.5},\tfrac{\text{œ}_2}{0.7},\tfrac{\text{œ}_3}{0.3},\tfrac{\text{œ}_4}{0.2}\right\},\\
\daleth_{16}&=\daleth(p_3,\overline{h}_1,0)=\left\{\tfrac{\text{œ}_1}{0.1},\tfrac{\text{œ}_2}{0.4},\tfrac{\text{œ}_3}{0.6},\tfrac{\text{œ}_4}{0.8}\right\},\\
\daleth_{17}&=\daleth(p_3,\overline{h}_2,0)=\left\{\tfrac{\text{œ}_1}{0.2},\tfrac{\text{œ}_2}{0.8},\tfrac{\text{œ}_3}{0.7},\tfrac{\text{œ}_4}{0.3}\right\},\\
\daleth_{18}&=\daleth(p_3,\overline{h}_3,0)=\left\{\tfrac{\text{œ}_1}{0.4},\tfrac{\text{œ}_2}{0.3},\tfrac{\text{œ}_3}{0.5},\tfrac{\text{œ}_4}{0.1}\right\}.
\end{aligned}$$

The construction of Fses (H,S) can be made as

$$(H,S)=\left\{\begin{aligned}
\daleth_1&=\daleth(p_1,\overline{h}_1,1)=\left\{\tfrac{\text{œ}_1}{0.1},\tfrac{\text{œ}_2}{0.7},\tfrac{\text{œ}_3}{0.4},\tfrac{\text{œ}_4}{0.1}\right\},\\
\daleth_2&=\daleth(p_1,\overline{h}_2,1)=\left\{\tfrac{\text{œ}_1}{0.3},\tfrac{\text{œ}_2}{0.8},\tfrac{\text{œ}_3}{0.1},\tfrac{\text{œ}_4}{0.2}\right\},\\
\daleth_3&=\daleth(p_1,\overline{h}_3,1)=\left\{\tfrac{\text{œ}_1}{0.6},\tfrac{\text{œ}_2}{0.4},\tfrac{\text{œ}_3}{0.5},\tfrac{\text{œ}_4}{0.3}\right\},\\
\daleth_4&=\daleth(p_2,\overline{h}_1,1)=\left\{\tfrac{\text{œ}_1}{0.8},\tfrac{\text{œ}_2}{0.4},\tfrac{\text{œ}_3}{0.6},\tfrac{\text{œ}_4}{0.3}\right\},\\
\daleth_5&=\daleth(p_2,\overline{h}_2,1)=\left\{\tfrac{\text{œ}_1}{0.4},\tfrac{\text{œ}_2}{0.8},\tfrac{\text{œ}_3}{0.3},\tfrac{\text{œ}_4}{0.2}\right\},\\
\daleth_6&=\daleth(p_2,\overline{h}_3,1)=\left\{\tfrac{\text{œ}_1}{0.5},\tfrac{\text{œ}_2}{0.3},\tfrac{\text{œ}_3}{0.6},\tfrac{\text{œ}_4}{0.8}\right\},\\
\daleth_7&=\daleth(p_3,\overline{h}_1,1)=\left\{\tfrac{\text{œ}_1}{0.2},\tfrac{\text{œ}_2}{0.9},\tfrac{\text{œ}_3}{0.4},\tfrac{\text{œ}_4}{0.5}\right\},\\
\daleth_8&=\daleth(p_3,\overline{h}_2,1)=\left\{\tfrac{\text{œ}_1}{0.4},\tfrac{\text{œ}_2}{0.6},\tfrac{\text{œ}_3}{0.7},\tfrac{\text{œ}_4}{0.9}\right\},\\
\daleth_9&=\daleth(p_3,\overline{h}_3,1)=\left\{\tfrac{\text{œ}_1}{0.7},\tfrac{\text{œ}_2}{0.3},\tfrac{\text{œ}_3}{0.5},\tfrac{\text{œ}_4}{0.2}\right\},\\
\daleth_{10}&=\daleth(p_1,\overline{h}_1,0)=\left\{\tfrac{\text{œ}_1}{0.3},\tfrac{\text{œ}_2}{0.2},\tfrac{\text{œ}_3}{0.4},\tfrac{\text{œ}_4}{0.1}\right\},\\
\daleth_{11}&=\daleth(p_1,\overline{h}_2,0)=\left\{\tfrac{\text{œ}_1}{0.1},\tfrac{\text{œ}_2}{0.9},\tfrac{\text{œ}_3}{0.6},\tfrac{\text{œ}_4}{0.2}\right\},\\
\daleth_{12}&=\daleth(p_1,\overline{h}_3,0)=\left\{\tfrac{\text{œ}_1}{0.2},\tfrac{\text{œ}_2}{0.1},\tfrac{\text{œ}_3}{0.3},\tfrac{\text{œ}_4}{0.5}\right\},\\
\daleth_{13}&=\daleth(p_2,\overline{h}_1,0)=\left\{\tfrac{\text{œ}_1}{0.7},\tfrac{\text{œ}_2}{0.3},\tfrac{\text{œ}_3}{0.5},\tfrac{\text{œ}_4}{0.7}\right\},\\
\daleth_{14}&=\daleth(p_2,\overline{h}_2,0)=\left\{\tfrac{\text{œ}_1}{0.7},\tfrac{\text{œ}_2}{0.2},\tfrac{\text{œ}_3}{0.9},\tfrac{\text{œ}_4}{0.4}\right\},\\
\daleth_{15}&=\daleth(p_2,\overline{h}_3,0)=\left\{\tfrac{\text{œ}_1}{0.5},\tfrac{\text{œ}_2}{0.7},\tfrac{\text{œ}_3}{0.3},\tfrac{\text{œ}_4}{0.2}\right\},
\end{aligned}\right.$$

$$\urcorner_{16} = \urcorner(p_3, \bar{h}_1, 0) = \left\{\frac{\alpha_1}{0.1}, \frac{\alpha_2}{0.4}, \frac{\alpha_3}{0.6}, \frac{\alpha_4}{0.8}\right\},$$
$$\urcorner_{17} = \urcorner(p_3, \bar{h}_2, 0) = \left\{\frac{\alpha_1}{0.2}, \frac{\alpha_2}{0.8}, \frac{\alpha_3}{0.7}, \frac{\alpha_4}{0.3}\right\},$$
$$\urcorner_{18} = \urcorner(p_3, \bar{h}_3, 0) = \left\{\frac{\alpha_1}{0.4}, \frac{\alpha_2}{0.3}, \frac{\alpha_3}{0.5}, \frac{\alpha_4}{0.1}\right\}\Bigg\}.$$

Example 3.9. Referring to the Example (3.8). Suppose
$S_1 = \{\urcorner(p_1, \bar{h}_1, 1), \urcorner(p_1, \bar{h}_2, 1), \urcorner(p_3, \bar{h}_2, 0)\}$, and
$S_2 = \{\urcorner(p_1, \bar{h}_1, 1), \urcorner(p_1, \bar{h}_2, 1), \urcorner(p_3, \bar{h}_2, 1)\}$.
Let (H_1, S_1) and (H_2, S_2) be two fuzzy soft expert sets such that

$$(H_1, S_1) = \Bigg\{ \urcorner_1 = \urcorner(p_1, \bar{h}_1, 1) = \left\{\frac{\alpha_1}{0.1}, \frac{\alpha_2}{0.7}, \frac{\alpha_3}{0.4}, \frac{\alpha_4}{0.1}\right\},$$
$$\urcorner_2 = \urcorner(p_1, \bar{h}_2, 1) = \left\{\frac{\alpha_1}{0.3}, \frac{\alpha_2}{0.8}, \frac{\alpha_3}{0.1}, \frac{\alpha_4}{0.2}\right\},$$
$$\urcorner_{17} = \urcorner(p_3, \bar{h}_2, 0) = \left\{\frac{\alpha_1}{0.2}, \frac{\alpha_2}{0.8}, \frac{\alpha_3}{0.7}, \frac{\alpha_4}{0.3}\right\}\Bigg\}.$$

and

$$(H_2, S_2) = \Bigg\{ \urcorner_1 = \urcorner(p_1, \bar{h}_1, 1) = \left\{\frac{\alpha_1}{0.1}, \frac{\alpha_2}{0.8}, \frac{\alpha_3}{0.4}, \frac{\alpha_4}{0.2}\right\},$$
$$\urcorner_2 = \urcorner(p_1, \bar{h}_2, 1) = \left\{\frac{\alpha_1}{0.5}, \frac{\alpha_2}{0.7}, \frac{\alpha_3}{0.2}, \frac{\alpha_4}{0.3}\right\},$$
$$\urcorner_8 = \urcorner(p_3, \bar{h}_2, 1) = \left\{\frac{\alpha_1}{0.3}, \frac{\alpha_2}{0.4}, \frac{\alpha_3}{0.6}, \frac{\alpha_4}{0.8}\right\}\Bigg\}.$$

By applying the union operation (max), we get

$$(H_3, S_3) = \Bigg\{ \urcorner_1 = \urcorner(p_1, \bar{h}_1, 1) = \left\{\frac{\alpha_1}{0.1}, \frac{\alpha_2}{0.8}, \frac{\alpha_3}{0.4}, \frac{\alpha_4}{0.2}\right\},$$
$$\urcorner_2 = \urcorner(p_1, \bar{h}_2, 1) = \left\{\frac{\alpha_1}{0.5}, \frac{\alpha_2}{0.8}, \frac{\alpha_3}{0.2}, \frac{\alpha_4}{0.3}\right\},$$
$$\urcorner_8 = \urcorner(p_3, \bar{h}_2, 1) = \left\{\frac{\alpha_1}{0.3}, \frac{\alpha_2}{0.4}, \frac{\alpha_3}{0.6}, \frac{\alpha_4}{0.8}\right\},$$
$$\urcorner_{17} = \urcorner(p_3, \bar{h}_2, 0) = \left\{\frac{\alpha_1}{0.2}, \frac{\alpha_2}{0.8}, \frac{\alpha_3}{0.7}, \frac{\alpha_4}{0.3}\right\}\Bigg\}.$$

Example 3.10. Referring to the Example (3.8), and taking the same sets as described above, i.e., (H_1, S_1) and (H_2, S_2). By applying the intersection operation (min), we get

$$(H_3, S_3) = \Bigg\{ \urcorner_1 = \urcorner(p_1, \bar{h}_1, 1) = \left\{\frac{\alpha_1}{0.1}, \frac{\alpha_2}{0.7}, \frac{\alpha_3}{0.4}, \frac{\alpha_4}{0.1}\right\},$$
$$\urcorner_2 = \urcorner(p_1, \bar{h}_2, 1) = \left\{\frac{\alpha_1}{0.3}, \frac{\alpha_2}{0.7}, \frac{\alpha_3}{0.1}, \frac{\alpha_4}{0.2}\right\}\Bigg\}.$$

Definition 3.11. Concave fuzzy soft expert set (Ihsan *et al.*, 2021) The fuzzy soft expert set on S becomes a concave fuzzy soft expert set if

$$h_{F_{ses}}(\lambda\mu_1 + (1-\lambda)\mu_2) \subseteq h_{F_{ses}}(\mu_1) \cup h_{F_{ses}}(\mu_2)$$

for each $\mu_1, \mu_2 \in S$ and $\lambda \in \Gamma^{\bullet}$.

4. (λ, ψ)-CONVEX AND CONCAVE FUZZY SOFT EXPERT SETS

This section explains the definitions of the (λ, ψ)-convex and (λ, ψ)-concave fuzzy soft expert sets, along with examples to illustrate each.

Definition 4.1. Let S be a discrete parameter set of an Fses. We define a parameter discretization operator R that maps an intermediate parameter vector to its nearest admissible parameter vector in S, componentwise. Thus, for $\mu = (\mu_1, \mu_2, \ldots, \mu_n)$,

$$R(\mu) = (r(\mu_1), r(\mu_2), \ldots, r(\mu_n)),$$

where $r(\mu_i)$ denotes the nearest admissible value in S. For example, $R(0.75, 1, 0.75) = (1, 1, 1)$. Accordingly, when a convex combination produces a parameter vector outside S, the corresponding Fses is evaluated as $h_{F_{ses}}(\lambda\mu_1 + \psi(1-\lambda)\mu_2)$. To make sure the final parameter vector is a part of the discrete parameter space, the operator R is introduced. The convexity condition is therefore verified after admissible-parameter discretization; R itself is not assumed to preserve convexity.

Definition 4.2. A convex fuzzy soft expert set becomes a (λ, ψ)-convex fuzzy soft expert set if

$$h_{F_{ses}}(\lambda\mu_1 + \psi(1-\lambda)\mu_2) \supseteq h_{F_{ses}}(\mu_1) \cap h_{F_{ses}}(\mu_2)$$

for $\psi \in [0, 1]$, $\lambda \in (0, 1]$ and $\mu_1, \mu_2 \in S$.

Example 4.3. Referring to Example (3.8) with $S = \{a_1, a_2, a_3\} = \{1, 2, 3\}$, and $Y = \{s, t, u\} = \{1, 2, 3\}$,

$$a_1 = h_{F_{ses}}(1, 1, 1) = \left\{\frac{\alpha_1}{0.1}, \frac{\alpha_2}{0.7}, \frac{\alpha_3}{0.4}, \frac{\alpha_4}{0.1}\right\},$$
$$a_2 = h_{F_{ses}}(1, 2, 1) = \left\{\frac{\alpha_1}{0.3}, \frac{\alpha_2}{0.8}, \frac{\alpha_3}{0.1}, \frac{\alpha_4}{0.2}\right\}.$$

Take $\lambda = 0.5$, $\psi = 1$, and $\mu_1 = (1, 1, 1)$, $\mu_2 = (1, 2, 1)$. Then

$$\begin{aligned} h_{F_{ses}}(\lambda\mu_1 + \psi(1-\lambda)\mu_2) &= h_{F_{ses}}(0.5(1, 1, 1) + (1 - 0.5)(1, 2, 1)) \\ &= h_{F_{ses}}((0.5, 0.5, 0.5) + (0.5, 1.0, 0.5)) \\ &= h_{F_{ses}}(1, 1.5, 1). \end{aligned}$$

With the help of round off property, we get $(1, 2, 1)$. So

$$h_{F_{ses}}(1, 2, 1) = \left\{\frac{\alpha_1}{0.3}, \frac{\alpha_2}{0.8}, \frac{\alpha_3}{0.1}, \frac{\alpha_4}{0.2}\right\}$$

and

$$\begin{aligned} h_{F_{ses}}(\mu_1) \cap h_{F_{ses}}(\mu_2) &= h_{F_{ses}}(1, 1, 1) \cap h_{F_{ses}}(1, 2, 1) \\ &= \left\{\frac{\alpha_1}{0.1}, \frac{\alpha_2}{0.7}, \frac{\alpha_3}{0.4}, \frac{\alpha_4}{0.1}\right\} \\ &\quad \cap \left\{\frac{\alpha_1}{0.3}, \frac{\alpha_2}{0.8}, \frac{\alpha_3}{0.1}, \frac{\alpha_4}{0.2}\right\} \\ &= \left\{\frac{\alpha_1}{0.1}, \frac{\alpha_2}{0.7}, \frac{\alpha_3}{0.1}, \frac{\alpha_4}{0.1}\right\}. \end{aligned}$$

It is clear that

$$h_{F_{ses}}(\lambda\mu_1 + \psi(1-\lambda)\mu_2) \supseteq h_{F_{ses}}(\mu_1) \cap h_{F_{ses}}(\mu_2).$$

Definition 4.4. A concave fuzzy soft expert set is named as a (λ, ψ)-concave fuzzy soft expert set if

$$h_{F_{ses}}(\lambda\mu_1 + \psi(1-\lambda)\mu_2) \subseteq h_{F_{ses}}(\mu_1) \cup h_{F_{ses}}(\mu_2)$$

for $\psi \in [0, 1]$, $\lambda \in (0, 1]$ and $\mu_1, \mu_2 \in S$.

Definition 4.5. Suppose Ω be a Fses over $\mathcal{Z}$ and ρ be subset of $\mathcal{Z}$. Subsequently ρ-inclusion of Ω is defined as

$$\Omega^{\rho} = \{\mu \in F : g_{\Omega}(\mu) \supseteq \rho\}.$$

Theorem 4.6. *$h_1 \cap h_2$ is a (λ, ψ)-convex fuzzy soft expert sets while h_1 and h_2 are (λ, ψ)-convex Fsess.*
Proof. Assume that $\mu_1, \mu_2 \in S$ and $\lambda \in (0, 1]$, and $h_3 = h_1 \cap h_2$. Then

$$\begin{aligned} h_{F_{ses}(h_3)}(\lambda\mu_1 + \psi(1-\lambda)\mu_2) &= h_{F_{ses}(h_1)}(\lambda\mu_1 + \psi(1-\lambda)\mu_2) \\ &\quad \cap h_{F_{ses}(h_2)}(\lambda\mu_1 + \psi(1-\lambda)\mu_2). \end{aligned}$$

Since h_1 and h_2 are (λ, ψ)-convex Fsess, so

$$h_{F_{ses}(h_1)}(\lambda\mu_1 + \psi(1-\lambda)\mu_2) \supseteq h_{F_{ses}(h_1)}(\mu_1) \cap h_{F_{ses}(h_1)}(\mu_2) \quad (1)$$

$$h_{F_{ses}(h_2)}(\lambda\mu_1 + \psi(1-\lambda)\mu_2) \supseteq h_{F_{ses}(h_2)}(\mu_1) \cap h_{F_{ses}(h_2)}(\mu_2). \quad (2)$$

By using the operation intersection, we have

$$h_{F_{ses(h_3)}}(\lambda\mu_1+\psi(1-\lambda)\mu_2) \supseteq \big(h_{F_{ses(h_1)}}(\mu_1)\cap h_{F_{ses(h_1)}}(\mu_2)\big) \cap \big(h_{F_{ses(h_2)}}(\mu_1)\cap h_{F_{ses(h_2)}}(\mu_2)\big).$$

Then

$$h_{F_{ses(h_3)}}(\lambda\mu_1+\psi(1-\lambda)\mu_2) \supseteq h_{F_{ses(h_3)}}(\mu_1)\cap h_{F_{ses(h_3)}}(\mu_2).$$

□

Remark 4.7. If the collection of (λ,ψ)-convex Fsess is represented by $\{h_k : k \in \{1,2,3,\dots\}\}$, then their intersection represented by $\bigcap_{k\in I} h_k$ is also a (λ,ψ)-convex Fses.

Theorem 4.8. *z is a (λ,ψ)-convex Fses on S if and only if for each $\lambda\in(0,1]$ and $\rho\in P(\mathcal{Z})$, z^ρ is (λ,ψ)-convex Fses on S.*
Proof. Consider z as a (λ,ψ)-convex Fses. For $\mu_1,\mu_2\in S$ and $\rho\in P(\mathcal{Z})$, then $h_{F_{ses(z)}}(\mu_1)\supseteq\rho$ and $h_{F_{ses(z)}}(\mu_2)\supseteq\rho$. By the convexity of z, we have

$$h_{F_{ses(z)}}(\lambda\mu_1+\psi(1-\lambda)\mu_2) \supseteq h_{F_{ses(z)}}(\mu_1)\cap h_{F_{ses(z)}}(\mu_2),$$

so z^ρ is (λ,ψ)-convex Fses.

On converse, suppose that z^ρ is (λ,ψ)-convex Fses for each $\lambda\in(0,1]$. Then, for $\mu_1,\mu_2\in S$, z^ρ is (λ,ψ)-convex Fses, for $\rho = h_{F_{ses(z)}}(\mu_1)\cap h_{F_{ses(z)}}(\mu_2)$. Since $h_{F_{ses(z)}}(\mu_1)\supseteq\rho$ and $h_{F_{ses(z)}}(\mu_2)\supseteq\rho$, we have $\mu_1\in z^\rho$ and $\mu_2\in z^\rho$, hence $\lambda\mu_1+\psi(1-\lambda)\mu_2\in z^\rho$. Then

$$h_{F_{ses(z)}}(\lambda\mu_1+\psi(1-\lambda)\mu_2) \supseteq h_{F_{ses(z)}}(\mu_1)\cap h_{F_{ses(z)}}(\mu_2).$$

Thus, z is (λ,ψ)-convex Fses. □

Remark 4.9. Let $\mathcal{Z}$ be the underlying crisp universe, and let A and B be subsets of $\mathcal{Z}$. In this paper, $\mathcal{Z}\setminus A$ denotes the ordinary set-theoretic complement of A with respect to $\mathcal{Z}$. This notation is used only to reverse the direction of set inclusion. In particular,

$$A\subseteq B \Leftrightarrow \mathcal{Z}\setminus B\supseteq\mathcal{Z}\setminus A.$$

Therefore, $\mathcal{Z}\setminus A$ does not represent a new complement operation on the fuzzy membership values of an Fses. It is used only as a standard set-theoretic device in the inclusion arguments presented below.

Theorem 4.10. *χ' is (λ,ψ)-concave Fses while χ is (λ,ψ)-convex Fses.*
Proof. Assume that $\mu_1,\mu_2\in S,\lambda\in(0,1]$. Since χ is (λ,ψ)-convex Fses,

$$h_{F_{ses(\chi)}}(\lambda\mu_1+\psi(1-\lambda)\mu_2) \supseteq h_{F_{ses(\chi)}}(\mu_1)\cap h_{F_{ses(\chi)}}(\mu_2)$$

or

$$\mathcal{Z}\setminus h_{F_{ses(\chi)}}(\lambda\mu_1+\psi(1-\lambda)\mu_2) \subseteq \mathcal{Z}\setminus\big\{h_{F_{ses(\chi)}}(\mu_1)\cap h_{F_{ses(\chi)}}(\mu_2)\big\}.$$

If $h_{F_{ses(\chi)}}(\mu_1)\supseteq h_{F_{ses(\chi)}}(\mu_2)$, then we may write

$$\mathcal{Z}\setminus h_{F_{ses(\chi)}}(\lambda\mu_1+\psi(1-\lambda)\mu_2) \subseteq \mathcal{Z}\setminus h_{F_{ses(\chi)}}(\mu_2). \quad (3)$$

If $h_{F_{ses(\chi)}}(\mu_1)\subseteq h_{F_{ses(\chi)}}(\mu_2)$, then we may write

$$\mathcal{Z}\setminus h_{F_{ses(\chi)}}(\lambda\mu_1+\psi(1-\lambda)\mu_2) \subseteq \mathcal{Z}\setminus h_{F_{ses(\chi)}}(\mu_1). \quad (4)$$

From equations (3) and (4), we have

$$\mathcal{Z}\setminus h_{F_{ses(\chi)}}(\lambda\mu_1+\psi(1-\lambda)\mu_2) \subseteq \big\{\mathcal{Z}\setminus h_{F_{ses(\chi)}}(\mu_1)\cup\mathcal{Z}\setminus h_{F_{ses(\chi)}}(\mu_2)\big\}. \quad (5)$$

So, χ' is (λ,ψ)-concave Fses. □

Theorem 4.11. *ζ' is (λ,ψ)-convex Fses while ζ is (λ,ψ)-concave Fses.*
Proof. Assume that $\mu_1,\mu_2\in S,\lambda\in(0,1]$. Since ζ is (λ,ψ)-concave Fses,

$$h_{F_{ses(\zeta)}}(\lambda\mu_1+\psi(1-\lambda)\mu_2) \subseteq h_{F_{ses(\zeta)}}(\mu_1)\cup h_{F_{ses(\zeta)}}(\mu_2)$$

or

$$\mathcal{Z}\setminus h_{F_{ses(\zeta)}}(\lambda\mu_1+\psi(1-\lambda)\mu_2) \supseteq \mathcal{Z}\setminus\big\{h_{F_{ses(\zeta)}}(\mu_1)\cup h_{F_{ses(\zeta)}}(\mu_2)\big\}$$

If $h_{F_{ses(\zeta)}}(\mu_1)\supseteq h_{F_{ses(\zeta)}}(\mu_2)$ then we may write

$$\mathcal{Z}\setminus h_{F_{ses(\zeta)}}(\lambda\mu_1+(1-\lambda)\mu_2) \supseteq \mathcal{Z}\setminus h_{F_{ses(\zeta)}}(\mu_1). \quad (6)$$

If $h_{F_{ses(\zeta)}}(\mu_1)\subseteq h_{F_{ses(\zeta)}}(\mu_2)$. Then

$$\mathcal{Z}\setminus h_{F_{ses(\zeta)}}(\lambda\mu_1+\psi(1-\lambda)\mu_2) \supseteq \mathcal{Z}\setminus h_{F_{ses(\zeta)}}(\mu_2). \quad (7)$$

From (6) and (7), we have

$$\mathcal{Z}\setminus h_{F_{ses(\zeta)}}(\lambda\mu_1+\psi(1-\lambda)\mu_2) \supseteq \big\{\mathcal{Z}\setminus h_{F_{ses(\zeta)}}(\mu_1)\cap\mathcal{Z}\setminus h_{F_{ses(\zeta)}}(\mu_2)\big\}. \quad (8)$$

Hence, ζ' is (λ,ψ)-convex Fses. □

Theorem 4.12. *Π is (λ,ψ)-concave Fses on S if and only if for each $\lambda\in(0,1]$ and $\rho\in P(\mathcal{Z})$, Π^ρ is (λ,ψ)-concave Fses on S.*
Proof. Suppose that Π is (λ,ψ)-concave Fses. If $\mu_1,\mu_2\in S$ and $\rho\in P(\mathcal{Z})$, then $h_{F_{ses(\Pi)}}(\mu_1)\supseteq\rho$ and $h_{F_{ses(\Pi)}}(\mu_2)\supseteq\rho$. It implies that $h_{F_{ses(\Pi)}}(\mu_1)\cup h_{F_{ses(\Pi)}}(\mu_2)\supseteq\rho$. So we have

$$\rho\subseteq h_{F_{ses(\Pi)}}(\mu_1)\cap h_{F_{ses(\Pi)}}(\mu_2)\subseteq h_{F_{ses(\Pi)}}(\lambda\mu_1+\psi(1-\lambda)\mu_2) \subseteq h_{F_{ses(\Pi)}}(\mu_1)\cup h_{F_{ses(\Pi)}}(\mu_2)$$

That is $\rho\subseteq h_{F_{ses(\Pi)}}(\lambda\mu_1+\psi(1-\lambda)\mu_2)$.
Hence Π^ρ is a (λ,ψ)-concave Fses.

Conversely, suppose that Π^ρ is (λ,ψ)-concave Fses for each $\lambda\in(0,1]$. Subsequently, for $\mu_1,\mu_2\in S$, Π^ρ is (λ,ψ)-concave Fses for $\rho = h_{F_{ses(\Pi)}}(\mu_1)\cup h_{F_{ses(\Pi)}}(\mu_2)$. Then $h_{F_{ses(\Pi)}}(\mu_1)\supseteq\rho$ and $h_{F_{ses(\Pi)}}(\mu_2)\supseteq\rho$, we have $\mu_1\in\Pi^\rho$ and $\mu_2\in\Pi^\rho$, hence $\lambda\mu_1+\psi(1-\lambda)\mu_2\in\Pi^\rho$. Hence, $h_{F_{ses(\Pi)}}(\lambda\mu_1+\psi(1-\lambda)\mu_2)\subseteq\rho$. So,

$$h_{F_{ses(\Pi)}}(\lambda\mu_1+\psi(1-\lambda)\mu_2) \subseteq h_{F_{ses(\Pi)}}(\mu_1)\cup h_{F_{ses(\Pi)}}(\mu_2),$$

it is clear that Π is (λ,ψ)-concave Fses. □

Theorem 4.13. *$G_1\cup G_2$ is a (λ,ψ)-concave Fses while G_1 and G_2 are (λ,ψ)-concave Fsess.*
Proof. Assume $\mu_1,\mu_2\in S$ and $\lambda\in(0,1]$, and $G_3 = G_1\cup G_2$.

$$h_{F_{ses(G_3)}}(\lambda\mu_1+\psi(1-\lambda)\mu_2) = h_{F_{ses(G_1)}}(\lambda\mu_1+\psi(1-\lambda)\mu_2) \cup h_{F_{ses(G_2)}}(\lambda\mu_1+\psi(1-\lambda)\mu_2).$$

Since G_1 and G_2 are (λ,ψ)-concave Fsess, so

$$h_{F_{ses(G_1)}}(\lambda\mu_1+\psi(1-\lambda)\mu_2) \subseteq h_{F_{ses(G_1)}}(\mu_1)\cup h_{F_{ses(G_1)}}(\mu_2) \quad (9)$$

$$h_{F_{ses(G_2)}}(\lambda\mu_1+\psi(1-\lambda)\mu_2) \subseteq h_{F_{ses(G_2)}}(\mu_1)\cup h_{F_{ses(G_2)}}(\mu_2). \quad (10)$$

By applying union operation, we have

$$h_{F_{ses(G_3)}}(\lambda\mu_1+\psi(1-\lambda)\mu_2) \subseteq \big\{\big(h_{F_{ses(G_1)}}(\mu_1)\cup h_{F_{ses(G_1)}}(\mu_2)\big) \cup \big(h_{F_{ses(G_2)}}(\mu_1)\cup h_{F_{ses(G_2)}}(\mu_2)\big)\big\}.$$

Hence,

$$h_{F_{ses(G_3)}}(\lambda\mu_1+\psi(1-\lambda)\mu_2) \subseteq h_{F_{ses(G_3)}}(\mu_1)\cup h_{F_{ses(G_3)}}(\mu_2).$$

□

5. (λ,ψ)-CONVEX AND (λ,ψ)-CONCAVE FUZZY SOFT EXPERT SETS IN FIRST AND SECOND SENSE

This section will introduce the first and second sense concepts of (λ,ψ)-convex and (λ,ψ)-concave fuzzy soft expert sets, along with some demonstrated outcomes.

Definition 5.1. A fuzzy soft expert set becomes a (λ,ψ)-convex fuzzy soft expert set in first sense if

$h_{F_{ses}}(\lambda\mu_1 + \psi(1-\lambda^{\delta})\mu_2) \supseteq h_{F_{ses}}(\mu_1) \cap h_{F_{ses}}(\mu_2)$, for $\lambda, \delta \in (0,1]$, $\psi \in [0,1]$ and $\mu_1, \mu_2 \in S$.

Example 5.2. Referring to Example (3.8),

$$a_1 = h_{F_{ses}}(1,1,1) = \left\{\frac{\alpha_1}{0.1}, \frac{\alpha_2}{0.7}, \frac{\alpha_3}{0.4}, \frac{\alpha_4}{0.1}\right\},$$
$$a_2 = h_{F_{ses}}(1,2,1) = \left\{\frac{\alpha_1}{0.3}, \frac{\alpha_2}{0.8}, \frac{\alpha_3}{0.1}, \frac{\alpha_4}{0.2}\right\}.$$

Take $\lambda = 0.5$, $\psi = 1$, $\delta = 0.5$ and $\mu_1 = (1,1,1)$, $\mu_2 = (1,2,1)$ then

$$\begin{aligned} h_{F_{ses}}(\lambda\mu_1 + \psi(1-\lambda^{\delta})\mu_2) &= h_{F_{ses}}(0.5(1,1,1) \\ &\quad + (1-(0.5)^{0.5})(1,2,1)) \\ &= h_{F_{ses}}((0.5,0.5,0.5) + 0.29(1,2,1)) \\ &= h_{F_{ses}}((0.5,0.5,0.5) \\ &\quad + (0.29,0.58,0.29)) \\ &= h_{F_{ses}}(0.79,1.08,0.79). \end{aligned}$$

Taking help from round off, $(1,1,1)$ and we see that

$$h_{F_{ses}}(1,1,1) = \left\{\frac{\alpha_1}{0.1}, \frac{\alpha_2}{0.7}, \frac{\alpha_3}{0.4}, \frac{\alpha_4}{0.1}\right\}$$

and

$$\begin{aligned} h_{F_{ses}}(\mu_1) \cap h_{F_{ses}}(\mu_2) &= h_{F_{ses}}(1,1,1) \cap h_{F_{ses}}(1,2,1) \\ &= \left\{\frac{\alpha_1}{0.1}, \frac{\alpha_2}{0.7}, \frac{\alpha_3}{0.4}, \frac{\alpha_4}{0.1}\right\} \\ &\quad \cap \left\{\frac{\alpha_1}{0.3}, \frac{\alpha_2}{0.8}, \frac{\alpha_3}{0.1}, \frac{\alpha_4}{0.2}\right\} \\ &= \left\{\frac{\alpha_1}{0.1}, \frac{\alpha_2}{0.7}, \frac{\alpha_3}{0.1}, \frac{\alpha_4}{0.1}\right\}. \end{aligned}$$

It is clear that

$h_{F_{ses}}(\lambda\mu_1 + \psi(1-\lambda^{\delta})\mu_2) \supseteq h_{F_{ses}}(\mu_1) \cap h_{F_{ses}}(\mu_2)$.

Definition 5.3. A fuzzy soft expert set is named as a (λ,ψ)-concave fuzzy soft expert set in first sense if

$h_{F_{ses}}(\lambda\mu_1 + \psi(1-\lambda^{\delta})\mu_2) \subseteq h_{F_{ses}}(\mu_1) \cap h_{F_{ses}}(\mu_2)$, for $\lambda, \delta \in (0,1]$, $\psi \in [0,1]$ and $\mu_1, \mu_2 \in S$.

Definition 5.4. A fuzzy soft expert set is named as a (λ,ψ)-convex fuzzy soft expert set in second sense if

$h_{F_{ses}}(\lambda\mu_1 + \psi(1-\lambda)^{\delta}\mu_2), \supseteq h_{F_{ses}}(\mu_1) \cap h_{F_{ses}}(\mu_2)$ for $\lambda, \delta \in (0,1]$, $\psi \in [0,1]$ and $\mu_1, \mu_2 \in S$.

Example 5.5. Referring to Example (3.8),

$$a_1 = h_{F_{ses}}(1,1,1) = \left\{\frac{\alpha_1}{0.1}, \frac{\alpha_2}{0.7}, \frac{\alpha_3}{0.4}, \frac{\alpha_4}{0.1}\right\},$$
$$a_2 = h_{F_{ses}}(1,2,1) = \left\{\frac{\alpha_1}{0.3}, \frac{\alpha_2}{0.8}, \frac{\alpha_3}{0.1}, \frac{\alpha_4}{0.2}\right\}.$$

Take $\lambda = 0.5$, $\psi = 1$, $\delta = 0.5$ and $\mu_1 = (1,1,1)$, $\mu_2 = (1,2,1)$, then

$$\begin{aligned} h_{F_{ses}}(\lambda\mu_1 + \psi(1-\lambda)^{\delta}\mu_2) &= h_{F_{ses}}(0.5(1,1,1) \\ &\quad + (1-0.5)^{0.5}(1,2,1)) \\ &= h_{F_{ses}}((0.5,0.5,0.5) + 0.71(1,2,1)) \\ &= h_{F_{ses}}((0.5,0.5,0.5) \\ &\quad + (0.71,1.42,0.71)) \\ &= h_{F_{ses}}(1.21,1.92,1.21). \end{aligned}$$

Using the round off property, we have $\{1.21,1.92,1.21\} = \{1,2,1\}$,

$$h_{F_{ses}}(1,2,1) = \left\{\frac{\alpha_1}{0.3}, \frac{\alpha_2}{0.8}, \frac{\alpha_3}{0.1}, \frac{\alpha_4}{0.2}\right\}$$

and

$$\begin{aligned} h_{F_{ses}}(\mu_1) \cap h_{F_{ses}}(\mu_2) &= h_{F_{ses}}(1,1,1) \cap h_{F_{ses}}(1,2,1) \\ &= \left\{\frac{\alpha_1}{0.1}, \frac{\alpha_2}{0.7}, \frac{\alpha_3}{0.4}, \frac{\alpha_4}{0.1}\right\} \\ &\quad \cap \left\{\frac{\alpha_1}{0.3}, \frac{\alpha_2}{0.8}, \frac{\alpha_3}{0.1}, \frac{\alpha_4}{0.2}\right\} \\ &= \left\{\frac{\alpha_1}{0.1}, \frac{\alpha_2}{0.7}, \frac{\alpha_3}{0.1}, \frac{\alpha_4}{0.1}\right\}. \end{aligned}$$

It is clear that

$h_{F_{ses}}(\lambda\mu_1 + \psi(1-\lambda)^{\delta}\mu_2) \supseteq h_{F_{ses}}(\mu_1) \cap h_{F_{ses}}(\mu_2)$.

Definition 5.6. A fuzzy soft expert set is named as a (λ,ψ)-concave fuzzy soft expert set in second sense if

$h_{F_{ses}}(\lambda\mu_1 + \psi(1-\lambda)^{\delta}\mu_2), \subseteq h_{F_{ses}}(\mu_1) \cap h_{F_{ses}}(\mu_2)$. for $\lambda, \delta \in (0,1]$, $\psi \in [0,1]$ and $\mu_1, \mu_2 \in S$.

Theorem 5.7. *$G_1 \cap G_2$ is a (λ,ψ)-convex fuzzy soft expert set in first sense while G_1 and G_2 are (λ,ψ)-convex Fsess in first sense.*
Proof. Assume $\mu_1, \mu_2 \in S$ and $\lambda \in (0,1]$, and $G_3 = G_1 \cap G_2$.

$h_{F_{ses}(G_3)}(\lambda\mu_1 + \psi(1-\lambda^{\delta})\mu_2) =$
$h_{F_{ses}(G_1)}(\lambda\mu_1 + \psi(1-\lambda^{\delta})\mu_2) \cap h_{F_{ses}(G_2)}(\lambda\mu_1 + \psi(1-\lambda^{\delta})\mu_2)$.
Since G_1 and G_2 are (λ,ψ)-convex Fsess in first sense, so

$$h_{F_{ses}(G_1)}(\lambda\mu_1 + \psi(1-\lambda^{\delta})\mu_2) \supseteq h_{F_{ses}(G_1)}(\mu_1) \cap h_{F_{ses}(G_1)}(\mu_2). \quad (11)$$

$$h_{F_{ses}(G_2)}(\lambda\mu_1 + \psi(1-\lambda^{\delta})\mu_2) \supseteq h_{F_{ses}(G_2)}(\mu_1) \cap h_{F_{ses}(G_2)}(\mu_2). \quad (12)$$

By using the intersection operation, we get

$$\begin{aligned} h_{F_{ses}(G_3)}(\lambda\mu_1 + \psi(1-\lambda^{\delta})\mu_2) &\supseteq \left(h_{F_{ses}(G_1)}(\mu_1) \cap h_{F_{ses}(G_1)}(\mu_2)\right) \\ &\quad \cap \left(h_{F_{ses}(G_2)}(\mu_1) \cap h_{F_{ses}(G_2)}(\mu_2)\right). \end{aligned}$$

$h_{F_{ses}(G_3)}(\lambda\mu_1 + \psi(1-\lambda^{\delta})\mu_2) \supseteq h_{F_{ses}(G_3)}(\mu_1) \cap h_{F_{ses}(G_3)}(\mu_2)$.
Hence the result is proved. □

Remark 5.8. If the collection of (λ,ψ)-convex Fsess in first sense is represented by $\{G_k : k \in \{1,2,3,\dots\}\}$, then their intersection represented by $\bigcap_{k\in I} G_k$ is a (λ,ψ)-convex Fses in first sense.

Theorem 5.9. *$G_1 \cap G_2$ is a (λ,ψ)-convex fuzzy soft expert set in 2nd sense while G_1 and G_2 are (λ,ψ)-convex Fsess in 2nd sense.*
Proof. Assume $\mu_1, \mu_2 \in S$ and $\lambda \in (0,1]$, and $G_3 = G_1 \cap G_2$.

$$\begin{aligned} h_{F_{ses}(G_3)}(\lambda\mu_1 + \psi(1-\lambda)^{\delta}\mu_2) &= h_{F_{ses}(G_1)}(\lambda\mu_1 + \psi(1-\lambda)^{\delta}\mu_2) \\ &\quad \cap h_{F_{ses}(G_2)}(\lambda\mu_1 + \psi(1-\lambda)^{\delta}\mu_2). \end{aligned}$$

Since G_1 and G_2 are (λ,ψ)-convex Fsess in 2nd sense, so

$$h_{F_{ses}(G_1)}(\lambda\mu_1 + \psi(1-\lambda)^{\delta}\mu_2) \supseteq h_{F_{ses}(G_1)}(\mu_1) \cap h_{F_{ses}(G_1)}(\mu_2). \quad (13)$$

$$h_{F_{ses}(G_2)}(\lambda\mu_1 + \psi(1-\lambda)^{\delta}\mu_2) \supseteq h_{F_{ses}(G_2)}(\mu_1) \cap h_{F_{ses}(G_2)}(\mu_2). \quad (14)$$

Applying intersection on both equations above, we get

$$\begin{aligned} h_{F_{ses}(G_3)}(\lambda\mu_1 + (1-\lambda)^{\delta}\mu_2) &\supseteq \left(h_{F_{ses}(G_1)}(\mu_1) \cap h_{F_{ses}(G_1)}(\mu_2)\right) \\ &\quad \cap h_{F_{ses}(G_2)}(\mu_1) \cap h_{F_{ses}(G_2)}(\mu_2). \end{aligned}$$

$h_{F_{ses}(G_3)}(\lambda\mu_1 + (1-\lambda)^{\delta}\mu_2) \supseteq h_{F_{ses}(G_3)}(\mu_1) \cap h_{F_{ses}(G_3)}(\mu_2)$

hence the result is proved. □

Remark 5.10. If the collection of (λ,ψ)-convex Fsess in 2nd sense is represented by $\{G_k : k \in \{1,2,3,\dots\}\}$, then their intersection represented by $\bigcap_{k\in I} G_k$ is also a (λ,ψ)-convex Fses in 2nd sense.

Theorem 5.11. *Suppose Θ represents a (λ,ψ)-convex Fses in first sense if and only if for every $\lambda \in (0,1]$, $\rho \in P(\mathcal{Z})$, Θ^ρ is (λ,ψ)-convex Fses in first sense on S.*
Proof. Given that Θ is (λ,ψ)-convex Fses. Then for $\mu_1,\mu_2 \in S$ and $\rho \in P(\mathcal{Z})$, then $h_{F_{ses}(\Theta)}(\mu_1) \supseteq \rho$ and $h_{F_{ses}(\Theta)}(\mu_2) \supseteq \rho$. By the convexity of Θ, we have

$$h_{F_{ses}(\Theta)}(\lambda\mu_1 + \psi(1-\lambda^\delta)\mu_2) \supseteq h_{F_{ses}(\Theta)}(\mu_1) \cap h_{F_{ses}(\Theta)}(\mu_2).$$

So Θ^ρ is (λ,ψ)-convex Fses in first sense.

On converse way, suppose that Θ^ρ is (λ,ψ)-convex Fses in first sense for each $\lambda \in (0,1]$. Then, for $\mu_1,\mu_2 \in S$, Θ^ρ is (λ,ψ)-convex Fses in first sense for $\rho = h_{F_{ses}(\Theta)}(\mu_1) \cap h_{F_{ses}(\Theta)}(\mu_2)$. Since $h_{F_{ses}(\Theta)}(\mu_1) \supseteq \rho$ and $h_{F_{ses}(\Theta)}(\mu_2) \supseteq \rho$, we have $\mu_1 \in \Theta^\rho$ and $\mu_2 \in \Theta^\rho$, hence $\lambda\mu_1 + \psi(1-\lambda^\delta)\mu_2 \in \Theta^\rho$. Hence,

$$h_{F_{ses}(\Theta)}(\lambda\mu_1 + \psi(1-\lambda^\delta)\mu_2) \supseteq h_{F_{ses}(\Theta)}(\mu_1) \cap h_{F_{ses}(\Theta)}(\mu_2),$$

Therefore, Θ is (λ,ψ)-convex Fses in first sense. □

Theorem 5.12. *Θ is a (λ,ψ)-convex fuzzy soft expert set (Fses) in 2nd sense on S if and only if for each $\lambda \in (0,1]$ and $\rho \in P(\mathcal{Z})$, Θ^ρ is (λ,ψ)-convex Fses in 2nd sense on S.*
Proof. Suppose that Θ is (λ,ψ)-convex Fses in 2nd sense. If $\mu_1,\mu_2 \in S$ and $\rho \in P(\mathcal{Z})$, then $h_{F_{ses}(\Theta)}(\mu_1) \supseteq \rho$ and $h_{F_{ses}(\Theta)}(\mu_2) \supseteq \rho$. By the convexity of Θ, we have

$$h_{F_{ses}(\Theta)}(\lambda\mu_1 + \psi(1-\lambda)^\delta\mu_2) \supseteq h_{F_{ses}(\Theta)}(\mu_1) \cap h_{F_{ses}(\Theta)}(\mu_2),$$

so Θ^ρ is (λ,ψ)-convex Fses in 2nd sense.

On converse way, suppose that Θ^ρ is (λ,ψ)-convex Fses in 2nd sense for each $\lambda \in (0,1]$. Then, for $\mu_1,\mu_2 \in S$, Θ^ρ is (λ,ψ)-convex Fses in 2nd sense. For $\rho = h_{F_{ses}(\Theta)}(\mu_1) \cap h_{F_{ses}(\Theta)}(\mu_2)$. Since $h_{F_{ses}(\Theta)}(\mu_1) \supseteq \rho$ and $h_{F_{ses}(\Theta)}(\mu_2) \supseteq \rho$, we have $\mu_1 \in \Theta^\rho$ and $\mu_2 \in \Theta^\rho$, hence $\lambda\mu_1 + \psi(1-\lambda)^\delta\mu_2 \in \Theta^\rho$. Hence,

$$h_{F_{ses}(\Theta)}(\lambda\mu_1 + \psi(1-\lambda)^\delta\mu_2) \supseteq h_{F_{ses}(\Theta)}(\mu_1) \cap h_{F_{ses}(\Theta)}(\mu_2),$$

Therefore, Θ is (λ,ψ)-convex Fses in 2nd sense. □

Theorem 5.13. *Ψ' is (λ,ψ)-concave Fses in first sense while Ψ is (λ,ψ)-convex Fses in first sense.*
Proof. Assume that $\mu_1,\mu_2 \in S, \lambda \in (0,1]$. Since Ψ is (λ,ψ)-convex Fses in first sense,

$$h_{F_{ses}(\Psi)}(\lambda\mu_1 + \psi(1-\lambda^\delta)\mu_2) \supseteq h_{F_{ses}(\Psi)}(\mu_1) \cap h_{F_{ses}(\Psi)}(\mu_2)$$

or

$$\mathcal{Z}\backslash h_{F_{ses}(\Psi)}(\lambda\mu_1+\psi(1-\lambda^\delta)\mu_2) \subseteq \mathcal{Z}\backslash\{h_{F_{ses}(\Psi)}(\mu_1)\cap h_{F_{ses}(\Psi)}(\mu_2)\}$$

If $h_{F_{ses}(\Psi)}(\mu_1) \supseteq h_{F_{ses}(\Psi)}(\mu_2)$, then we may write

$$\mathcal{Z} \setminus h_{F_{ses}(\Psi)}(\lambda\mu_1 + \psi(1-\lambda^\delta)\mu_2) \subseteq \mathcal{Z} \setminus h_{F_{ses}(\Psi)}(\mu_2). \quad (15)$$

If $h_{F_{ses}(\Psi)}(\mu_1) \subseteq h_{F_{ses}(\Psi)}(\mu_2)$, then we may write

$$\mathcal{Z} \setminus h_{F_{ses}(\Psi)}(\lambda\mu_1 + \psi(1-\lambda^\delta)\mu_2) \subseteq \mathcal{Z} \setminus h_{F_{ses}(\Psi)}(\mu_1). \quad (16)$$

From equations (15) and (16), we have

$$\mathcal{Z} \setminus h_{F_{ses}(\Psi)}(\lambda\mu_1 + \psi(1-\lambda^\delta)\mu_2) \subseteq \{\mathcal{Z} \setminus h_{F_{ses}(\Psi)}(\mu_1) \cup \mathcal{Z} \setminus h_{F_{ses}(\Psi)}(\mu_2)\}. \quad (17)$$

So, Ψ' is (λ,ψ)-concave Fses in first sense. □

Theorem 5.14. *Ψ' is (λ,ψ)-concave Fses in 2nd sense while Ψ is (λ,ψ)-convex Fses in 2nd sense.*
Proof. Assume that $\mu_1,\mu_2 \in S, \lambda \in (0,1]$. Since Ψ is (λ,ψ)-convex Fses in 2nd sense,

$$h_{F_{ses}(\Psi)}(\lambda\mu_1 + \psi(1-\lambda)^\delta\mu_2) \supseteq h_{F_{ses}(\Psi)}(\mu_1) \cap h_{F_{ses}(\Psi)}(\mu_2)$$

or

$$\mathcal{Z}\backslash h_{F_{ses}(\Psi)}(\lambda\mu_1+\psi(1-\lambda)^\delta\mu_2) \subseteq \mathcal{Z}\backslash\{h_{F_{ses}(\Psi)}(\mu_1)\cap h_{F_{ses}(\Psi)}(\mu_2)\}.$$

If $h_{F_{ses}(\Psi)}(\mu_1) \supseteq h_{F_{ses}(\Psi)}(\mu_2)$, then we may write

$$\mathcal{Z} \setminus h_{F_{ses}(\Psi)}(\lambda\mu_1 + \psi(1-\lambda)^\delta\mu_2) \subseteq \mathcal{Z} \setminus h_{F_{ses}(\Psi)}(\mu_2). \quad (18)$$

If $h_{F_{ses}(\Psi)}(\mu_1) \subseteq h_{F_{ses}(\Psi)}(\mu_2)$, then we may write

$$\mathcal{Z} \setminus h_{F_{ses}(\Psi)}(\lambda\mu_1 + \psi(1-\lambda)^\delta\mu_2) \subseteq \mathcal{Z} \setminus h_{F_{ses}(\Psi)}(\mu_1). \quad (19)$$

From equations (18) and (19), we have

$$\mathcal{Z} \setminus h_{F_{ses}(\Psi)}(\lambda\mu_1 + \psi(1-\lambda)^\delta\mu_2) \subseteq \{\mathcal{Z} \setminus h_{F_{ses}(\Psi)}(\mu_1) \cup \mathcal{Z} \setminus h_{F_{ses}(\Psi)}(\mu_2)\}. \quad (20)$$

So, Ψ' is (λ,ψ)-concave Fses in 2nd sense. □

Theorem 5.15. *Υ' is (λ,ψ)-convex Fses in first sense while Υ is (λ,ψ)-concave Fses in first sense.*
Proof. Assume that $\mu_1,\mu_2 \in S$, $\lambda \in (0,1]$. Since Υ is (λ,ψ)-concave Fses in first sense,

$$h_{F_{ses}(\Upsilon)}(\lambda\mu_1 + \psi(1-\lambda^\delta)\mu_2) \subseteq h_{F_{ses}(\Upsilon)}(\mu_1) \cup h_{F_{ses}(\Upsilon)}(\mu_2)$$

or

$$\mathcal{Z}\backslash h_{F_{ses}(\Upsilon)}(\lambda\mu_1+\psi(1-\lambda^\delta)\mu_2). \supseteq \mathcal{Z}\backslash\{h_{F_{ses}(\Upsilon)}(\mu_1)\cup h_{F_{ses}(\Upsilon)}(\mu_2)\}$$

If $h_{F_{ses}(\Upsilon)}(\mu_1) \supseteq h_{F_{ses}(\Upsilon)}(\mu_2)$ then we may write

$$\mathcal{Z} \setminus h_{F_{ses}(\Upsilon)}(\lambda\mu_1 + (1-\lambda^\delta)\mu_2) \supseteq \mathcal{Z} \setminus h_{F_{ses}(\Upsilon)}(\mu_1). \quad (21)$$

If $h_{F_{ses}(\Upsilon)}(\mu_1) \subseteq h_{F_{ses}(\Upsilon)}(\mu_2)$, therefore

$$\mathcal{Z} \setminus h_{F_{ses}(\Upsilon)}(\lambda\mu_1 + \psi(1-\lambda^\delta)\mu_2) \supseteq \mathcal{Z} \setminus h_{F_{ses}(\Upsilon)}(\mu_2). \quad (22)$$

From (21) and (22), we have

$$\mathcal{Z} \setminus h_{F_{ses}(\Upsilon)}(\lambda\mu_1 + \psi(1-\lambda^\delta)\mu_2) \supseteq \{\mathcal{Z} \setminus h_{F_{ses}(\Upsilon)}(\mu_1) \cap \mathcal{Z} \setminus h_{F_{ses}(\Upsilon)}(\mu_2)\} \quad (23)$$

Thus, Υ' is (λ,ψ)-convex Fses in first sense. □

Theorem 5.16. *Υ' is (λ,ψ)-convex Fses in 2nd sense while Υ is (λ,ψ)-concave Fses in 2nd sense.*
Proof. Assume that $\mu_1,\mu_2 \in S$, $\lambda \in (0,1]$. Since Υ is (λ,ψ)-concave Fses in 2nd sense,

$$h_{F_{ses}(\Upsilon)}(\lambda\mu_1 + \psi(1-\lambda)^\delta\mu_2) \subseteq h_{F_{ses}(\Upsilon)}(\mu_1) \cup h_{F_{ses}(\Upsilon)}(\mu_2)$$

or

$$\mathcal{Z}\backslash h_{F_{ses}(\Upsilon)}(\lambda\mu_1+\psi(1-\lambda)^\delta\mu_2) \supseteq \mathcal{Z}\backslash\{h_{F_{ses}(\Upsilon)}(\mu_1)\cup h_{F_{ses}(\Upsilon)}(\mu_2)\}.$$

If $h_{F_{ses}(\Upsilon)}(\mu_1) \supseteq h_{F_{ses}(\Upsilon)}(\mu_2)$ then we may write

$$\mathcal{Z} \setminus h_{F_{ses}(\Upsilon)}(\lambda\mu_1 + (1-\lambda)^\delta\mu_2) \supseteq \mathcal{Z} \setminus h_{F_{ses}(\Upsilon)}(\mu_1). \quad (24)$$

If $h_{F_{ses}(\Upsilon)}(\mu_1) \subseteq h_{F_{ses}(\Upsilon)}(\mu_2)$. Then

$$\mathcal{Z} \setminus h_{F_{ses}(\Upsilon)}(\lambda\mu_1 + \psi(1-\lambda)^\delta\mu_2) \supseteq \mathcal{Z} \setminus h_{F_{ses}(\Upsilon)}(\mu_2). \quad (25)$$

From (24) and (25), we have

$$\mathcal{Z} \setminus h_{F_{ses}(\Upsilon)}(\lambda\mu_1 + \psi(1-\lambda)^\delta\mu_2) \supseteq \{\mathcal{Z} \setminus h_{F_{ses}(\Upsilon)}(\mu_1) \cap \mathcal{Z} \setminus h_{F_{ses}(\Upsilon)}(\mu_2)\}. \quad (26)$$

Therefore, Υ' is (λ,ψ)-convex Fses in 2nd sense. □

Theorem 5.17. *Λ is (λ,ψ)-concave Fses in first sense on S if and only if for each $\lambda \in (0,1]$ and $\rho \in P(\mathcal{Z})$, Λ^ρ is (λ,ψ)-concave Fses*

in first sense on S.
Proof. Firstly, consider Λ as a (λ,ψ)-concave Fses in first sense. Then for $\mu_1,\mu_2 \in S$ and $\rho \in P(\mathcal{Z})$, then $h_{F_{ses}(\Lambda)}(\mu_1) \supseteq \rho$ and $h_{F_{ses}(\Lambda)}(\mu_2) \supseteq \rho$. By the (λ,ψ)-concavity of Λ in first sense, then

$$h_{F_{ses}(\Lambda)}(\lambda\mu_1 + \psi(1-\lambda^{\delta})\mu_2) \subseteq h_{F_{ses}(\Lambda)}(\mu_1) \cup h_{F_{ses}(\Lambda)}(\mu_2).$$

Hence Λ^{ρ} is a (λ,ψ)-concave Fses in first sense.

On the converse way, suppose that Λ^{ρ} is (λ,ψ)-concave Fses in first sense for each $\lambda \in (0,1]$. Subsequently, for $\mu_1,\mu_2 \in S$, Λ^{ρ} is (λ,ψ)-concave Fses in first sense for $\rho = h_{F_{ses}(\Lambda)}(\mu_1) \cup h_{F_{ses}(\Lambda)}(\mu_2)$. So, $h_{F_{ses}(\Lambda)}(\mu_1) \supseteq \rho$ and $h_{F_{ses}(\Lambda)}(\mu_2) \supseteq \rho$, we have $\mu_1 \in \Lambda^{\rho}$ and $\mu_2 \in \Lambda^{\rho}$, hence $\lambda\mu_1 + \psi(1-\lambda^{\delta})\mu_2 \in \Lambda^{\rho}$. Hence,

$$h_{F_{ses}(\Lambda)}(\lambda\mu_1 + \psi(1-\lambda^{\delta})\mu_2) \subseteq h_{F_{ses}(\Lambda)}(\mu_1) \cup h_{F_{ses}(\Lambda)}(\mu_2),$$

it is clear that Λ is (λ,ψ)-concave Fses in first sense. □

Theorem 5.18. *Λ is (λ,ψ)-concave Fses in 2nd sense on $S \Leftrightarrow$ for each $\lambda \in (0,1]$ and $\rho \in P(\mathcal{Z})$, Λ^{ρ} is (λ,ψ)-concave Fses in 2nd sense on S.*
Proof. Suppose that Λ is (λ,ψ)-concave Fses in 2nd sense. If $\mu_1,\mu_2 \in S$ and $\rho \in P(\mathcal{Z})$, then $h_{F_{ses}(\Lambda)}(\mu_1) \supseteq \rho$ and $h_{F_{ses}(\Lambda)}(\mu_2) \supseteq \rho$. By the (λ,ψ)-concavity of Λ in 2nd sense, we have

$$h_{F_{ses}(\Lambda)}(\lambda\mu_1 + \psi(1-\lambda)^{\delta}\mu_2) \subseteq h_{F_{ses}(\Lambda)}(\mu_1) \cup h_{F_{ses}(\Lambda)}(\mu_2),$$

hence Λ^{ρ} is a (λ,ψ)-concave Fses in 2nd sense.

On the other hand, we now consider Λ^{ρ} as a (λ,ψ)-concave Fses in 2nd sense for each $\lambda \in (0,1]$. Subsequently, for $\mu_1,\mu_2 \in S$, Λ^{ρ} is (λ,ψ)-concave Fses in 2nd sense for $\rho = h_{F_{ses}(\Lambda)}(\mu_1) \cup h_{F_{ses}(\Lambda)}(\mu_2)$. Thus, $h_{F_{ses}(\Lambda)}(\mu_1) \supseteq \rho$ and $h_{F_{ses}(\Lambda)}(\mu_2) \supseteq \rho$, we have $\mu_1 \in \Lambda^{\rho}$ and $\mu_2 \in \Lambda^{\rho}$, hence $\lambda\mu_1 + \psi(1-\lambda)^{\delta}\mu_2 \in \Lambda^{\rho}$. Hence,

$$h_{F_{ses}(\Lambda)}(\lambda\mu_1 + \psi(1-\lambda)^{\delta}\mu_2) \subseteq h_{F_{ses}(\Lambda)}(\mu_1) \cup h_{F_{ses}(\Lambda)}(\mu_2),$$

it is clear that Λ is (λ,ψ)-concave Fses in 2nd sense. □

Theorem 5.19. *$\Phi_1 \cup \Phi_2$ is a (λ,ψ)-concave fuzzy soft expert set in first sense while Φ_1 and Φ_2 are (λ,ψ)-concave Fsess in first sense.*
Proof. Assume $\mu_1,\mu_2 \in S$ and $\lambda \in (0,1]$, and $\Phi_3 = \Phi_1 \cup \Phi_2$.

$$h_{F_{ses}(\Phi_3)}(\lambda\mu_1 + \psi(1-\lambda^{\delta})\mu_2) = h_{F_{ses}(\Phi_1)}(\lambda\mu_1 + \psi(1-\lambda^{\delta})\mu_2) \cup h_{F_{ses}(\Phi_2)}(\lambda\mu_1 + \psi(1-\lambda^{\delta})\mu_2).$$

Since Φ_1 and Φ_2 are (λ,ψ)-concave Fsess in first sense, so

$$h_{F_{ses}(\Phi_1)}(\lambda\mu_1 + \psi(1-\lambda^{\delta})\mu_2) \subseteq h_{F_{ses}(\Phi_1)}(\mu_1) \cup h_{F_{ses}(\Phi_1)}(\mu_2) \quad (27)$$

$$h_{F_{ses}(\Phi_2)}(\lambda\mu_1 + \psi(1-\lambda^{\delta})\mu_2) \subseteq h_{F_{ses}(\Phi_2)}(\mu_1) \cup h_{F_{ses}(\Phi_2)}(\mu_2). \quad (28)$$

With the usage of union operation, we get

$$h_{F_{ses}(\Phi_3)}(\lambda\mu_1 + \psi(1-\lambda^{\delta})\mu_2) \subseteq \big(h_{F_{ses}(\Phi_1)}(\mu_1) \cup h_{F_{ses}(\Phi_1)}(\mu_2)\big) \cup \big(h_{F_{ses}(\Phi_2)}(\mu_1) \cup h_{F_{ses}(\Phi_2)}(\mu_2)\big).$$

Therefore,

$$h_{F_{ses}(\Phi_3)}(\lambda\mu_1 + \psi(1-\lambda^{\delta})\mu_2) \subseteq h_{F_{ses}(\Phi_3)}(\mu_1) \cup h_{F_{ses}(\Phi_3)}(\mu_2).$$

□

Theorem 5.20. *$\Phi_1 \cup \Phi_2$ is a (λ,ψ)-concave fuzzy soft expert set in 2nd sense while Φ_1 and Φ_2 are (λ,ψ)-concave Fsess in 2nd sense.*
Proof. Assume $\mu_1,\mu_2 \in S$ and $\lambda \in (0,1]$, and $\Phi_3 = \Phi_1 \cup \Phi_2$.

$$h_{F_{ses}(\Phi_3)}(\lambda\mu_1 + \psi(1-\lambda)^{\delta}\mu_2) = h_{F_{ses}(\Phi_1)}(\lambda\mu_1 + \psi(1-\lambda)^{\delta}\mu_2) \cup h_{F_{ses}(\Phi_2)}(\lambda\mu_1 + \psi(1-\lambda)^{\delta}\mu_2).$$

Since Φ_1 and Φ_2 are (λ,ψ)-concave Fsess in 2nd sense, so

$$h_{F_{ses}(\Phi_1)}(\lambda\mu_1 + \psi(1-\lambda)^{\delta}\mu_2) \subseteq h_{F_{ses}(\Phi_1)}(\mu_1) \cup h_{F_{ses}(\Phi_1)}(\mu_2). \quad (29)$$

$$h_{F_{ses}(\Phi_2)}(\lambda\mu_1 + \psi(1-\lambda)^{\delta}\mu_2) \subseteq h_{F_{ses}(\Phi_2)}(\mu_1) \cup h_{F_{ses}(\Phi_2)}(\mu_2). \quad (30)$$

With the help of union of above equations, we have

$$h_{F_{ses}(\Phi_3)}(\lambda\mu_1 + \psi(1-\lambda)^{\delta}\mu_2) \subseteq \big\{h_{F_{ses}(\Phi_1)}(\mu_1) \cup h_{F_{ses}(\Phi_1)}(\mu_2)\big\} \cup \big\{h_{F_{ses}(\Phi_2)}(\mu_1) \cup h_{F_{ses}(\Phi_2)}(\mu_2)\big\}. \quad (31)$$

So,

$$h_{F_{ses}(\Phi_3)}(\lambda\mu_1 + \psi(1-\lambda)^{\delta}\mu_2) \subseteq h_{F_{ses}(\Phi_3)}(\mu_1) \cup h_{F_{ses}(\Phi_3)}(\mu_2).$$

□

6. STRONGLY CONVEX AND STRICTLY CONVEX FUZZY SOFT EXPERT SETS

In this study, we define strongly convex fuzzy soft expert set, strictly convex fuzzy soft expert set, strongly concave fuzzy soft expert set, strictly concave fuzzy soft expert set and describe some of their characteristics.

Definition 6.1. The fuzzy soft expert set over S is said to be a strongly convex fuzzy soft expert set if

$$h_{F_{ses}}(\lambda\mu_1 + (1-\lambda)\mu_2) \supset h_{F_{ses}}(\mu_1) \cap h_{F_{ses}}(\mu_2),$$

for each $\mu_1,\mu_2 \in S$, $\mu_1 \neq \mu_2$ and $\lambda \in \Gamma' = (0,1)$.

Definition 6.2. The fuzzy soft expert set over S is said to be a strongly concave fuzzy soft expert set if

$$h_{F_{ses}}(\lambda\mu_1 + (1-\lambda)\mu_2) \subset h_{F_{ses}}(\mu_1) \cup h_{F_{ses}}(\mu_2)$$

for each $\mu_1,\mu_2 \in S$, $\mu_1 \neq \mu_2$ and $\lambda \in \Gamma'$.

Definition 6.3. The fuzzy soft expert set over S is said to be a strictly convex fuzzy soft expert set if

$$h_{F_{ses}}(\lambda\mu_1 + (1-\lambda)\mu_2) \supset h_{F_{ses}}(\mu_1) \cap h_{F_{ses}}(\mu_2),$$

for each $\mu_1,\mu_2 \in S$, $h_{F_{ses}}(\mu_1) \neq h_{F_{ses}}(\mu_2)$ and $\lambda \in \Gamma'$.

Definition 6.4. The fuzzy soft expert set over S is said to be a strictly concave fuzzy soft expert set if

$$h_{F_{ses}}(\lambda\mu_1 + (1-\lambda)\mu_2) \subset h_{F_{ses}}(\mu_1) \cup h_{F_{ses}}(\mu_2),$$

for each $\mu_1,\mu_2 \in S$, $h_{F_{ses}}(\mu_1) \neq h_{F_{ses}}(\mu_2)$ and $\lambda \in \Gamma'$.

Theorem 6.5. *Suppose $(h_{F_{ses}}, S)$ be a strictly convex fuzzy soft expert set. If there exists $\lambda \in \Gamma'$ for all $\mu_1,\mu_2 \in S$ such that*

$$h_{F_{ses}}(\lambda\mu_1 + (1-\lambda)\mu_2) \supseteq h_{F_{ses}}(\mu_1) \cap h_{F_{ses}}(\mu_2),$$

then $(h_{F_{ses}}, S)$ is a convex fuzzy soft expert set.
Proof. Assume that $h_{F_{ses}}(\mu_1) \subseteq h_{F_{ses}}(\mu_2)$ and there exist $\mu_1,\mu_2 \in S$, $\delta \in \Gamma'$,

$$\mathcal{Z} \setminus h_{F_{ses}(T)}(\delta\mu_1 + (1-\delta)\mu_2) \supseteq \mathcal{Z} \setminus \big\{h_{F_{ses}(T)}(\mu_1) \cap h_{F_{ses}(T)}(\mu_2)\big\}. \quad (32)$$

If $h_{F_{ses}}(\mu_1) \subset h_{F_{ses}}(\mu_2)$, then the given statement contradicts that $(h_{F_{ses}}, S)$ is a strictly convex fuzzy soft expert set. Now if we take $h_{F_{ses}}(\mu_1) = h_{F_{ses}}(\mu_2)$ and $\delta \in [0,\lambda]$, suppose $\mu_3 = \frac{\delta}{\lambda}\mu_1 + \big(1-\frac{\delta}{\lambda}\big)\mu_2$ and $\tau = \big(\frac{1}{\delta}-1\big)\big(\frac{1}{\lambda}-1\big)^{-1}$. So

$$\begin{aligned} h_{F_{ses}(T)}(\delta\mu_1 + (1-\delta)\mu_2) &= h_{F_{ses}(T)}(\lambda\Big(\tfrac{\delta}{\lambda}\mu_1 + \big(1-\tfrac{\delta}{\lambda}\big)\mu_2\Big) \\ &\quad + (1-\delta)\mu_2) \\ &= h_{F_{ses}(T)}(\lambda\mu_3 + (1-\lambda)\mu_2) \end{aligned}$$

$$\supseteq \{h_{F_{ses}(T)}(\mu_1) \cap h_{F_{ses}(T)}(\mu_2)\}.$$

Therefore,

$$h_{F_{ses}(T)}(\delta\mu_1 + (1-\delta)\mu_2) \supseteq \{h_{F_{ses}(T)}(\mu_1) \cap h_{F_{ses}(T)}(\mu_2)\}. \quad (33)$$

Now

$$\begin{aligned} h_{F_{ses}(T)}(\mu_3) &= h_{F_{ses}(T)}(\tfrac{\delta}{\lambda}\mu_1 + \big(1-\tfrac{\delta}{\lambda}\big)\mu_2) \\ &= h_{F_{ses}(T)}(\tau\mu_1 + (1-\tau)\big(\delta\mu_1 + (1-\delta)\mu_2\big)). \end{aligned} \quad (34)$$

From (32), (33), and $h_{F_{ses}}(\mu_1) = h_{F_{ses}}(\mu_2)$, we have

$$h_{F_{ses}(T)}(\delta\mu_1 + (1-\delta)\mu_2) \supseteq h_{F_{ses}}(\mu_3). \quad (35)$$

Also from (31), (32), and $h_{F_{ses}}(\mu_1) = h_{F_{ses}}(\mu_2)$ and strictly convex fuzzy soft expert set condition, we have

$$h_{F_{ses}(T)}(\delta\mu_1 + (1-\delta)\mu_2) \subset h_{F_{ses}}(\mu_3), \quad (36)$$

or

$$\mathcal{Z} \setminus h_{F_{ses}(T)}(\delta\mu_1 + (1-\delta)\mu_2) \supseteq \mathcal{Z} \setminus h_{F_{ses}}(\mu_3), \quad (37)$$

thus (34) and (36) contradict the fact.

If $h_{F_{ses}}(\mu_1) = h_{F_{ses}}(\mu_2)$ and $\delta \in [\lambda, 1]$,
suppose $\mu_4 = \frac{\delta-\lambda}{1-\lambda}\mu_1 + \frac{1-\delta}{1-\lambda}\mu_2$, then

$$\begin{aligned} h_{F_{ses}(T)}(\delta\mu_1 + (1-\delta)\mu_2) &= h_{F_{ses}(T)}(\lambda\mu_1 + (1-\lambda)\mu_4) \\ &\supseteq \{h_{F_{ses}(T)}(\mu_1) \cap h_{F_{ses}(T)}(\mu_4)\}. \end{aligned}$$

So,

$$h_{F_{ses}(T)}(\delta\mu_1 + (1-\delta)\mu_2) \supseteq \{h_{F_{ses}(T)}(\mu_1) \cap h_{F_{ses}(T)}(\mu_4).\} \quad (38)$$

From (32), (37) and $h_{F_{ses}}(\mu_1) = h_{F_{ses}}(\mu_2)$, we have

$$h_{F_{ses}(T)}(\delta\mu_1 + (1-\delta)\mu_2) \supseteq h_{F_{ses}}(\mu_4), \quad (39)$$

Also, $(\delta\mu_1 + (1-\delta)\mu_2) = (\lambda\mu_1 + (1-\lambda)\mu_4)$ becomes

$$\begin{aligned} \mu_4 &= \tfrac{1}{1-\lambda}(\delta\mu_1 + (1-\delta)\mu_2) - \tfrac{\lambda}{1-\lambda}\mu_1 \\ &= \tfrac{1}{1-\lambda}(\delta\mu_1 + (1-\delta)\mu_2) \\ &\quad - \tfrac{\lambda}{1-\lambda}\Big(\tfrac{1}{\delta}(\delta\mu_1 + (1-\delta)\mu_2) - \tfrac{1-\delta}{\delta}\mu_2\Big) \\ &= \tfrac{\delta-\lambda}{(1-\lambda)\delta}(\delta\mu_1 + (1-\delta)\mu_2) + \Big(1 - \tfrac{\delta-\lambda}{(1-\lambda)\delta}\Big)\mu_2. \end{aligned}$$

So,

$$\mu_4 = \tfrac{\delta-\lambda}{(1-\lambda)\delta}(\delta\mu_1 + (1-\delta)\mu_2) + \Big(1 - \tfrac{\delta-\lambda}{(1-\lambda)\delta}\Big)\mu_2. \quad (40)$$

Now from (32), (39), and $h_{F_{ses}}(\mu_1) = h_{F_{ses}}(\mu_2)$ and strictly convex fuzzy soft expert set condition, we have

$$\mathcal{Z} \setminus h_{F_{ses}(T)}(\delta\mu_1 + (1-\delta)\mu_2) \supseteq \mathcal{Z} \setminus h_{F_{ses}}(\mu_1). \quad (41)$$

Hence equations (39) and (41) contradict the fact. □

Theorem 6.6. *Suppose $(h_{F_{ses}}, S)$ be a strictly concave fuzzy soft expert set. If there exists $\lambda \in \Gamma'$ for all $\mu_1, \mu_2 \in S$ such that*

$$h_{F_{ses}}(\lambda\mu_1 + (1-\lambda)\mu_2) \subseteq h_{F_{ses}}(\mu_1) \cap h_{F_{ses}}(\mu_2),$$

then $(h_{F_{ses}}, S)$ is a concave fuzzy soft expert set.

Proof. Assume that $h_{F_{ses}}(\mu_1) \supseteq h_{F_{ses}}(\mu_2)$ and there exist $\mu_1, \mu_2 \in S, \delta \in \Gamma'$,

$$\mathcal{Z} \setminus h_{F_{ses}(T)}(\delta\mu_1 + (1-\delta)\mu_2) \subseteq \mathcal{Z} \setminus \{h_{F_{ses}(T)}(\mu_1) \cap h_{F_{ses}(T)}(\mu_2)\} \quad (42)$$

if $h_{F_{ses}}(\mu_1) \supset h_{F_{ses}}(\mu_2)$, then (41) contradicts that $(h_{F_{ses}}, S)$ is a strictly concave fuzzy soft expert set. Now if we take $h_{F_{ses}}(\mu_1) = h_{F_{ses}}(\mu_2)$ and $\delta \in [0, \lambda]$,
suppose $\mu_3 = \frac{\delta}{\lambda}\mu_1 + \big(1-\frac{\delta}{\lambda}\big)\mu_2$ and $\tau = \big(\frac{1}{\delta} - 1\big)\big(\frac{1}{\lambda} - 1\big)^{-1}$. So

$$\begin{aligned} h_{F_{ses}(T)}(\delta\mu_1 + (1-\delta)\mu_2) &= h_{F_{ses}(T)}(\lambda\Big(\tfrac{\delta}{\lambda}\mu_1 + \big(1-\tfrac{\delta}{\lambda}\big)\mu_2\Big) \\ &\quad + (1-\delta)\mu_2) \\ &= h_{F_{ses}(T)}(\lambda\mu_3 + (1-\lambda)\mu_2) \\ &\subseteq \{h_{F_{ses}(T)}(\mu_1) \cap h_{F_{ses}(T)}(\mu_2)\} \end{aligned}$$

Therefore,

$$h_{F_{ses}(T)}(\delta\mu_1 + (1-\delta)\mu_2) \subseteq \{h_{F_{ses}(T)}(\mu_1) \cap h_{F_{ses}(T)}(\mu_2)\}. \quad (43)$$

Now

$$\begin{aligned} h_{F_{ses}(T)}(\mu_3) &= h_{F_{ses}(T)}(\tfrac{\delta}{\lambda}\mu_1 + \big(1-\tfrac{\delta}{\lambda}\big)\mu_2) \\ &= h_{F_{ses}(T)}(\tau\mu_1 + (1-\tau)\big(\delta\mu_1 + (1-\delta)\mu_2\big)). \end{aligned} \quad (44)$$

From (42), (43) and $h_{F_{ses}}(\mu_1) = h_{F_{ses}}(\mu_2)$, we have

$$h_{F_{ses}(T)}(\delta\mu_1 + (1-\delta)\mu_2) \subseteq h_{F_{ses}}(\mu_3), \quad (45)$$

now from (42), (43), and $h_{F_{ses}}(\mu_1) = h_{F_{ses}}(\mu_2)$ and strictly concave soft expert set condition, we have

$$h_{F_{ses}(T)}(\delta\mu_1 + (1-\delta)\mu_2) \supset h_{F_{ses}}(\mu_3), \quad (46)$$

or

$$\mathcal{Z} \setminus h_{F_{ses}(T)}(\delta\mu_1 + (1-\delta)\mu_2) \subseteq \mathcal{Z} \setminus h_{F_{ses}}(\mu_3), \quad (47)$$

Thus, both above equations do not agree with given fact.

If $h_{F_{ses}}(\mu_1) = h_{F_{ses}}(\mu_2)$ and $\delta \in [\lambda, 1]$,
suppose $\mu_4 = \frac{\delta-\lambda}{1-\lambda}\mu_1 + \frac{1-\delta}{1-\lambda}\mu_2$ then

$$\begin{aligned} h_{F_{ses}(T)}(\delta\mu_1 + (1-\delta)\mu_2) &= h_{F_{ses}(T)}(\lambda\mu_1 + (1-\lambda)\mu_4) \\ &\subseteq h_{F_{ses}(T)}(\mu_1) \cap h_{F_{ses}(T)}(\mu_4). \end{aligned}$$

So,

$$h_{F_{ses}(T)}(\delta\mu_1 + (1-\delta)\mu_2) \subseteq \{h_{F_{ses}(T)}(\mu_1) \cap h_{F_{ses}(T)}(\mu_4)\} \quad (48)$$

From (42), (47) and $h_{F_{ses}}(\mu_1) = h_{F_{ses}}(\mu_2)$, we have

$$h_{F_{ses}(T)}(\delta\mu_1 + (1-\delta)\mu_2) \subseteq h_{F_{ses}}(\mu_4), \quad (49)$$

Also, $(\delta\mu_1 + (1-\delta)\mu_2) = (\lambda\mu_1 + (1-\lambda)\mu_4)$ becomes

$$\begin{aligned} \mu_4 &= \tfrac{1}{1-\lambda}(\delta\mu_1 + (1-\delta)\mu_2) - \tfrac{\lambda}{1-\lambda}\mu_1 \\ &= \tfrac{1}{1-\lambda}(\delta\mu_1 + (1-\delta)\mu_2) \\ &\quad - \tfrac{\lambda}{1-\lambda}\Big(\tfrac{1}{\delta}(\delta\mu_1 + (1-\delta)\mu_2) - \tfrac{1-\delta}{\delta}\mu_2\Big) \\ &= \tfrac{\delta-\lambda}{(1-\lambda)\delta}(\delta\mu_1 + (1-\delta)\mu_2) + \Big(1 - \tfrac{\delta-\lambda}{(1-\lambda)\delta}\Big)\mu_2. \end{aligned}$$

Hence,

$$\mu_4 = \tfrac{\delta-\lambda}{(1-\lambda)\delta}(\delta\mu_1 + (1-\delta)\mu_2) + \Big(1 - \tfrac{\delta-\lambda}{(1-\lambda)\delta}\Big)\mu_2. \quad (50)$$

Now from (42), (49), $h_{F_{ses}}(\mu_1) = h_{F_{ses}}(\mu_2)$ and strictly concave fuzzy soft expert set condition, we have

$$\mathcal{Z} \setminus h_{F_{ses}(T)}(\delta\mu_1 + (1-\delta)\mu_2) \subseteq \mathcal{Z} \setminus h_{F_{ses}}(\mu_4). \quad (51)$$

Hence above equations against the fact. □

Theorem 6.7. *Suppose $(h_{F_{ses}}, S)$ be a convex fuzzy soft expert set. If there exists $\lambda \in \Gamma'$ for all $\mu_1, \mu_2 \in S$ with $h_{F_{ses}}(\mu_1) \neq h_{F_{ses}}(\mu_2)$*

$$h_{F_{ses}}(\lambda\mu_1 + (1-\lambda)\mu_2) \supseteq h_{F_{ses}}(\mu_1) \cap h_{F_{ses}}(\mu_2)$$

then $(h_{F_{ses}}, S)$ is a strictly convex fuzzy soft expert set.

Proof. Assume that $\mu_1, \mu_2 \in S, \delta \in \Gamma'$ such that

$$\mathcal{Z} \setminus h_{F_{ses}}(\delta\mu_1 + (1-\delta)\mu_2) \supset \mathcal{Z} \setminus \{h_{F_{ses}}(\mu_1) \cap h_{F_{ses}}(\mu_2)\}. \quad (52)$$

If $h_{F_{ses}}(\mu_1) \supset h_{F_{ses}}(\mu_2)$, then above equation becomes

$$\mathcal{Z} \setminus h_{F_{ses}}(\delta\mu_1 + (1-\delta)\mu_2) \supset \mathcal{Z} \setminus h_{F_{ses}}(\mu_1). \quad (53)$$

But by following the condition of convexity

$$h_{F_{ses}}(\delta\mu_1 + (1-\delta)\mu_2) \supseteq \{h_{F_{ses}}(\mu_1) \cap h_{F_{ses}}(\mu_2)\}. \quad (54)$$

By equations (53) and (54), we get

$$h_{F_{ses}}(\delta\mu_1 + (1-\delta)\mu_2) = \{h_{F_{ses}}(\mu_1) \cap h_{F_{ses}}(\mu_2)\}. \quad (55)$$

Continuing with $h_{F_{ses}}(\mu_1) \supset h_{F_{ses}}(\mu_2)$, we get

$$h_{F_{ses}}(\delta\mu_1 + (1-\delta)\mu_2) = h_{F_{ses}}(\mu_2) \quad (56)$$

or

$$h_{F_{ses}}(\delta\mu_1 + (1-\delta)\mu_2) \subset h_{F_{ses}}(\mu_1). \quad (57)$$

By supposition and (56), we have

$$h_{F_{ses}}(\lambda\mu_1 + (1-\lambda)(\delta\mu_1 + (1-\delta)\mu_2)) \supset h_{F_{ses}}(\delta\mu_1 + (1-\delta)\mu_2). \quad (58)$$

We generalize it for taking $n \in \{1, 2, 3, \dots\}$

$$h_{F_{ses}}(\lambda^n\mu_1 + (1-\lambda^n)(\delta\mu_1 + (1-\delta)\mu_2)) \supset h_{F_{ses}}(\delta\mu_1 + (1-\delta)\mu_2). \quad (59)$$

Suppose $\mu_3 = (\tau\mu_1 + (1-\tau)\mu_2)$ with $\tau = \delta - \lambda^n\delta + \lambda^n \in \Gamma'$ for any value of n. Then we see from above equation

$$\begin{aligned} h_{F_{ses}}(\mu_3) &= h_{F_{ses}}(\tau\mu_1 + (1-\tau)\mu_2) \\ &= h_{F_{ses}}(\lambda^n\mu_1 + (1-\lambda^n)(\delta\mu_1 + (1-\delta)\mu_2)) \\ &\supset h_{F_{ses}}(\delta\mu_1 + (1-\delta)\mu_2) \end{aligned}$$

Hence,

$$h_{F_{ses}}(\mu_3) \supset h_{F_{ses}}(\delta\mu_1 + (1-\delta)\mu_2). \quad (60)$$

Also, suppose $\mu_4 = (\gamma\mu_1 + (1-\gamma)\mu_2)$ with $\gamma = \delta - \lambda^n + \frac{1}{1-\lambda} + \frac{\delta\lambda^n}{1-\lambda} \in \Gamma'$ for any n. we have

$$h_{F_{ses}}(\delta\mu_1 + (1-\delta)\mu_2) = h_{F_{ses}}(\lambda\mu_3 + (1-\lambda)\mu_4). \quad (61)$$

Also when, $h_{F_{ses}}(\mu_3) \subseteq h_{F_{ses}}(\mu_4)$, subsequently by using the convexity of fuzzy soft expert set, we have $\mathcal{Z} \setminus h_{F_{ses}}(\mu_3) \supset \mathcal{Z} \setminus h_{F_{ses}}(\delta\mu_1 + (1-\delta)\mu_2)$ which contradicts equation (52). Now when $\mathcal{Z} \setminus h_{F_{ses}}(\mu_3) \subseteq \mathcal{Z} \setminus h_{F_{ses}}(\mu_4)$, subsequently by making use of the statement of theorem, we have

$$\begin{aligned} h_{F_{ses}}(\delta\mu_1 + (1-\delta)\mu_2) &\supset \{h_{F_{ses}}(\mu_3) \cap h_{F_{ses}}(\mu_4)\} \\ &\supseteq \{h_{F_{ses}}(\mu_1) \cap h_{F_{ses}}(\mu_2)\} \cap \{h_{F_{ses}}(\mu_1) \cap h_{F_{ses}}(\mu_2)\} \\ &= h_{F_{ses}}(\mu_1) \end{aligned}$$

Thus, $h_{F_{ses}}(\delta\mu_1 + (1-\delta)\mu_2) \supset h_{F_{ses}}(\mu_1)$ which contradicts equation (52). □

Corollary 6.8. *Suppose $(h_{F_{ses}}, S)$ be a concave fuzzy soft expert set. If there exists $\lambda \in \Gamma'$ for all $\mu_1, \mu_2 \in S$ with $h_{F_{ses}}(\mu_1) \neq h_{F_{ses}}(\mu_2)$,*

$$h_{F_{ses}}(\lambda\mu_1 + (1-\lambda)\mu_2) \subset h_{F_{ses}}(\mu_1) \cap h_{F_{ses}}(\mu_2),$$

then $(h_{F_{ses}}, S)$ is a strictly concave fuzzy soft expert set.
Proof. This is similar to the above theorem just with help of the above equations. □

Theorem 6.9. *Suppose $(h_{F_{ses}}, S)$ be a strongly convex soft fuzzy expert set. If there exists $\lambda \in \Gamma'$ for all $\mu_1, \mu_2 \in S$ with*

$$h_{F_{ses}}(\lambda\mu_1 + (1-\lambda)\mu_2) \supseteq h_{F_{ses}}(\mu_1) \cap h_{F_{ses}}(\mu_2) \quad (62)$$

then $(h_{F_{ses}}, S)$ is a convex fuzzy soft expert set.
Proof. Assume that $\mu_1, \mu_2 \in S$, $\delta \in \Gamma'$ such that

$$\mathcal{Z} \setminus h_{F_{ses}}(\delta\mu_1 + (1-\delta)\mu_2) \supseteq \mathcal{Z} \setminus \{h_{F_{ses}}(\mu_1) \cap h_{F_{ses}}(\mu_2)\}. \quad (63)$$

If $\mu_1 \neq \mu_2$ then equation (62) contradicts the fact that $(h_{F_{ses}}, S)$ is a convex fuzzy soft expert set. If $\mu_1 = \mu_2$, select $\delta \neq \delta_1 \in \Gamma'$ with $\delta = \lambda\delta_1 + (1-\lambda)\delta_1$. Suppose $\mu_1 = \delta_1\mu_1 + (1-\delta_1)\mu_2$, $\mu_2 = \delta_2\mu_1 + (1-\delta_2)\mu_2$. Equation (62) becomes

$$\mathcal{Z} \setminus h_{F_{ses}}(\mu_1) \supseteq \mathcal{Z} \setminus \{h_{F_{ses}}(\mu_1) \cap h_{F_{ses}}(\mu_2)\}. \quad (64)$$

$$\mathcal{Z} \setminus h_{F_{ses}}(\mu_2) \supseteq \mathcal{Z} \setminus \{h_{F_{ses}}(\mu_1) \cap h_{F_{ses}}(\mu_2)\}. \quad (65)$$

By equations (62), (63) and (64), we have

$$\begin{aligned} h_{F_{ses}}(\lambda\mu_1 + (1-\lambda)\mu_2) &\supseteq \{h_{F_{ses}}(\mu_1) \cap h_{F_{ses}}(\mu_2)\} \\ &\subset \{h_{F_{ses}}(\mu_1) \cap h_{F_{ses}}(\mu_2)\} \cap \{h_{F_{ses}}(\mu_1) \cap h_{F_{ses}}(\mu_2)\} \\ &= h_{F_{ses}}(\mu_1) \cap h_{F_{ses}}(\mu_2). \end{aligned}$$

And this contradicts that $(h_{F_{ses}}, S)$ is a strongly convex fuzzy soft expert set. □

Corollary 6.10. *Suppose $(h_{F_{ses}}, S)$ be a strongly concave fuzzy soft expert set. If there exists $\lambda \in \Gamma'$ for all $\mu_1, \mu_2 \in S$ with*

$$h_{F_{ses}}(\lambda\mu_1 + (1-\lambda)\mu_2) \subseteq h_{F_{ses}}(\mu_1) \cup h_{F_{ses}}(\mu_2) \quad (66)$$

then $(h_{F_{ses}}, S)$ is a concave fuzzy soft expert set.
Proof. The proof is similar to the above theorem. □

Theorem 6.11. *Suppose $(h_{F_{ses}}, S)$ be a convex fuzzy soft expert set. If there exists $\lambda \in \Gamma'$ for all μ_1, μ_2 such that $\mu_1 \neq \mu_2 \in S$ then*

$$h_{F_{ses}}(\lambda\mu_1 + (1-\lambda)\mu_2) \supset h_{F_{ses}}(\mu_1) \cap h_{F_{ses}}(\mu_2)$$

then $(h_{F_{ses}}, S)$ is a strongly convex fuzzy soft expert set.
Proof. Assume $\mu_1 \neq \mu_2, \mu_1, \mu_2 \in S$, $\delta \in \Gamma'$ such that

$$\mathcal{Z} \setminus h_{F_{ses}}(\delta\mu_1 + (1-\delta)\mu_2) \supset \mathcal{Z} \setminus \{h_{F_{ses}}(\mu_1) \cap h_{F_{ses}}(\mu_2)\}. \quad (67)$$

By using the above equation and with the condition of convex Fses

$$h_{F_{ses}}(\delta\mu_1 + (1-\delta)\mu_2) = \{h_{F_{ses}}(\mu_1) \cap h_{F_{ses}}(\mu_2)\}. \quad (68)$$

Moreover, it can be written as

$$(\lambda\mu_1 + (1-\lambda)\mu_2) = (\delta\mu_1 + (1-\delta)\mu_2). \quad (69)$$

While $\mu_1 = (\delta\mu_1 + (1-\delta)\mu_2)$, $\mu_2 = (\delta\mu_1 + (1-\delta)\mu_2)$ with selecting $\delta \in \Gamma'$. With the usage of definition of convexity of Fses and above defining μ_1, μ_2, we get

$$h_{F_{ses}}(\mu_1) \supseteq \{h_{F_{ses}}(\mu_1) \cap h_{F_{ses}}(\mu_2)\}. \quad (70)$$

$$h_{F_{ses}}(\mu_2) \supseteq \{h_{F_{ses}}(\mu_1) \cap h_{F_{ses}}(\mu_2)\}. \quad (71)$$

Using the given statement with three equations (68), (69), and (70), we have

$$\begin{aligned} h_{F_{ses}}(\delta\mu_1 + (1-\delta)\mu_2) &= h_{F_{ses}}(\lambda\mu_1 + (1-\lambda)\mu_2) \\ &\supset \{h_{F_{ses}}(\mu_1) \cap h_{F_{ses}}(\mu_2)\} \\ &\supseteq \{h_{F_{ses}}(\mu_1) \cap h_{F_{ses}}(\mu_2)\} \\ &\quad \cap \{h_{F_{ses}}(\mu_1) \cap h_{F_{ses}}(\mu_2)\} \\ &= h_{F_{ses}}(\mu_1) \cap h_{F_{ses}}(\mu_2) \end{aligned}$$

which is totally against the (67). □

Corollary 6.12. *Suppose $(h_{F_{ses}}, S)$ be a concave fuzzy soft expert set. If there exists $\lambda \in \Gamma'$ for all μ_1, μ_2 such that $\mu_1 \neq \mu_2 \in S$ then*

$$h_{F_{ses}}(\lambda\mu_1 + (1-\lambda)\mu_2) \subset h_{F_{ses}}(\mu_1) \cap h_{F_{ses}}(\mu_2)$$

then $(h_{F_{ses}}, S)$ is a strongly concave fuzzy soft expert set.
Proof. The proof is similar to the above theorem. □

7. CONVEX HULL AND CONVEX CONE

With the aid of the previously established concept of convex hull and cone on soft set, a new concept of convex hull and cone for fuzzy soft expert set has been constructed in this part. Fuzzy soft expert sets will be collected using the notation $X(\mathcal{Z})$.

Definition 7.1. The convex hull of a fuzzy soft expert set (H, S) is

$$\mathbf{Cnh}(H, S) = \bigcap_{(D,S) \supseteq (H,S)} \{(D, S) : (D, S) \in X(\mathcal{Z}) \text{ are Fses}\}.$$

Theorem 7.2. *Let $(D,S)\in X(\mathcal{Z})$, then its convex hull is given as*

$$\mathbf{Cnh}H(\mu)=\hat{\cup}_{r\in\mathbb{N}}\,\hat{\cup}_{P\in K(\mu,r)}\bar{\cap}\{H(v):v\in P\}$$

where

$$K(\mu,r)=\Big\{\{\mu_1,\mu_2,\ldots,\mu_r\}\subset S:\exists\,\pi_i\in[0,1],\ \sum_{i=1}^{r}\pi_i=1,\ \sum_{i=1}^{r}\pi_i\mu_i=\mu\Big\}.$$

Proof. By the usage of def. of convex hull, we have $\mathbf{Cnh}H(\mu)=\bigcap D(\mu):H(\mu)\supseteq D(\mu)$ such that $(D,S)\in X(\mathcal{Z})$. Now define the fuzzy soft expert set as

$$\mathbf{Cnh}\widetilde{H}(\mu)=\hat{\cup}_{r\in\mathbb{N}}\,\hat{\cup}_{P\in K(\mu,r)}\bar{\cap}\{H(v):v\in P\}$$

now we prove that $\mathbf{Cnh}H(\mu)=\mathbf{Cnh}\widetilde{H}(\mu)$. For this, we will do $\mathbf{Cnh}H(\mu)\subseteq\mathbf{Cnh}\widetilde{H}(\mu)$, and, then $\mathbf{Cnh}H(\mu)\supseteq\widetilde{H}(\mu)$.

Since every fuzzy soft expert set is a convex fuzzy soft expert set. So by convex fuzzy soft expert set, we have

$$D(\mu)\supseteq\hat{\cup}_{r\in\mathbb{N}}\,\hat{\cup}_{P\in K(\mu,r)}\bar{\cap}\{H(v):v\in P\}\supseteq\mathbf{Cnh}\widetilde{H}(\mu).$$

By using the intersection operation on left side, we get

$$\bar{\cap}_{H(\mu)\supseteq D(\mu)}D(\mu)\supseteq\mathbf{Cnh}\widetilde{H}(\mu).$$

Thus, $\mathbf{Cnh}H(\mu)\supseteq\mathbf{Cnh}\widetilde{H}(\mu)$.

Now, we will prove that $\mathbf{Cnh}H(\mu)\subseteq\mathbf{Cnh}\widetilde{H}(\mu)$. It is enough to show that $(\widetilde{H},S)$ is a convex Fses. Actually, since $\mathbf{Cnh}(H,S)$ is the smallest convex fuzzy soft expert set containing (H,S), $\mathbf{Cnh}(H,S)\subseteq(\widetilde{H},S)$. Suppose $\mu=\sum_{i=1}^{q}v_i\mu_i$ and $\mu'=\sum_{i=1}^{l}\gamma_i\mu_i'$ such that $\{\mu_1,\mu_2,\ldots,\mu_q\}\in K(\mu,q)$ and $\{\mu_1',\mu_2',\ldots,\mu_l'\}\in K(\mu',l)$, with $\sum_{i=1}^{q}\mu_i=1$ and $\sum_{i=1}^{l}\mu_i'=1$. Therefore,

$$\widetilde{H}\big(\lambda\mu+(1-\lambda)\mu'\big)=\widetilde{H}\Big(\lambda\sum_{i=1}^{q}v_i\mu_i+(1-\lambda)\sum_{i=1}^{l}\gamma_i\mu_i'\Big)$$

such that $\lambda\sum_{i=1}^{q}v_i\mu_i+(1-\lambda)\sum_{i=1}^{l}\gamma_i\mu_i'=1$ and $\{\mu_1,\mu_2,\mu_3,\ldots,\mu_q,\mu_1',\mu_2',\mu_3',\ldots,\mu_l'\}\in C(\lambda\mu+(1-\lambda)\mu',l+q)$. As $S=\mathbb{R}^n$, so $\lambda\mu+(1-\lambda)\mu'\in S=\mathbb{R}^n$.

Using the definition of fuzzy soft expert set described above to μ,μ' and to $x=\lambda\mu+(1-\lambda)\mu'$, we have

$$\hat{\cup}_{l+q\in\mathbb{N}}\,\hat{\cup}_{D\cup V\in K(x,l+q)}\,\bar{\cap}_{i=1}^{l+q}\{H(x_i):x_i\in D\cup V\}\supseteq\hat{\cup}_{q\in\mathbb{N}}\,\hat{\cup}_{D\in K(\mu,q)}\,\bar{\cap}_{i=1}^{q}\{H(\mu_i):\mu_i\in D\}\cap\hat{\cup}_{l\in\mathbb{N}}\,\hat{\cup}_{V\in K(\mu,l)}\,\bar{\cap}_{i=1}^{l}\{H(\mu_i):\mu_i\in V\},$$

That is,

$$\widetilde{H}\big(\lambda\mu+(1-\lambda)\mu'\big)\supseteq\widetilde{H}(\mu)\cap\widetilde{H}(\mu').$$

Hence proved. □

Definition 7.3. Let $(\Im,S)$ be fuzzy soft expert set, then it is called a cone if

$$\Im(\lambda\mu)\supseteq\Im(\mu)$$

for all $\mu\in S$ and $\lambda>0$. If the Fses is a convex, then it becomes convex cone.

Theorem 7.4. *A fuzzy soft expert set $(\Im,S)$ is said to be convex cone if and only if for every $\mu,\mu'\in S$ and $\lambda>0$*

- $\Im(\lambda\mu)\supseteq\Im(\mu)$.
- $\Im(\mu+\mu')\supseteq\Im(\mu)\cap\Im(\mu')$.

Proof. Suppose $(\Im,S)$ be a fuzzy soft expert set, then obviously (1) is proved. Now for 2nd point, take $\lambda=\frac{1}{2}$ and for any $\mu,\mu'\in S$, taking help from convex cone definition

$$\Im\big(\tfrac{1}{2}\mu+\tfrac{1}{2}\mu'\big)\supseteq\Im(\mu)\cap\Im(\mu')$$

and

$$\Im\Big(2\big(\tfrac{1}{2}\mu+\tfrac{1}{2}\mu'\big)\Big)\supseteq\Im\big(\tfrac{1}{2}\mu+\tfrac{1}{2}\mu'\big)$$

making use of above equations, we have

$$\begin{aligned}\Im(\mu+\mu')&=\Im\Big(2\big(\tfrac{1}{2}\mu+\tfrac{1}{2}\mu'\big)\Big)\\&\supseteq\Im\big(\tfrac{1}{2}\mu+\tfrac{1}{2}\mu'\big)\\&\supseteq\Im(\mu)\cap\Im(\mu')\end{aligned}$$

$$\Im(\mu+\mu')\supseteq\Im(\mu)\cap\Im(\mu')$$

hence proved. On converse way, we prove $(\Im,S)$ is a convex fuzzy soft expert set. By first condition $(\Im,S)$ is cone. With the help of 1st and 2nd points and $\lambda\in[0,1]$, it is obviously that $(\Im,S)$ is convex fuzzy soft expert set. i.e.,

$$\begin{aligned}\Im\big(\lambda\mu+(1-\lambda)\mu'\big)&\supseteq\Im(\lambda\mu)\cap\Im\big((1-\lambda)\mu'\big)\\&\supseteq\Im(\mu)\cap\Im(\mu')\end{aligned}$$

Therefore, $\Im\big(\lambda\mu+(1-\lambda)\mu'\big)\supseteq\Im(\mu)\cap\Im(\mu')$. So, $(\Im,S)$ is convex fuzzy soft expert set. □

Corollary 7.5. *If $(\Im,S)$ is a fuzzy soft expert set then it is a convex cone if and only if*

$$\Im\Big(\sum_{i=1}^{m}\lambda_i\mu_i\Big)\supseteq\bigcap_{i=1}^{m}\Im(\mu_i),$$

for each $\mu_i\in S$ and $\lambda_i>0$.

8. COMPARISON AND DISCUSSION

In this study, the convexity on fuzzy soft expert sets has been compared with the convexity on different models, such as soft sets, fuzzy soft sets, and soft expert sets.

(1) **Convexity in Soft Sets (Deli, 2019; Salih & Sabir, 2020; Majeed, 2016; Rahman *et al.*, 2021b):** Convexity has primarily been introduced in a crisp and parameter-based settings in classical soft set theory, where membership grades are not used to represent uncertainty.

- By taking into account combinations of soft elements based on parameterized subsets, Deli (2019) developed soft convex sets, primarily concentrating on set inclusion under convex combinations of elements.
- By examining the structural characteristics of soft sets under algebraic operations and highlighting the preservation of convexity under union and intersection-like operations, Salih and Sabir (2020) expanded the concepts of convexity.
- Convex soft structures in decision-making contexts were further studied by Majeed (2016) with an emphasis on the stability of soft alternatives under convex combinations and viable region development.

In classical soft set theory, different researchers, as described above, tried to define convexity in a crisp and parameterized framework, where uncertainty is handled only through set inclusion. It provides a basic algebraic structure for studying combinations of soft elements under convex-type conditions. However, it does not include fuzziness or membership grades, so gradual uncertainty cannot be modeled. Moreover, expert opinions are not considered, which limits its applicability in complex decision-making problems. Hence, its scope remains mainly theoretical, with restricted real-world modeling capability.

(2) **Convexity in Soft Expert Sets (Ihsan *et al.*, 2021):** Convexity is expanded to incorporate expert-based evaluations in soft expert sets, in which several experts provide parameterized judgments.

- Expert parameter agreement and inclusion relationships are used to determine convexity.
- Focus is on ensuring that combinations of expert evaluations remain within the acceptable decision space.

Soft expert sets incorporate expert opinions into the decision-making framework, allowing parameterized uncertainty to be evaluated through expert-based roles such as agree/disagree or yes/no responses, which makes the model more realistic and applicable than classical soft sets. However, the structure remains crisp in nature and does not include fuzzy membership grades, so it cannot represent partial belonging or gradual preference levels. As a result, the concept of convexity in soft expert sets is still restricted to set inclusion logic without capturing smooth or graded transitions between alternatives. Therefore, while more expressive than soft sets, it still has limitations in modeling complex uncertainty.

(3) **Convexity in Fuzzy Soft Sets (Rahman *et al.*, 2021a):** Convexity in fuzzy soft sets, where membership degrees are used to indicate uncertainty, was presented by Rahman *et al.*

- Convexity is defined using fuzzy membership inequalities under convex combinations.
- If two fuzzy soft elements belong to a set, their weighted average also belongs with appropriate membership condition.

Fuzzy soft sets handle fuzziness through membership degrees, allowing the representation of gradual uncertainty in decision-making problems. They are effective in modeling partial belonging and support weighted aggregation of information, which improves flexibility in combining uncertain data. However, they do not incorporate expert opinions or multi-expert decision-making structures, which limits their use in expert-driven environments. As a result, convexity in fuzzy soft sets is purely based on membership values rather than expert judgments or opinion-based interactions.

(4) **Convexity in Fuzzy Soft Expert Sets (proposed work):** Fuzzy soft expert sets provide a comprehensive framework by integrating fuzzy membership, parameterization, and expert opinions into a single structure, making them highly suitable for complex decision-making under uncertainty. This model strengthens the geometric and algebraic foundation by introducing advanced convexity concepts such as standard convexity, strict (strong) convexity, and (λ, ψ)-convexity, along with convex hull and convex cone structures. These extensions allow greater flexibility in modeling different degrees and types of convex behavior in expert-based fuzzy environments. As a result, the framework supports both algebraic analysis and optimization-based decision-making, offering a more powerful and general approach compared to earlier soft and fuzzy soft models.

The model's comparison analysis is displayed in Table 1. Some salient features of current models have been compared with the suggested structure in this table. It is evident from Table 1 that our suggested framework is more comprehensive than the ones mentioned before.

Table 1: Comparison with features.

Features	SS	SES	FSS	FSES
Uncertain type	Crisp	Crisp + expert	Fuzzy	Fuzzy + expert
Parameterization	Yes	Yes	Yes	Yes
Experts opinions	No	Yes	No	Yes
Fuzzy membership	No	No	Yes	Yes
Classical convexity	Yes	Limited	Yes	Yes
Strict/strong convexity	No	No	Limited	Fully developed
(λ, ψ) Convexity	No	No	Limited	Fully developed
Convexity in Senses	No	No	Limited	Fully developed
Convex hull	No	No	Limited	Fully developed
Convex cone	No	No	Limited	Fully developed

Convexity in fuzzy soft expert sets provides a unified and more general framework that simultaneously incorporates fuzziness, parameterization, and expert knowledge while also extending classical convexity into stronger and more flexible forms, such as strict convexity and (λ, ψ)-convexity. Moreover, the introduction of convex hull and convex cone structures significantly enhances the algebraic and geometric richness of the model.

(1) It is more useful as compared to soft-like structures because, through this structure, multi-decisive opinions of different experts can easily be found.

(2) It has a good environment for uncertain types of data by applying fuzzy membership values to every entity communicating with every parameter.

8.1 Practical Application of (λ, ψ)-Convexity in Group Decision-Making

Consider a group decision-making problem involving the selection of a supplier for a construction project. Three criteria are used to evaluate the choice: μ_1 = Cost, μ_2 = Quality, and μ_3 = Delivery Reliability. Suppose that four experts, $\omega_1, \omega_2, \omega_3$, and ω_4, participate in the evaluation. At first, each expert expresses a binary view, with 0 denoting a bad opinion and 1 denoting a good one. The following two supplier assessment profiles are the outcome of incorporating these expert opinions into the suggested Fses framework:

$$A_1 = h_{F_{ses}}(1,1,1) = \{\omega_1/0.1, \omega_2/0.7, \omega_3/0.4, \omega_4/0.1\}, \text{ and}$$
$$A_2 = h_{F_{ses}}(1,2,1) = \{\omega_1/0.3, \omega_2/0.8, \omega_3/0.1, \omega_4/0.2\}.$$

Here, A_1 and A_2 represent two supplier evaluation profiles under the considered decision parameters, while the values associated with $\omega_1, \omega_2, \omega_3$, and ω_4, represent the corresponding degrees of expert support generated by the FSES framework. Let $\lambda = 0.5$, $\psi = 1$, and $\delta = 0.5$, and consider $\mu_1 = (1,1,1)$ and $\mu_2 = (1,2,1)$. According to the proposed definition of (λ, ψ)-convexity,

$$h_{F_{ses}}(\lambda\mu_1 + \psi(1-\lambda)\mu_2) \supseteq h_{F_{ses}}(\mu_1) \cap h_{F_{ses}}(\mu_2).$$

The combined decision profile is first obtained as follows:

$$\begin{aligned} h_{F_{ses}}(\lambda\mu_1 + \psi(1-\lambda)\mu_2) &= 0.5(1,1,1) + (1)(1-0.5)(1,2,1) \\ &= 0.5(1,1,1) + 0.5(1,2,1) \\ &= (1, 1.5, 1). \end{aligned}$$

The resulting profile represents a combined assessment obtained by giving equal weight to the two supplier evaluation profiles. The common information contained in the two supplier profiles is obtained from $h_{F_{ses}}(\mu_1) \cap h_{F_{ses}}(\mu_2) = h_{F_{ses}}(1,1,1) \cap h_{F_{ses}}(1,2,1)$. Thus,

$$h_{F_{ses}}(\mu_1) \cap h_{F_{ses}}(\mu_2) = \{\omega_1/0.1, \omega_2/0.7, \omega_3/0.1, \omega_4/0.1\}.$$

Therefore, the (λ, ψ)-convexity condition is expressed as

$$h_{F_{ses}}(1, 1.5, 1) \supseteq \{\omega_1/0.1, \omega_2/0.7, \omega_3/0.1, \omega_4/0.1\}.$$

The combined profile is read in accordance with the relevant parameter evaluation under the parameter representation used in the proposed FSES framework, and the final assessment retains the shared expert-supported information.

A decision-making interpretation of (λ, ψ)-convexity in supplier selection is given by the aforementioned expression. Different supplier assessment scenarios based on cost, quality, and delivery dependability are represented by the two FSES profiles. The four experts provide their evaluations, and the resulting expert-support degrees are incorporated into the FSES representation. The set $h_{F_{ses}}(\mu_1) \cap h_{F_{ses}}(\mu_2)$ represents the expert-supported information that is common to the two assessment profiles. The expression $(\lambda\mu_1 + \psi(1 - \lambda)\mu_2)$ represents a compromise between the two supplier profiles, where λ controls the contribution of the first profile and ψ regulates the contribution of the second profile. Therefore, a mathematical method for determining whether two supplier assessment profiles' shared expert-supported knowledge is maintained when the profiles are joined is provided by (λ, ψ)-convexity. This feature can be used to examine compromise assessments resulting from various supplier evaluation profiles in a collaborative decision-making setting. As a result, the suggested (λ, ψ)-convex FSES framework goes beyond straightforward numerical verification by offering a mathematical depiction of compromise and shared expert knowledge in real-world decision-making issues like supplier selection.

9. CONCLUSION

After a deep analysis of the literature, we observe that fuzzy soft set-like structures describe the opinion of a single expert in a single model. However, there are certain situations (a) when we require multi-decisive opinions of different experts in a single model. Therefore, in order to address these limitations, certain features of convexity (concavity) have been discussed with fine results. In this article, different types of convexity with different senses, convex hull, and convex cone are developed for fuzzy soft expert sets. Some of their essential characteristics, i.e., complement, intersection, and union, are proved. It may also include the extension of this work by considering the modified versions of different aggregation operators. It has been observed through the literature that fuzzy soft set-like structures have certain limitations. After the analysis of the literature, when we talk about the parameters, the proposed structure has limitations for hypersoft expert sets, along with fuzzy, intuitionistic fuzzy, and neutrosophic settings in the domain. The proposed study also has certain limitations for certain situations (a) when information has an interval or (b) periodic form. It also has certain limitations for structures dealing with non-membership and neutral degrees. It cannot handle data of a periodic nature, interval type, or two-dimensional space type data (intuitionistic fuzzy and neutrosophic). The upcoming task (future directions) may involve the extension of the proposed framework to more general settings.

(1) Development of different types of convexities with their properties for fuzzy set-like structures, such as intuitionistic, Pythagorean, neutrosophic, and so forth.

(2) Characterization of convex hull and convex cone on fuzzy set-like structures, such as intuitionistic, Pythagorean, neutrosophic, and so forth.

(3) Development of many other variants of convexity, such as φ-convexity, triangular convexity, etc., for fuzzy set-like structures.

(4) The extension of this work to other structures, such as intuitionistic fuzzy soft set, neutrosophic soft set, hypersoft set, fuzzy hypersoft set, intuitionistic fuzzy hypersoft set, neutrosophic hypersoft set, etc., under multi-decisive opinions.

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Ethical Statement

This study does not involve human participants or animals. Ethical approval is therefore not required.

Conflict of Interest

The authors declare that they have no conflict of interest.

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Author Contributions

M.I. conceived and conducted the study, analyzed and interpreted the data, reviewed the literature, drafted the manuscript, and supervised the research. M.A. provided guidance and revised the manuscript.

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