

Original Article

LOCAL FROBENIUS ANALYSIS OF THE SEQUENTIAL CONFORMABLE HEUN EQUATION AT THREE FINITE α -SINGULAR POINTS

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ABSTRACT

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In this study, a specific Frobenius framework is proposed for the sequential conformable Heun equation in the original variable. The corresponding substitution $z = x^\alpha$, $y(x) = Y(z)$ determines the relevant scaled parameters and establishes precise equivalence with the ordinary general Heun equation. The local analysis in x is carried out using the sequential operator, $T_\alpha^2 = T_\alpha \circ T_\alpha$ with conformable steps x^α , $x^\alpha - 1$, and $x^\alpha - a$. For $0 < \alpha \leq 1$, $a > 0$, and $a \neq 1$, the finite singular points $x = 0$, $x = 1$, and $x = a^{\frac{1}{\alpha}}$ are proved to be regular α -singular. Each center yields characteristic exponents, the indicial equation, a three-term coefficient recurrence, nonresonant Frobenius fundamental conditions, and a guaranteed local convergence domain. To solve resonant cases, a recursive compatibility condition is used: if the condition is satisfied, a second ordinary conformable Frobenius series occurs; alternatively, the second independent solution contains a logarithmic term. Furthermore, sufficient and necessary termination conditions are obtained for finite conformable Frobenius solutions, as well as a determinant condition regarding the accessory parameter and an exact terminating example based on an identically zero residual. A numerical example shows how to calculate local coefficient sequences and evaluate the reduced series with a normalized conformable residual. When $\alpha = 1$, the equation is reduced to the normal general Heun-Frobenius structure. The framework that is generated gives a unified original-variable description of the local solution framework at all three finite singular points. In the numerical verification, increasing the truncation order from $N = 6$ to $N = 20$ reduces the maximum normalized residual by factors of about $4.9 \times 10^3 - 9.0 \times 10^3$ across the six local branches; the $N = 20$ residuals range from 1.147×10^{-6} to 6.745×10^{-5} on the stated interior comparison interval.

Keywords: Conformable derivative; sequential conformable equation; Heun equation; Frobenius method; regular α -singular point; recurrence relation.

1. INTRODUCTION

For second-order linear Fuchsian equations, the general Heun equation has four regular singular points. Three of these points occur at 0 , 1 , a in conventional normalization, with the fourth at ∞ . This singularity structure underlies the appearance of Heun equations in a variety of mathematical physics contexts, including wave propagation, quantum mechanics, general relativity, and boundary-value and spectral problems (Coddington & Levinson, 1955; Hortaçsu, 2018; Ince, 1956; NIST Digital Library of Mathematical Functions, n.d.; Ronveaux, 1995; Slavyanov & Lay, 2000). Their local analysis is therefore closely related to Frobenius expansions and the extension of solutions from one regular singular point to another (Cheb-Terrab, 2004; A. M. Ishkhanyan, 2019, 2020; A. M. Ishkhanyan & Suominen, 2003; T. A. Ishkhanyan *et al.*, 2018; Maier, 2005; Manukyan *et al.*, 2014). Transformations among local Heun solutions and their singular-point representations are treated systematically in (Maier, 2007).

Several studies provide the foundation for the two lines of research brought together here. Ronveaux's volume provided a system-

atic treatment of the classical Heun equation (Ronveaux, 1995). Maier investigated transformations among its local solutions (Maier, 2007), and Hortaçsu surveyed its physical applications (Hortaçsu, 2018). In conformable calculus, Khalil *et al.* introduced the derivative (Khalil *et al.*, 2014), Abdeljawad developed its main calculus rules (Abdeljawad, 2015), and Ünal, Gökdoğan, and Çelik developed series solutions for sequential conformable equations at ordinary and regular α -singular points and established existence and uniqueness results (Gökdoğan *et al.*, 2016; Ünal *et al.*, 2015, 2017). Chung *et al.* showed that conformable Schrödinger equations can occur in wave functions defined as Heun functions (Chung *et al.*, 2017). These results provide a natural foundation for the direct Frobenius analysis provided in this paper.

From an applied viewpoint, Heun-type equations arise in quantum-mechanical and wave problems, and conformable Schrödinger models with Heun-function solutions have also been reported (Hortaçsu, 2018; Chung *et al.*, 2017). Accordingly, the local formulas developed here can be used when such a model is written directly in a conformable coordinate, and local behavior is required near a finite singular point.

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On the positive real axis, the conformable derivative of a differentiable function can be represented as its conventional derivative. As a result, the modification of the variable $z = x^\alpha$ changes the order conformable Heun equation into the ordinary general Heun equation with the appropriate modification of the parameters. This interaction is essential for understanding the classical structure and confirming the $\alpha = 1$ limit (Abdelhakim, 2019; Anderson *et al.*, 2019).

Related work by Abu Hammad *et al.* applied the conformable Frobenius method to the confluent α -hypergeometric equation near the regular α -singular point $x = 0$ (Abu Hammad *et al.*, 2022).

Their study was restricted to one singular point of a confluent equation. In contrast, this study examines the Heun equation at all three finite α -singular points. It offers a unified treatment of recurrence relations, resonance cases, convergence domains, and series termination conditions.

Several studies have used series-based and numerical strategies to solve fractional and nonlinear differential equations, such as residual power-series approximations and numerical treatments for fractional-order models (Hasan, 2018; Manaa *et al.*, 2021; Zekri, 2021).

Even though this transformation determines the underlying classical equation, it does not change the value of a direct expression in the original variable. In particular, powers of an increment are closed by the sequential operator, which allows the indicial equations and coefficient recurrences to be determined without constantly converting the local series between two variables. The direct method clarifies the points of the α -singular points, particularly the right-hand treatment of the boundary point $x = 0$.

The change $z = x^\alpha$ demonstrates that the sequential conformable equation is comparable to the classical general Heun equation after reducing the parameters. As a result, the equation itself is not presented as a new special function. This study uses the local Frobenius analysis in the original variable x . The classical Heun concept is typically expressed in the modified variable z (Cheb-Terrab, 2004; Coddington & Levinson, 1955; Hortaçsu, 2018; Ince, 1956; A. M. Ishkhanyan, 2019, 2020; A. M. Ishkhanyan & Suominen, 2003; T. A. Ishkhanyan *et al.*, 2018; Maier, 2005, 2007; Manukyan *et al.*, 2014; NIST Digital Library of Mathematical Functions, n.d.; Ronveaux, 1995; Slavyanov & Lay, 2000), whereas Refs. (Gökdoğan *et al.*, 2016; Ünal *et al.*, 2015, 2017) establish the general foundations for series solutions to sequential conformable equations. Combining these viewpoints, we obtain the local expansions at $x = 0$, $x = 1$, and $x = a^{\frac{1}{\alpha}}$ by the natural increments x^α , $x^\alpha - 1$, and $x^\alpha - a$. The unified method simplifies branch conventions, resonance conditions, recurrence relations, convergence domains, and termination conditions for all three finite singular points.

This paper presents a unified Frobenius analysis of the sequential conformable Heun equation in the original variable x . We first establish its exact connection to the classical general Heun equation; we use the transformation $z = x^\alpha$ and scale the exponent and accessory parameters correctly. The study is performed directly at the three finite regular α -singular points $x = 0$, $x = 1$, and $x = a^{1/\alpha}$. The corresponding domains, branch conventions, and right-hand interpretation at $x = 0$ are fully provided. At each point, we construct the indicial equation and the matching three-term coefficient recurrence and represent the three local recurrences within a uniform framework. This formulation specifies the compatibility condition that identifies whether a resonant second solution remains an ordinary conformable Frobenius series or includes a logarithmic term. Additionally, local convergence domains are produced from the points of the classical Heun singularities. The article establishes necessary and sufficient criteria for series termination, uses a finite determinant to describe the accessory-parameter requirement, and supports its findings with an accurate terminating example and normalized numerical residuals.

At each center x_0 , the construction is carried out in the conformable increment $\xi = x^\alpha - x_0^\alpha$. We substitute $y = \xi^r \sum_{n=0}^{\infty} u_n \xi^n$, apply $T_\alpha(\xi^m) = \alpha m \xi^{m-1}$ and $T_\alpha^2(\xi^m) = \alpha^2 m(m-1) \xi^{m-2}$, expand the polynomial coefficients in ξ , and first equate the lowest power to obtain the indicial equation. Equating each subsequent power then gives the three-term recurrence. The methodological point of this formulation is that the same original-variable calculation is used at all three finite centers, so the branch convention, resonance

obstruction, convergence domain, and termination conditions remain visible in x rather than being reconstructed separately after solving in $z = x^\alpha$.

The remaining sections of the paper are structured as follows. Section 2 summarizes the conformable derivative, sequential operator, domain conventions, and general indicial formulas. Section 3 discusses the sequential conformability. The Heun equation is compared to the classical generic Heun equation and classified into three finite regular α -singular locations. Sections 4–6 calculate the local Frobenius expansions at the three centers. Section 7 looks at nonresonant and resonant local bases, as well as termination and determinant criteria for finite conformable Frobenius solutions. Section 8 presents a numerical demonstration and direct residual verification, while Section 9 summarizes the paper.

2. PRELIMINARIES

This section recalls the conformable derivative and its sequential operator, fixes the domain and branch conventions used at the three finite α -singular points, and states the regular α -singular-point criterion and the general indicial equation used in the subsequent local Frobenius analysis.

CONFORMABLE DERIVATIVE AND SEQUENTIAL OPERATOR

Let $0 < \alpha \leq 1$ and $x > 0$. The conformable derivative of a function f is defined by (Khalil *et al.*, 2014)

$$T_\alpha f(x) = \lim_{h \rightarrow 0} \frac{f(x + hx^{1-\alpha}) - f(x)}{h}. \quad (1)$$

This definition is valid when the limit exists. If f is differentiable at $x > 0$, then

$$T_\alpha f(x) = x^{1-\alpha} f'(x). \quad (2)$$

We apply the sequential operator:

$$T_\alpha^2 f := T_\alpha(T_\alpha f), \quad (3)$$

Note that this is not a derivative of order 2α .

If x_0 is a point satisfying $x_0 \geq 0$, and m is an arbitrary exponent, then direct application of the conformable derivative yields

$$T_\alpha(x^\alpha - x_0^\alpha)^m = \alpha m(x^\alpha - x_0^\alpha)^{m-1}, \quad T_\alpha^2(x^\alpha - x_0^\alpha)^m = \alpha^2 m(m-1)(x^\alpha - x_0^\alpha)^{m-2}. \quad (4)$$

These identities represent the fundamental algebraic foundation of the direct conformable Frobenius method. Since this operator is expressible through the ordinary derivative for differentiable functions, it is local rather than a nonlocal fractional operator; the term conformable is retained here in its established literature sense (Abdelhakim, 2019; Anderson *et al.*, 2019).

DOMAIN AND BRANCH CONVENTIONS

The conformable equation is considered to be on the positive real axis $x > 0$, with $0 < \alpha \leq 1$. We assume that $a > 0$ and $a \neq 1$, and define

$$x_a = a^{1/\alpha}.$$

Thus, the three finite α -singular points are $x = 0$, $x = 1$, and $x = x_a$. The expansion about $x = 0$ is interpreted in the limit as x approaches zero from the right, since the conformable derivative for $x > 0$ is defined. Consequently, $x = 0$ is treated as a regular α -singular point on the right-hand boundary.

For a noninteger exponent r , the branch of $(x^\alpha - x_0^\alpha)^r$ is chosen on a one-sided punctured neighborhood in which the conformable increment has a fixed sign. In the present real-valued analysis, right-hand neighborhoods of the singular points are used. The absolute-value inequalities shown below describe the entire convergence radii in the conformable coordinate, although the real branches are understood to be restricted to $x > x_0$.

For complex r on $x > 0$ we use $x^r = \exp(r \operatorname{Log} x)$, where $\operatorname{Log} x$ is the real logarithm on the positive axis. Thus x^r is single-valued on each right-hand punctured neighborhood of $x = 0$ and $|x^r| = x^{\operatorname{Re}(r)}$. The Frobenius expressions near $x = 0$ are therefore interpreted for $x \rightarrow 0^+$ on $0 < x < \eta$; no continuation through $x = 0$ or across the negative real axis is used.

REGULAR α -SINGULAR POINTS

Consider the sequential conformable differential equation.

$$T_\alpha^2 y + p(x) T_\alpha y + q(x) y = 0. \quad (5)$$

Definition 2.1. If (Ünal *et al.*, 2017)

$$(x^\alpha - x_0^\alpha) p(x) \quad \text{and} \quad (x^\alpha - x_0^\alpha)^2 q(x)$$

have convergent power-series expansions in $x^\alpha - x_0^\alpha$ close to $x = x_0$, then $x = x_0$ is a regular α -singular point of (5).

Proposition 2.1. Assume that close to $x = x_0$,

$$p(x) = \frac{p_{-1}}{x^\alpha - x_0^\alpha} + \sum_{m=0}^{\infty} p_m (x^\alpha - x_0^\alpha)^m,$$

$$q(x) = \frac{q_{-2}}{(x^\alpha - x_0^\alpha)^2} + \frac{q_{-1}}{x^\alpha - x_0^\alpha} + \sum_{m=0}^{\infty} q_m (x^\alpha - x_0^\alpha)^m.$$

For a solution in direct conformable Frobenius form,

$$y(x) = \sum_{n=0}^{\infty} c_n (x^\alpha - x_0^\alpha)^{n+r}, \quad c_0 \neq 0,$$

Therefore, the indicial equation is

The calculation uses the ordinary product rule in its conformable form, $T_\alpha(fg) = fT_\alpha g + gT_\alpha f$. For $0 < \alpha \leq 1$, this identity is applied on $x > 0$ where the factors are differentiable. For the sequential operator $T_\alpha^2 = T_\alpha \circ T_\alpha$, y and $T_\alpha y$ are required to be differentiable on the one-sided punctured neighborhood considered. Near $x = 0$ the Frobenius powers are handled only for $x > 0$ and termwise differentiation is used inside the local convergence interval; the endpoint $x = 0$ is reached only as a right-hand limit.

$$\alpha^2 r(r-1) + \alpha p_{-1} r + q_{-2} = 0. \quad (6)$$

Proof. Using (4),

$$T_\alpha y = \alpha \sum_{n=0}^{\infty} (n+r) c_n (x^\alpha - x_0^\alpha)^{n+r-1},$$

$$T_\alpha^2 y = \alpha^2 \sum_{n=0}^{\infty} (n+r)(n+r-1) c_n (x^\alpha - x_0^\alpha)^{n+r-2}.$$

The lowest power coefficient $(x^\alpha - x_0^\alpha)^{r-2}$ is

Consequently, $[\alpha^2 r(r-1) + \alpha p_{-1} r + q_{-2}] c_0 = 0$, and since $c_0 \neq 0$, the bracketed factor must vanish.

Equation (6) follows since $c_0 \neq 0$.

3. THE SEQUENTIAL CONFORMABLE HEUN EQUATION

Consider

$$T_\alpha^2 y + \left(\frac{\gamma}{x^\alpha} + \frac{\delta}{x^\alpha - 1} + \frac{\varepsilon}{x^\alpha - a} \right) T_\alpha y + \frac{Ax^\alpha - B}{x^\alpha (x^\alpha - 1)(x^\alpha - a)} y = 0, \quad (7)$$

where $0 < \alpha \leq 1$, $a > 0$, $a \neq 1$, and $\gamma, \delta, \varepsilon, A$, and B are constants.

RELATION WITH THE CLASSICAL GENERAL HEUN EQUATION

Before carrying out the direct singular-point calculations, it is useful to clarify the relation between equation (7) and the classical general Heun equation. Introduce the change of variable:

$$z = x^\alpha, \quad y(x) = Y(z).$$

For $x > 0$, equation (2) gives

$$T_\alpha y = \alpha \frac{dY}{dz}, \quad T_\alpha^2 y = \alpha^2 \frac{d^2 Y}{dz^2}.$$

Substitution into equation (7), followed by division by α^2 , yields

$$\frac{d^2 Y}{dz^2} + \left(\frac{\gamma/\alpha}{z} + \frac{\delta/\alpha}{z-1} + \frac{\varepsilon/\alpha}{z-a} \right) \frac{dY}{dz} + \frac{(A/\alpha^2)z - B/\alpha^2}{z(z-1)(z-a)} Y = 0.$$

Define the scaled parameters:

$$\tilde{\gamma} = \frac{\gamma}{\alpha}, \quad \tilde{\delta} = \frac{\delta}{\alpha}, \quad \tilde{\varepsilon} = \frac{\varepsilon}{\alpha}, \quad \tilde{A} = \frac{A}{\alpha^2}, \quad \tilde{B} = \frac{B}{\alpha^2}.$$

The transformed equation can then be written in the standard general Heun form (Cheb-Terrab, 2004; Maier, 2005, 2007; NIST Digital Library of Mathematical Functions, n.d.)

$$Y'' + \left(\frac{\tilde{\gamma}}{z} + \frac{\tilde{\delta}}{z-1} + \frac{\tilde{\varepsilon}}{z-a} \right) Y' + \frac{\lambda\mu z - q}{z(z-1)(z-a)} Y = 0,$$

where

$$\lambda\mu = \tilde{A}, \quad q = \tilde{B}, \quad \lambda + \mu + 1 = \tilde{\gamma} + \tilde{\delta} + \tilde{\varepsilon}.$$

Thus, equation (7) is related to the classical general Heun equation through the reparametrization $z = x^\alpha$. Under this transformation, the singular points $z = 0$, $z = 1$, and $z = a$ correspond respectively to

$$x = 0, \quad x = 1, \quad x = a^{1/\alpha}.$$

This correspondence is stated only to identify the underlying classical structure. In the following sections, the singular-point classifications, indicial equations, coefficient recurrence relations, and local Frobenius expansions are obtained directly in the original variable x , using the conformable increments.

$$x^\alpha, \quad x^\alpha - 1, \quad x^\alpha - a.$$

Remark 3.1. When $\alpha = 1$, the transformation becomes the identity $z = x$, and the scaled parameters reduce to the original parameters, while the local exponents and recurrence relations reduce to their classical Heun counterparts.

Because $z = x^\alpha$ is an exact pullback on $x > 0$, the conformable formulation does not by itself introduce nonlocal memory or a new physical interaction. Its practical modeling advantage is instead that equations posed in a conformable coordinate can be analyzed without leaving the original variable x : α remains explicit as a scaling parameter, the locations $x = 0$, 1 , and x_a retain their direct interpretation, and boundary or local-series conditions can be imposed in the same variable used by the model. This is useful for conformable Schrödinger- and wave-type formulations, while remaining fully consistent with the classical Heun structure (Chung *et al.*, 2017; Anderson *et al.*, 2019).

DIRECT CLASSIFICATION OF THE FINITE α -INDICIAL EQUATION SINGULAR POINTS

The coefficient functions in (7) are

$$p(x) = \frac{\gamma}{x^\alpha} + \frac{\delta}{x^\alpha - 1} + \frac{\varepsilon}{x^\alpha - a}, q(x) = \frac{Ax^\alpha - B}{x^\alpha(x^\alpha - 1)(x^\alpha - a)}.$$

At $x = 0$,

$$x^\alpha p(x) = \gamma + \frac{\delta x^\alpha}{x^\alpha - 1} + \frac{\varepsilon x^\alpha}{x^\alpha - a},$$

$$(x^\alpha)^2 q(x) = \frac{x^\alpha(Ax^\alpha - B)}{(x^\alpha - 1)(x^\alpha - a)}.$$

Both expressions possess convergent expansions in powers of x^α near $x = 0$.

At $x = 1$,

$$(x^\alpha - 1)p(x) = \frac{\gamma(x^\alpha - 1)}{x^\alpha} + \delta + \frac{\varepsilon(x^\alpha - 1)}{x^\alpha - a},$$

$$(x^\alpha - 1)^2 q(x) = \frac{(x^\alpha - 1)(Ax^\alpha - B)}{x^\alpha(x^\alpha - a)}.$$

Both expressions possess convergent expansions in powers of $x^\alpha - 1$ near $x = 1$.

At $x = x_a = a^{1/\alpha}$,

$$(x^\alpha - a)p(x) = \frac{\gamma(x^\alpha - a)}{x^\alpha} + \frac{\delta(x^\alpha - a)}{x^\alpha - 1} + \varepsilon,$$

$$(x^\alpha - a)^2 q(x) = \frac{(x^\alpha - a)(Ax^\alpha - B)}{x^\alpha(x^\alpha - 1)}.$$

Both expressions possess convergent expansions in powers of $x^\alpha - a$ near $x = x_a$. Hence, $x = 0$, $x = 1$, and $x = x_a$ are regular α -singular points of (7).

Multiplying (7) by $x^\alpha(x^\alpha - 1)(x^\alpha - a)$ gives

$$x^\alpha(x^\alpha - 1)(x^\alpha - a)T_\alpha y$$

$$+ [\gamma(x^\alpha - 1)(x^\alpha - a) + \delta x^\alpha(x^\alpha - a) + \varepsilon x^\alpha(x^\alpha - 1)]T_\alpha y$$

$$+ (Ax^\alpha - B)y = 0.$$

For convenience, define

$$S = \gamma + \delta + \varepsilon.$$

Remark 3.2. Convergence of the local series. Since the transformation $z = x^\alpha$ converts equation (7) into the classical general Heun equation, the classical Frobenius convergence theorem shows that each local series converges at least up to the nearest singular point in the z -plane (Coddington & Levinson, 1955; Ince, 1956). Therefore,

The same radii follow directly from the coefficient recurrences by a ratio argument. Let $\rho = \lim u_n/u_{n-1}$ when the limit exists and retain the n^2 terms of the three-term recurrence. About $x = 0$ one obtains $a\rho^2 - (1+a)\rho + 1 = 0$, with roots 1 and $1/a$; about $x = 1$ one obtains $(1-a)\rho^2 + (2-a)\rho + 1 = 0$, with roots -1 and $-1/(1-a)$; and about $x = x_a$ one obtains $a(a-1)\rho^2 + (2a-1)\rho + 1 = 0$, with roots $-1/a$ and $-1/(a-1)$. Hence the ratio test gives $R_0 = \min\{1, a\}$, $R_1 = \min\{1, |1-a|\}$, and $R_a = \min\{a, |a-1|\}$ in the conformable increment. These agree with the nearest-singularity radii and provide an explicit recurrence-based derivation.

$$R_0 = \min\{1, a\}, R_1 = \min\{1, |1-a|\}, R_a = \min\{a, |a-1|\},$$

and the corresponding convergence domain about $x = x_0$ is

$$0 < |x^\alpha - x_0^\alpha| < R_{x_0}.$$

At $x_0 = 0$, this domain is interpreted from the right.

4. FROBENIUS SOLUTIONS ABOUT $x = 0$

This section constructs the two local Frobenius branches at the right-hand α -singular point $x = 0$. The indicial roots are obtained first, followed by the coefficient recurrence and the nonresonant local basis.

Seek a solution of the form

$$y(x) = \sum_{n=0}^{\infty} c_n x^{\alpha(n+r)}, \quad c_0 \neq 0. \quad (9)$$

Then

$$T_\alpha y = \alpha \sum_{n=0}^{\infty} (n+r) c_n x^{\alpha(n+r-1)},$$

$$T_\alpha^2 y = \alpha^2 \sum_{n=0}^{\infty} (n+r)(n+r-1) c_n x^{\alpha(n+r-2)}.$$

Let

$$L_0 = (1+a)\gamma + a\delta + \varepsilon.$$

In (8), the polynomial factors satisfy

For clarity, the first-derivative coefficient is expanded as $\gamma(t-1)(t-a) + \delta t(t-a) + \varepsilon t(t-1) = St^2 - [(1+a)\gamma + a\delta + \varepsilon]t + a\gamma$, with $t = x^\alpha$ and $S = \gamma + \delta + \varepsilon$. Thus $L_0 = (1+a)\gamma + a\delta + \varepsilon$, which is the linear coefficient used in the recurrence.

$$x^\alpha(x^\alpha - 1)(x^\alpha - a) = x^{3\alpha} - (1+a)x^{2\alpha} + ax^\alpha,$$

and

$$\gamma(x^\alpha - 1)(x^\alpha - a) + \delta x^\alpha(x^\alpha - a) + \varepsilon x^\alpha(x^\alpha - 1) = Sx^{2\alpha} - L_0x^\alpha + a\gamma.$$

The coefficient of the lowest conformable power $x^{\alpha(r-1)}$ is

After factoring out the nonzero constant a , the relevant leading coefficient is $\alpha r[\alpha(r-1) + \gamma]c_0$; therefore, the extra factor a does not enter the indicial equation.

$$\alpha r[\alpha(r-1) + \gamma]c_0.$$

Therefore, the roots and indicial equation are

$$r[\alpha(r-1) + \gamma] = 0, \quad r_1^{(0)} = 0, \quad r_2^{(0)} = 1 - \frac{\gamma}{\alpha}. \quad (10)$$

RECURRENCE RELATION

Assume $c_{-1} = c_{-2} = 0$. For $n \geq 0$, compute

$$D_n^{(0)}(r) = a\alpha(n+r)[\alpha(n+r-1) + \gamma],$$

$$F_n^{(0)}(r) = -\alpha^2(1+a)(n+r-1)(n+r-2) + \alpha L_0(n+r-1) + B,$$

and

$$G_n(r) = \alpha^2(n+r-2)(n+r-3) + \alpha S(n+r-2) + A.$$

When the coefficient of $x^{\alpha(n+r-1)}$ is equated, it yields

$$D_n^{(0)}(r)c_n + F_n^{(0)}(r)c_{n-1} + G_n(r)c_{n-2} = 0.$$

Hence, for $D_n^{(0)}(r) \neq 0$ and $n \geq 1$,

$$c_n = -\frac{F_n^{(0)}(r)c_{n-1} + G_n(r)c_{n-2}}{D_n^{(0)}(r)}. \quad (11)$$

For $\gamma \neq 0$ and $r = 0$,

$$c_1 = \frac{B}{a\alpha\gamma}c_0. \quad (12)$$

TWO NONRESONANT LOCAL SOLUTIONS

Suppose that

$$1 - \frac{\gamma}{\alpha} \notin \mathbb{Z}.$$

The two linearly independent local conformable Frobenius solutions are

$$\begin{aligned} y_{0,1}(x) &= \sum_{n=0}^{\infty} c_n^{(0,1)} x^{n\alpha}, \\ y_{0,2}(x) &= x^{\alpha-\gamma} \sum_{n=0}^{\infty} c_n^{(0,2)} x^{n\alpha}. \end{aligned} \quad (13)$$

The recurrences are evaluated at $r = 0$ and $r = 1 - \gamma/\alpha$, respectively. By Remark 3.2, both conformable Frobenius solutions $y_{0,1}(x)$ and $y_{0,2}(x)$ converge at least in the conformable domain

$$0 < |x^\alpha| < R_0, \quad R_0 = \min\{1, a\}.$$

5. FROBENIUS SOLUTIONS ABOUT $x = 1$

INDICIAL EQUATION

Seek a solution directly in powers of $x^\alpha - 1$:

$$y(x) = \sum_{n=0}^{\infty} b_n (x^\alpha - 1)^{n+r}, \quad b_0 \neq 0. \quad (14)$$

By (4),

$$\begin{aligned} T_\alpha y &= \alpha \sum_{n=0}^{\infty} (n+r) b_n (x^\alpha - 1)^{n+r-1}, \\ T_\alpha^2 y &= \alpha^2 \sum_{n=0}^{\infty} (n+r)(n+r-1) b_n (x^\alpha - 1)^{n+r-2}. \end{aligned}$$

Define

$$L_1 = (1-a)\gamma + (2-a)\delta + \varepsilon.$$

In $x^\alpha - 1$, applying direct polynomial identities, equation (8) becomes

$$\begin{aligned} &[(x^\alpha - 1)^3 + (2-a)(x^\alpha - 1)^2 + (1-a)(x^\alpha - 1)] T_\alpha^2 y \\ &+ [S(x^\alpha - 1)^2 + L_1(x^\alpha - 1) + \delta(1-a)] T_\alpha y \\ &+ [A(x^\alpha - 1) + (A-B)] y = 0. \end{aligned} \quad (15) \quad \text{Then}$$

The coefficient of $(x^\alpha - 1)^{r-1}$ yields

$$r[\alpha(r-1) + \delta] = 0, \quad r_1^{(1)} = 0, \quad r_2^{(1)} = 1 - \frac{\delta}{\alpha}. \quad (16)$$

RECURRENCE RELATION

Assume $b_{-1} = b_{-2} = 0$ and identify

$$D_n^{(1)}(r) = (1-a)\alpha(n+r)[\alpha(n+r-1) + \delta],$$

$$F_n^{(1)}(r) = \alpha^2(2-a)(n+r-1)(n+r-2) + \alpha L_1(n+r-1) + (A-B).$$

The function $G_n(r)$ is identical to that found in Section 4.2. Calculating the coefficient of $(x^\alpha - 1)^{n+r-1}$ yields

$$D_n^{(1)}(r) b_n + F_n^{(1)}(r) b_{n-1} + G_n(r) b_{n-2} = 0.$$

Consequently, for $D_n^{(1)}(r) \neq 0$ and $n \geq 1$,

$$b_n = \frac{-F_n^{(1)}(r) b_{n-1} - G_n(r) b_{n-2}}{D_n^{(1)}(r)}. \quad (17)$$

For $\delta \neq 0$ and $r = 0$,

$$b_1 = \frac{A-B}{(a-1)\alpha\delta} b_0. \quad (18)$$

TWO NONRESONANT LOCAL SOLUTIONS

If

$$1 - \frac{\delta}{\alpha} \notin \mathbb{Z},$$

then the two linearly independent local solutions are

$$\begin{aligned} y_{1,1}(x) &= \sum_{n=0}^{\infty} b_n^{(1,1)} (x^\alpha - 1)^n, \\ y_{1,2}(x) &= (x^\alpha - 1)^{1-\delta/\alpha} \sum_{n=0}^{\infty} b_n^{(1,2)} (x^\alpha - 1)^n. \end{aligned} \quad (19)$$

The recurrences are evaluated at $r = 0$ and $r = 1 - \delta/\alpha$, respectively. By Remark 3.2, the two series converge at least in

$$0 < |x^\alpha - 1| < R_1, \quad R_1 = \min\{1, |1-a|\}.$$

6. FROBENIUS SOLUTIONS ABOUT $x_a = a^{1/\alpha}$

This section repeats the direct Frobenius construction at the third finite α -singular point $x = x_a$, using the local conformable increment $x^\alpha - a$. The indicial equation, recurrence, and nonresonant local basis are derived in the same form as at the first two centers.

INDICIAL EQUATION

Seek a direct solution in powers of $x^\alpha - a$:

$$y(x) = \sum_{n=0}^{\infty} e_n (x^\alpha - a)^{n+r}, \quad e_0 \neq 0. \quad (20)$$

Then

$$\begin{aligned} T_\alpha y &= \alpha \sum_{n=0}^{\infty} (n+r) e_n (x^\alpha - a)^{n+r-1}, \\ T_\alpha^2 y &= \alpha^2 \sum_{n=0}^{\infty} (n+r)(n+r-1) e_n (x^\alpha - a)^{n+r-2}. \end{aligned}$$

Define

$$L_a = (a-1)\gamma + a\delta + (2a-1)\varepsilon.$$

By applying direct polynomial identities in $x^\alpha - a$, equation (8) is transformed into

$$\begin{aligned} &[(x^\alpha - a)^3 + (2a-1)(x^\alpha - a)^2 + a(a-1)(x^\alpha - a)] T_\alpha^2 y \\ &+ [S(x^\alpha - a)^2 + L_a(x^\alpha - a) + \varepsilon a(a-1)] T_\alpha y \\ &+ [A(x^\alpha - a) + (Aa-B)] y = 0. \end{aligned} \quad (21)$$

The coefficient of $(x^\alpha - a)^{r-1}$ yields

$$r[\alpha(r-1) + \varepsilon] = 0, \quad r_1^{(a)} = 0, \quad r_2^{(a)} = 1 - \frac{\varepsilon}{\alpha}. \quad (22)$$

RECURRENCE RELATION

Assume that $e_{-1} = e_{-2} = 0$. Define

$$D_n^{(a)}(r) = a(a-1)\alpha(n+r)[\alpha(n+r-1) + \varepsilon],$$

$$F_n^{(a)}(r) = \alpha^2(2a-1)(n+r-1)(n+r-2) + \alpha L_a(n+r-1) + (Aa-B).$$

Calculating the coefficient of $(x^\alpha - a)^{n+r-1}$ yields

$$D_n^{(a)}(r)e_n + F_n^{(a)}(r)e_{n-1} + G_n(r)e_{n-2} = 0.$$

As a result, for $D_n^{(a)}(r) \neq 0$ and $n \geq 1$,

$$e_n = \frac{-F_n^{(a)}(r)e_{n-1} - G_n(r)e_{n-2}}{D_n^{(a)}(r)}. \quad (23)$$

For $\varepsilon \neq 0$ and $r = 0$,

$$e_1 = \frac{B - Aa}{a(a-1)\alpha\varepsilon}e_0. \quad (24)$$

TWO NONRESONANT LOCAL SOLUTIONS

If

$$1 - \frac{\varepsilon}{\alpha} \notin \mathbb{Z},$$

Table 1: Local exponents, nonresonance conditions, and verified conformable domains.

Point	Local power	Exponents	Condition	Domain
$x = 0$	x^α	$0, 1 - \gamma/\alpha$	$1 - \gamma/\alpha \notin \mathbb{Z}$	$0 < x^\alpha < R_0$
$x = 1$	$x^\alpha - 1$	$0, 1 - \delta/\alpha$	$1 - \delta/\alpha \notin \mathbb{Z}$	$0 < x^\alpha - 1 < R_1$
$x = x_a$	$x^\alpha - a$	$0, 1 - \varepsilon/\alpha$	$1 - \varepsilon/\alpha \notin \mathbb{Z}$	$0 < x^\alpha - a < R_a$

where $x_a = a^{1/\alpha}$, $R_0 = \min\{1, a\}$, $R_1 = \min\{1, |1 - a|\}$, and $R_a = \min\{a, |a - 1|\}$.

Theorem 7.1. Let $a > 0$, $a \neq 1$, and $0 < \alpha \leq 1$. Equation (7) admits a local conformable Frobenius basis in the nonresonant case for any finite regular α -singular point $x_0 \in \{0, 1, a^{1/\alpha}\}$. Table 1 lists the exponents, guaranteed conformable convergence domains, and the three-term recurrence corresponding to the chosen center, which determines the coefficients. The right-hand boundary interpretation of this statement is understood at $x_0 = 0$.

The entries of Table 1 follow directly from the three indicial equations: the exponent pairs are $\{0, 1 - \gamma/\alpha\}$, $\{0, 1 - \delta/\alpha\}$, and $\{0, 1 - \varepsilon/\alpha\}$. Nonresonance means that the difference of the two exponents is not an integer, which gives the three listed conditions. The domains are the ratio-test radii derived in Remark 3.2. When an exponent difference is an integer, the recurrence denominator for the smaller root vanishes at the resonant index, and the compatibility condition in (27) decides whether a second ordinary series exists.

RESONANCE OBSTRUCTION AND LOGARITHMIC ALTERNATIVE

The local recurrence at any center can be written in the unified form:

$$D_n(r)u_n + P_n(r)u_{n-1} + Q_n(r)u_{n-2} = 0, \quad u_{-1} = u_{-2} = 0. \quad (26)$$

The pairs of coefficients are

To remove ambiguity in the unified notation, $Q_n(r) = G_n(r)$ at all three centers, while $P_n(r) = F_n^{(0)}(r)$ for $x_0 = 0$, $P_n(r) = F_n^{(1)}(r)$ for $x_0 = 1$, and $P_n(r) = F_n^{(a)}(r)$ for $x_0 = x_a$.

then both independent local solutions are

$$y_{a,1}(x) = \sum_{n=0}^{\infty} e_n^{(a,1)}(x^\alpha - a)^n, \quad y_{a,2}(x) = (x^\alpha - a)^{1-\varepsilon/\alpha} \sum_{n=0}^{\infty} e_n^{(a,2)}(x^\alpha - a)^n. \quad (25)$$

The recurrences are evaluated at $r = 0$ and $r = 1 - \varepsilon/\alpha$, respectively. By Remark 3.2, the two series converge at least in

$$0 < |x^\alpha - a| < R_a, \quad R_a = \min\{a, |a - 1|\}.$$

7. UNIFIED STATEMENT AND RESONANT CASES

This section gathers the three local constructions into one notation, identifies the nonresonant bases, treats the resonance obstruction and logarithmic alternative, and states termination and determinant conditions for finite conformable Frobenius solutions.

NONRESONANT LOCAL BASES

Table 1 summarizes, for each finite α -singular center, the local conformable power, the two indicial exponents, the nonresonance condition, and the guaranteed convergence domain. The notation $x_a = a^{1/\alpha}$ is used consistently throughout.

$$(P_n, Q_n) = \begin{cases} (F_n^{(0)}, G_n), & x_0 = 0, \\ (F_n^{(1)}, G_n), & x_0 = 1, \\ (F_n^{(a)}, G_n), & x_0 = a^{1/\alpha}. \end{cases}$$

Let $r_L - r_S = N$, where $N \in \{0, 1, 2, \dots\}$. Denote the two indicial roots by r_L and r_S . The denominators factor using the indicial equation as

$$D_n(r_L) = C\alpha^2 n(n+N), \quad D_n(r_S) = C\alpha^2 n(n-N),$$

Here, at $x_0 = 0$, $C = a$, at $x_0 = 1$, $C = 1 - a$, and at $x_0 = a^{1/\alpha}$, $C = a(a-1)$.

The larger root r_L yields an unobstructed ordinary conformable Frobenius series if $N \geq 1$. The denominator vanishes only at $n = N$ for the smaller root r_S . After $u_1, \dots, u_{N-1}$ have been obtained iteratively, there is a second ordinary conformable Frobenius series only if

$$P_N(r_S)u_{N-1} + Q_N(r_S)u_{N-2} = 0. \quad (27)$$

If (27) holds, the $n = N$ equation does not restrict u_N ; one may set $u_N = 0$ because changing u_N adds a multiple of the r_L solution. When (27) fails, the second independent real local solution has the logarithmic form (Coddington & Levinson, 1955; Ince, 1956; NIST Digital Library of Mathematical Functions, n.d.)

In center-specific form, the resonant logarithmic alternative may be written as $y_2 = Ky_1 \log(x^\alpha) + x^{\alpha r_S} \sum d_n x^{n\alpha}$ at $x = 0$, $y_2 = Ky_1 \log|x^\alpha - 1| + (x^\alpha - 1)^{r_S} \sum d_n (x^\alpha - 1)^n$ at $x = 1$, and $y_2 = Ky_1 \log|x^\alpha - a| + (x^\alpha - a)^{r_S} \sum d_n (x^\alpha - a)^n$ at $x = x_a$. These are the three instances of (28).

$$y_2(x) = K y_1(x) \log|x^\alpha - x_0^\alpha| + (x^\alpha - x_0^\alpha)^{r_s} \sum_{n=0}^{\infty} d_n (x^\alpha - x_0^\alpha)^n, \quad (28)$$

The indicial root is repeated when $N = 0$. Up to normalization, the recurrence determines one ordinary conformable Frobenius series, and a logarithmic term occurs in a second independent solution. The repeated-root conditions are at $x = 0$, $\gamma = \alpha$, at $x = 1$, $\delta = \alpha$, and at $x = a^{1/\alpha}$, $\varepsilon = \alpha$. This is the standard Frobenius alternative for repeated or resonant exponents.

LOCAL SOLUTION SPACES

These three pairings do not represent six globally independent solutions; rather, they are local bases at different α -singular points. A nonsingular 2×2 connection matrix relates one basis to another on any connected regular interval where two local representations converge or can be continued. Such connection problems are classical in Heun theory (Maier, 2007; NIST Digital Library of Mathematical Functions, n.d.; Ronveaux, 1995; Slavyanov & Lay, 2000).

TERMINATING CONFORMABLE FROBENIUS SERIES AND FINITE SOLUTIONS

The recurrence relations derived at the three finite regular α -singular points also provide explicit conditions under which a local conformable Frobenius series terminates. Such termination produces an exact finite solution in the corresponding conformable increment (A. M. Ishkhanyan, 2019, 2020; T. A. Ishkhanyan *et al.*, 2018; Manukyan *et al.*, 2014).

Theorem 7.2. Let $x_0 \in \{0, 1, a^{1/\alpha}\}$, and let

$$y(x) = (x^\alpha - x_0^\alpha)^r \sum_{n=0}^{\infty} u_n (x^\alpha - x_0^\alpha)^n, \quad u_0 \neq 0,$$

be a local conformable Frobenius solution associated with an indicial root r . Assume that its coefficients satisfy the unified recurrence

$$D_n(r) u_n + P_n(r) u_{n-1} + Q_n(r) u_{n-2} = 0, \quad u_{-1} = u_{-2} = 0,$$

where $Q_n(r) = G_n(r)$. Let $N \geq 0$ and suppose that $D_n(r) \neq 0$ for $1 \leq n \leq N$. Then a terminating conformable Frobenius branch of exact degree N exists if and only if $u_N \neq 0$, $G_{N+2}(r) = 0$ and

$$P_{N+1}(r) u_N + G_{N+1}(r) u_{N-1} = 0.$$

Since

$$G_n(r) = \alpha^2(n+r-2)(n+r-3) + \alpha S(n+r-2) + A,$$

the first termination condition is equivalent to

$$\alpha^2(N+r)(N+r-1) + \alpha S(N+r) + A = 0.$$

Under these conditions, the resulting exact solution is

$$y_N(x) = (x^\alpha - x_0^\alpha)^r \sum_{n=0}^N u_n (x^\alpha - x_0^\alpha)^n.$$

Proof. Suppose first that the series terminates at degree N . Thus, $u_N \neq 0$ and $u_{N+1} = u_{N+2} = \dots = 0$. Substituting $n = N+1$ into the recurrence gives

$$P_{N+1}(r) u_N + G_{N+1}(r) u_{N-1} = 0.$$

Substituting $n = N+2$ gives

$$G_{N+2}(r) u_N = 0.$$

Because $u_N \neq 0$, it follows that $G_{N+2}(r) = 0$. Conversely, assume that these three conditions hold. The recurrence at $n = N+1$ is satisfied with $u_{N+1} = 0$, and the recurrence at $n = N+2$ is satisfied with $u_{N+2} = 0$. For every $n \geq N+3$, all coefficients occurring in the recurrence are zero, so the recurrence is satisfied identically. Therefore, the Frobenius expansion terminates at degree N . $\square$

For a fixed value of $N \geq 1$, the second termination condition can also be expressed as a finite determinant condition. In fact, the coefficients $u_0, \dots, u_N$ satisfy the homogeneous system

The tridiagonal system is indexed consistently with the recurrence: the first row contains P_1, D_1 ; the second row contains G_2, P_2, D_2 ; and, in general, row n contains G_n, P_n, D_n in the sub-diagonal, diagonal, and superdiagonal positions, respectively.

$$\begin{pmatrix} P_1 & D_1 & 0 & \cdots & 0 \\ G_2 & P_2 & D_2 & \ddots & \vdots \\ 0 & G_3 & P_3 & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & D_N \\ 0 & \cdots & 0 & G_{N+1} & P_{N+1} \end{pmatrix} \begin{pmatrix} u_0 \\ u_1 \\ u_2 \\ \vdots \\ u_N \end{pmatrix} = 0,$$

where all coefficient functions are evaluated at the chosen exponent r . Hence, a nontrivial terminating branch requires

$$\Delta_{N+1}(r) = \det \begin{pmatrix} P_1 & D_1 & 0 & \cdots & 0 \\ G_2 & P_2 & D_2 & \ddots & \vdots \\ 0 & G_3 & P_3 & \ddots & 0 \\ \vdots & \ddots & \ddots & \ddots & D_N \\ 0 & \cdots & 0 & G_{N+1} & P_{N+1} \end{pmatrix} = 0.$$

Because the accessory parameter B occurs in $P_n(r)$, the determinant condition gives an algebraic equation for the admissible values of B .

Example 7.1. Consider the expansion about $x = 0$ and choose

$$a = 2, \alpha = \frac{1}{2}, \gamma = \frac{1}{2}, \delta = 0, \varepsilon = 0, A = -\frac{1}{4}, B = -\frac{1}{4}.$$

For these parameters, the equation itself is $T_{1/2}^2 y + \frac{1}{4x^{1/2}} T_{1/2} y + \frac{1-x^{1/2}}{4x^{1/2}(x^{1/2}-1)(x^{1/2}-2)} y = 0$. At $x = 0$ the indicial equation reduces to $r^2/4 = 0$, so $r = 0$ is the repeated local exponent used for the terminating branch below.

Here $S = \gamma + \delta + \varepsilon = 1/2$, and the indicial exponent $r = 0$ is used. For a polynomial of degree $N = 1$, the first termination condition becomes

$$G_3(0) = \alpha S + A = \frac{1}{2} \cdot \frac{1}{2} - \frac{1}{4} = 0.$$

Taking $c_0 = 1$, the recurrence at $n = 1$ gives

$$c_1 = \frac{B}{\alpha\alpha\gamma} c_0 = \frac{-1/4}{2(1/2)(1/2)} = -\frac{1}{2}.$$

Moreover, $L_0 = (1+a)\gamma + a\delta + \varepsilon = 3/2$, and the terminal equation at $n = 2$ is

$$-(\alpha L_0 + B) c_1 + A c_0 = -\left(\frac{3}{4} - \frac{1}{4}\right) \left(-\frac{1}{2}\right) - \frac{1}{4} = 0.$$

Therefore, the local series terminates and gives the exact conformable Heun polynomial

$$y(x) = 1 - \frac{1}{2} x^{1/2}.$$

Indeed,

$$T_{1/2}y = -\frac{1}{4}, \quad T_{1/2}^2y = 0,$$

and direct substitution into equation (7) gives

$$-\frac{1}{8x^{1/2}} + \frac{1}{8x^{1/2}} = 0.$$

Thus, the residual vanishes identically, rather than only decreasing as the truncation degree increases. This example confirms that the recurrence relations can generate exact finite conformable Heun solutions for suitable parameter values.

8. NUMERICAL ILLUSTRATION AND DIRECT VERIFICATION

This section checks the preceding recurrences numerically for one illustrative parameter set. All coefficients, truncated-series curves,

Table 2: First four normalized coefficients for the six local branches.

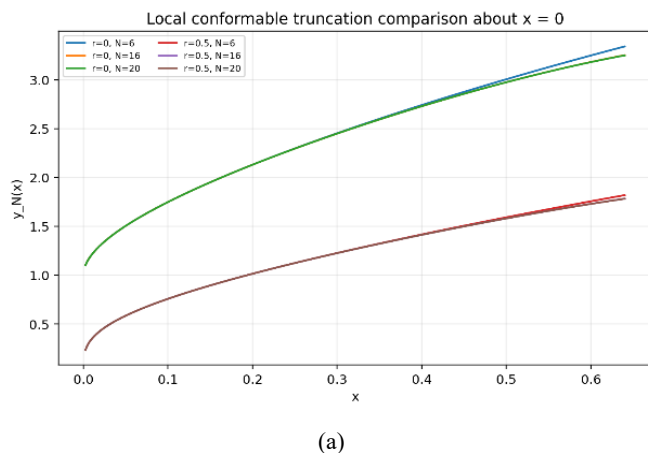
Center and exponent	u_0	u_1	u_2	u_3
$x = 0, r = 0$	1.000000	2.000000	1.077778	0.297235
$x = 0, r = 1/2$	1.000000	0.955556	0.416222	0.099760
$x = 1, r = 0$	1.000000	3.000000	2.910000	2.742250
$x = 1, r = 1/3$	1.000000	1.475000	1.511429	1.230686
$x = 4, r = 0$	1.000000	-7.500000	11.383929	-11.383309
$x = 4, r = 3/5$	1.000000	-2.218750	2.477965	-2.257824

The degree- N truncations at a general center x_0 , for the residual comparison, are

$$y_1^{(N)}(x) = \sum_{n=0}^N u_n (x^\alpha - x_0^\alpha)^n,$$

$$y_2^{(N)}(x) = (x^\alpha - x_0^\alpha)^{r_2} \sum_{n=0}^N v_n (x^\alpha - x_0^\alpha)^n.$$

To test truncation stability visually, Figure 1 compares $N = 6$, $N = 16$, and $N = 20$ for both local branches at each of the three centers. The higher-order curves become nearly indistinguishable in the interior of the convergence interval, while the $N = 6$ curves show the largest visible truncation effect.



and residual values reported below are computed directly from equations (11), (17), (23), and (29)–(30); no external numerical dataset is used.

Choose

$$a = 2, \alpha = \frac{1}{2}, \gamma = \frac{1}{4}, \varepsilon = \frac{1}{5}, \delta = \frac{1}{3}, A = 1, B = \frac{1}{2}.$$

The third finite α -singular point for these values is

$$x_a = a^{1/\alpha} = 2^2 = 4.$$

The indicial exponent pairs are $\{0, 1/2\}$ at $x = 0$, $\{0, 1/3\}$ at $x = 1$, and $\{0, 3/5\}$ at $x = 4$. The guaranteed conformable radii are $R_0 = R_1 = R_a = 1$, and all three cases are nonresonant.

The leading coefficient of each branch is set to one. Table 2 lists the first four coefficients derived from the recurrences.

Source: Authors' calculations from the recurrence relations (11), (17), and (23).

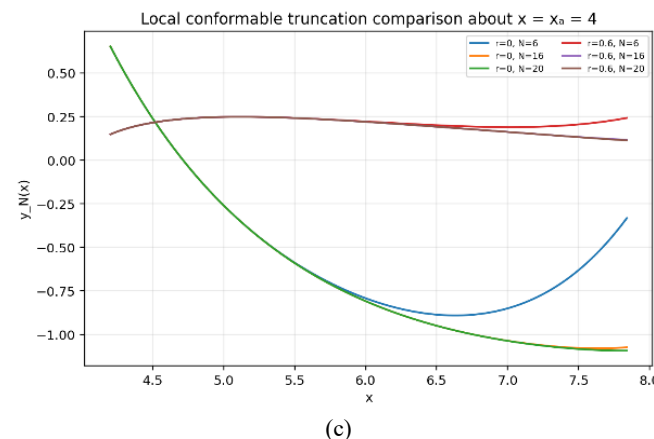
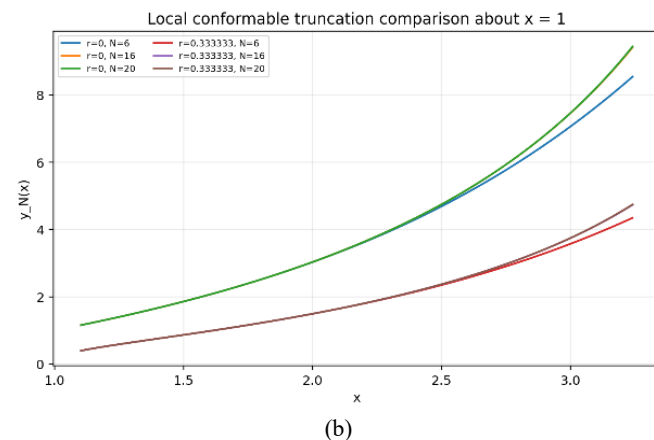


Figure 1 shows the corresponding comparative truncations of the local branches. Comparative degree- N ($N = 6, 16, 20$) local conformable solutions about: (a) $x = 0$; (b) $x = 1$; and (c) $x = a^{1/\alpha} = 4$.

DIRECT CONFORMABLE RESIDUAL

Define the residual of the multiplied conformable equation (8) for a truncated local series $y_N(x)$ by

$$\begin{aligned} \mathcal{R}_N(x) = & x^\alpha (x^\alpha - 1) (x^\alpha - a) T_\alpha^2 y_N(x) \\ & + [\gamma (x^\alpha - 1) (x^\alpha - a) + \delta x^\alpha (x^\alpha - a) + \varepsilon x^\alpha (x^\alpha - 1)] T_\alpha y_N(x) \\ & + (Ax^\alpha - B) y_N(x). \end{aligned} \quad (29)$$

For a uniformly balanced comparison, let

$$\mathcal{E}_N = \frac{\max_{0.05 \leq x^\alpha - x_0^\alpha \leq 0.5} |\mathcal{R}_N(x)|}{1 + |P_3(x) T_\alpha^2 y_N| + |P_2(x) T_\alpha y_N| + |(Ax^\alpha - B) y_N|}, \quad (30)$$

where

$$P_3(x) = x^\alpha (x^\alpha - 1) (x^\alpha - a)$$

and

$$P_2(x) = \gamma (x^\alpha - 1) (x^\alpha - a) + \delta x^\alpha (x^\alpha - a) + \varepsilon x^\alpha (x^\alpha - 1).$$

The maximum in (30) was determined using 2001 equally spaced grid points in the conformable coordinate. Table 3 displays a monotone residual decrease with increasing N .

The lower cutoff 0.05 is used to avoid evaluating the normalized residual at the singular center itself, where factors in the differential equation and derivatives of branches with noninteger exponents can be singular or numerically ill-conditioned. Thus $\mathcal{E}_N$ is an interior local-accuracy diagnostic, not a claim of uniform accuracy at the singular endpoint. The upper bound 0.5 remains well inside the guaranteed radius $R = 1$ for the present example, so the comparison is separated from the next singularity and reflects truncation error rather than boundary singular behavior.

Table 3: Maximum normalized residual $\mathcal{E}_N$ on $0.05 \leq x^\alpha - x_0^\alpha \leq 0.5$.

Center and exponent	$N = 6$	$N = 12$	$N = 16$	$N = 20$
$x = 0, r = 0$	2.567×10^{-2}	5.862×10^{-4}	4.205×10^{-5}	2.955×10^{-6}
$x = 0, r = 1/2$	1.032×10^{-2}	2.286×10^{-4}	1.636×10^{-5}	1.147×10^{-6}
$x = 1, r = 0$	4.465×10^{-2}	1.067×10^{-3}	7.884×10^{-5}	5.567×10^{-6}
$x = 1, r = 1/3$	3.261×10^{-2}	7.701×10^{-4}	5.661×10^{-5}	3.985×10^{-6}
$x = 4, r = 0$	3.283×10^{-1}	1.167×10^{-2}	9.144×10^{-4}	6.745×10^{-5}
$x = 4, r = 3/5$	2.461×10^{-1}	6.091×10^{-3}	4.658×10^{-4}	3.412×10^{-5}

Source: Authors' calculations from the recurrence relations and normalized residual (29)–(30).

Across all six branches, increasing N from 6 to 20 reduces $\mathcal{E}_N$ by approximately 4.9×10^3 to 9.0×10^3 . The smallest $N = 20$ residual is 1.147×10^{-6} for the $r = 1/2$ branch at $x = 0$, while the largest is 6.745×10^{-5} for the $r = 0$ branch at $x = x_a$. The slower decay for the $x = x_a$ branches is consistent with their larger early recurrence coefficients, which amplify finite-truncation effects before the asymptotic convergence regime dominates. Mathematically, the monotone decrease is expected because each added coefficient cancels the next unmatched powers generated by the differential operator; within the interior interval, this leaves progressively higher-order remainder terms. In applications to conformable wave or Schrödinger models, this residual decrease means that the truncated local series satisfies the governing differential equation more accurately over the interior interval; the present numerical example is therefore a mathematical local-accuracy test rather than a simulation of a specific physical system.

CONCLUSIONS

A direct Frobenius analysis of the sequential conformable Heun equation with original variables has been carried out at the three finite regular α -singular points: $x = 0$, $x = 1$, and $x = a^{1/\alpha}$. For every $0 < \alpha \leq 1$, equation (7) is a pullback of the classical general Heun equation with scaled Heun parameters, as demonstrated by the explicit reparametrization $z = x^\alpha$. Once this correspondence was established, the sequential operator T_α^2 was kept throughout the analytical derivations, and all local expansions were constructed directly from the conformable powers x^α , $x^\alpha - 1$, and $x^\alpha - a$.

Each center was discovered to have an indicial equation, two distinct exponents, a three-term coefficient recurrence, and a guaranteed conformable convergence domain. In nonresonant circumstances, denominator factorization produces two locally conformable Frobenius branches. A recursive compatibility parameter evaluates whether the second branch is an ordinary conformable series or whether it has a logarithmic term in resonant cases.

Furthermore, necessary and sufficient termination criteria and a finite determinant condition for the accessory parameter were identified, and an exact degree-one terminating solution was proven by an identically vanishing residual.

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AUTHORS' CONTRIBUTIONS:

I.N.: Conceptualization, methodology, mathematical analysis, investigation, and writing of the original draft. E.B.: Supervision, validation, critical review, and editing of the manuscript. Both authors read and approved the final manuscript.

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