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Science Journal of University of Zakho

Vol. 6, No. 2, pp. 64 – 69, Jun.-2018

SJUOZ

p-ISSN: 2410-7549
e-ISSN: 2414-6943

NUMERICAL SOLUTION OF THE BAGLEY-TORVIK EQUATION USING THE INTEGER-ORDER DERIVATIVES EXPANSION

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Received: Feb. 2018 / Accepted: Jun., 2018 / Published: Jun., 2018 *https://doi.org/10.25271/2018.6.2.437*

ABSTRACT:
Numerical solution of the well-known Bagley-Torvik equation is considered. The Bagley-Torvik equation is transformed into a system of first-order ordinary differential equations using the infinite series of integer-order derivatives expansion. The approximation of fractional-order derivative and the order of the truncated error are illustrated through some examples. Comparison between our result and exact solution are made by considering some examples of initial value problems for Bagley-Torvik equation with known analytical solutions to show the preciseness of our proposed approach.

KEYWORDS: Bagley-Torvik Equation, Riemann–Liouville Fractional Integral and Derivative, Numerical Solution, Infinite Series of Integer-Order Derivative Expansion.

1. INTRODUCTION

The present of fractional-order derivative in differential equations, ordinary and partial, has recently increased and the mathematical formulation of many applications ended up with fractional (non-integer) order of derivatives or integrals. Partial differential equations with space\time fractional-order derivative appear in the investigation of transport dynamics in complex systems (Metzler & Klafter, 2000). Also (Barkai, Metzler, & Klafter, 2000) used the fractional Fokker–Planck equation for anomalous diffusion in an external field and (Henry & Wearne, 2000) introduced the fractional advection diffusion equation for anomalous diffusion with sources and sinks. As to Ordinary Differential Equations (ODEs), the Bagley-Torvik equation (1) which governs some problems in applied science applications and engineering. This second order differential equation contains also a term with $3/2$-order derivative and it describes the plate, which is considered to be rigid, immersing in viscous Newtonian fluid. The Bagley-Torvik equation which contains a $1/2$-order derivative, for a damped oscillator describes its decaying motion. See (Torvik & Bagley, 1983) and (Torvik & Bagley, 1984).

For more fractional-order differential equations one can see the following. (Hilfer, 2000) for thermodynamics; (Benson, Wheatcraft & Meerschaert, 2000) for fractional advection dispersion equation; (Rousan, Malkawi, Rabei & Widyan, 2002) for some physical systems in gravity; (Chatterjee, 2005) for viscous fluid flows; (Bhrawy, Doha, Baleanu, Ezz-Eldien & Abdelkawy, 2015) for control problems; (Battaglia, 2002) for problems in heat conduction and etc.

In this paper we study the numerical solution of the Bagley-Torvik equation which has the following form

$$a_1\frac{d^2y(t)}{dt^2}+a_2\frac{d^{3/2}y(t)}{dt^{3/2}}+a_3y(t)=g(t). \qquad (1)$$

Equation (1) represents the motion of a large plate (thin and rigid) in a Newtonian fluid (Torvik & Bagley, 1984). In equation (1) $a_1\neq 0$, a_2 and a_3 are real constants, where a_1 represents the mass of the plate, $a_2=2a\sqrt{\mu\rho}$, with a is area of the part of the rigid plate immersed in the fluid, with μ is the viscosity of the fluid, and ρ is its density, and a_3 is the stiffness of the spring, (Podlubny, 1999), and $g(t)$ is any analytic function defined on the interval $[0,T], T>0$. The equation (1) is supplemented with initial conditions

$$y|_{t=0}=0, \qquad (2)$$

and

$$\frac{dy}{dt}|_{t=0}=0. \qquad (3)$$

The discussion on existence and uniqueness of the solution to the problem (1-3) is presented by (Luchko & Gorenflo, 1998), therefore we do not go into these matters in the present study.

As the problem (1-3) contains fractional-order derivative and due to the fact that the analytical solutions of such problems do not always exist and have a complicated form. Therefore, the solution to the initial-value problem (1-3) has developed recently by many researchers. In the following we try to mention some of the contributions which have been made for this particular problem.

Analytically a solution to the initial-value problem (1-3) is introduced by Podlubny, (Podlubny, 1999), in terms of Green's function expressed as an infinite sum of derivatives of Mittag-Leffler functions in a convolution integral. However this solution cannot be evaluated conveniently for general functions g(t). The previous solution considered for homogeneous conditions. But Luchko and Gorenflo, in their paper, (Luchko & Gorenflo, 1998), presented the analytical solution for the problem with inhomogeneous conditions in terms of multivariate generalizations of Mittag-Leffler functions which is long and complicated to be handled.

A numerical solution of the equation (1) proposed by Leszczynski and Ciesielski; by using the Abel integral equations, the equation (1) is written as a system of ordinary differential equations (Leszczyński & Ciesielski, 2001). Diethelm and Ford introduced a numerical scheme for the equation (1), (Diethelm & Ford, 2002). They converted the equation (1) into a system of $1/2$-order differential equations and then numerically solved the system. In addition, they used a different approach scheme which is called Adams-type predictor-corrector approach which has been previously developed by Diethelm and Freed, (Diethelm, & Freed, 1999). Under suitable assumption they found that the error behaves like $O(h^2)$ (h is the step size). The method of Adomian decomposition used and applied by Ray and Bera to the equation (1), (Ray & Bera, 2005), and they obtained the same solution as the Podlubny's solution by Green's function. Nouri, Elahi-Mehr and Torkzadeh, (Nouri, Elahi-Mehr, & Torkzadeh, 2016) used a numerical algorithm by inverse Laplace transform.

2. PRELIMINARIES AND FUNDAMENTAL RELATIONS

In this section some related definitions and properties which will be used throughout the present study are shown. For more details on the given definitions and properties see (Samko, Kilbas, & Marichev, 1993) and (Podlubny, 1999).

Definition 2.1: Let $\alpha > 0$, for $f(t) \in L^1(0,T)$, then the operator ${}_0^tI^\alpha$ defined as

$$ {}_0^tI^\alpha f(t) = \frac{1}{\Gamma(\alpha)}\int_0^t (t-\tau)^{\alpha-1} f(\tau)d\tau, \quad (4)$$

is called the Riemann-Liouville fractional integral operator of order $\alpha > 0$. For $\alpha = 0$, ${}_0^tI^\alpha f(t) = f(t)$.

Definition 2.2: For $f(t) \in C^n[0,T]$, then the operator ${}_0^tD^\alpha$ defined as

$$ {}_0^tD^\alpha f(t) = D^n\, {}_0^tI^{n-\alpha} f(t)$$

$$ = \frac{1}{\Gamma(n-\alpha)}\frac{d^n}{dt^n}\int_0^t (t-\tau)^{n-\alpha-1} f(\tau)d\tau, \quad (5)$$

is called the Riemann-Liouville fractional derivative operator of order $\alpha > 0, n-1 < \alpha < n$.

In equation (5), ${}_0^tI^{n-\alpha}$ is the Riemann-Liouville fractional integral of order $n-\alpha > 0$ which is defined in (4). Note that for $\alpha = n$, ${}_0^tD^\alpha f(t) = \frac{d^n f(t)}{dt^n}$.

One of the obstacles in fractional calculus is that we have more than one definition for fractional integrals and derivatives. In addition to the above definitions, Riemann-Liouville fractional integral and derivative, there are also different definitions like: Caputo fractional derivative, Erdelyi-Kober fractional integral and derivative, Hadamard part fractional integral and derivative, Riesz fractional integral and derivative, and Grünwald-Letnikov fractional derivative. All the above mentioned definitions play an important role in the fractional calculus field and its applications. In this paper only definitions (2.1-2.2) matter, for other definitions and relations between them see (Samko, Kilbas, & Marichev, 1993) and (Podlubny, 1999).

Definition 2.3: The classical Mittag-Leffler function, first introduced by Mittag-Leffler, (Mittag-Leffler, 1903), $E_\alpha(z)$, has the form

$$E_\theta(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\theta k + 1)}\,. \quad (6)$$

Where $\Gamma(\cdot)$ is the Gamma function which is intrinsically tied to fractional calculus and $z \in \mathbb{C}; Re(\theta) > 0$, and $\mathbb{C}$ is the set of all complex numbers. The Mittag-Leffler function is a great tool in the research of fractional calculus. We refer to the reader the book by Samko, Kilbas and Marichev, (Samko, Kilbas & Marichev, 1993), for more details on this important functions. The generalised Mittag-Leffler functions has the following form

$$E_{\theta,\beta}(z) = \sum_{k=0}^{\infty} \frac{z^k}{\Gamma(\theta k + \beta)}\,. \quad (7)$$

Where β is any real number.

Property 2.1: For $f(t) = t^\lambda$ with $\alpha, \lambda > 0$, we have

$$ {}_0^tD^\alpha t^\lambda = \frac{\Gamma(\lambda+1)}{\Gamma(\lambda+1-\alpha)} t^{\lambda-\alpha}. \quad (8)$$

Property 2.2: For $f(t) = e^{\lambda t}$ with $\alpha > 0$ and $\lambda \in \mathbb{R}$, we have

$$ {}_0^tD^\alpha e^{\lambda t} = t^{-\alpha}\; E_{1,1-\alpha}(\lambda t). \quad (9)$$

Definition 2.4: The integer-order derivative expansion for the Riemann–Liouville fractional derivative (5) is defined as, (Samko, Kilbas, & Marichev, 1993)

$$D^\alpha f(t) = \sum_{k=0}^{\infty} \binom{\alpha}{k} \frac{(t-b)^{k-\alpha}}{\Gamma(k-\alpha+1)} \frac{d^k f(t)}{dt^k}. \quad (10)$$

Where b is the base point of the Riemann–Liouville derivative. Also the generalized binomial coefficient, $\binom{\alpha}{k}$, which is valid for fractional order α, is defined as, see (Samko, Kilbas, & Marichev, 1993)

$$\binom{\alpha}{k} = \frac{\Gamma(\alpha+1)}{\Gamma(k+1)\Gamma(\alpha-k+1)} = \frac{(-1)^{k-1}\,\alpha\,\Gamma(k-\alpha)}{\Gamma(1-\alpha)\Gamma(k+1)}\,. \quad (11)$$

As we mentioned earlier that the different definitions of fractional integrals and derivatives are related to each other. The above definition, equation (11), is also applied for Caputo and Grünwald-Letnikov fractional derivative in (Gladkina, Shchedrin, Khawaja, & Carr, 2017).

The conversion of fractional derivative into infinite series of integer derivatives was first established by (Samko, Kilbas, & Marichev, 1993). Also some different representation for fractional derivatives in terms of infinite series introduced. For example see (Atanacković, Janev, Konjik, Pilipović, & Zorica, 2014) and (Atanackovic, & Stankovic, 2004). However the information about how many terms need to be computed numerically to reach the desired accuracy or the truncated error for the first finite number of terms was not studied. For this reason (Gladkina, Shchedrin, Khawaja, & Carr, 2017) gave in details the numerical computations for such infinite series. They showed that, for solving linear fractional differential equations, calculations based on the truncated integer-derivative expansion is a great tool.

3. DESCRIPTION OF THE METHOD

In this section we present our numerical scheme to find an approximate solution of the Bagley-Torvik equation (1) with attached homogeneous initial conditions (2-3).

Only when we have inhomogeneous conditions, the Riemann–Liouville (which is considered here) and Caputo definition for fractional derivative are different. Therefore, in the present study the use of Caputo fractional derivative will lead to the same result due to using homogeneous initial conditions (2-3) in this paper.

Using the definition 2.3, the second term in the Bagley-Torvik equation (1) can be written as

$$\frac{d^{3/2}y(t)}{dt^{3/2}} = \sum_{k=0}^{\infty} \binom{3/2}{k} \frac{t^{k-3/2}}{\Gamma(k-1/2)} \frac{d^k y(t)}{dt^k}. \quad (12)$$

Where the base point, b, here is chosen to be zero. The only restriction on the function $y(t)$ in equation (12) is to be analytic for all $t \in [0, T]$. The approximated form of the equation (12) is

$$\frac{d^{3/2}y(t)}{dt^{3/2}} \cong \sum_{k=0}^{N} \binom{3/2}{k} \frac{t^{k-3/2}}{\Gamma(k-1/2)} \frac{d^k y(t)}{dt^k}. \qquad (13)$$

Where N could be any non-negative integer. From the definition 2.4, equation (10), and some manipulation, the infinite series of integer-order expansion, equation (12), which represents $\frac{d^{3/2}y}{dt^{3/2}}$, can be rewritten as follow

$$\frac{d^{3/2}y(t)}{dt^{3/2}} = \frac{t^{-3/2}}{2\sqrt{\pi}} \left(-y(t) + 3t \frac{dy(t)}{dt} + 3/2\, t^2 \frac{d^2 y(t)}{dt^2} \right) + R(t). \qquad (14)$$

Where $R(t)$ represent the remaining terms of the infinite series in (10), (Gladkina, Shchedrin, Khawaja, & Carr, 2017). Although the behaviour of the truncated error is presented in (Gladkina, Shchedrin, Khawaja, & Carr, 2017), we also show through the following figures the approximated value for the fractional derivatives of some functions.

For polynomials of degree $\lambda = 2, 3$, equation (13) is used to represent ${}_0^tD^{3/2}t^\lambda$ on the interval $[0, 1]$. In this paper $N = 2$ is considered (the first three terms), however as λ and α increase, one notices that $N = 2$ may not be enough.

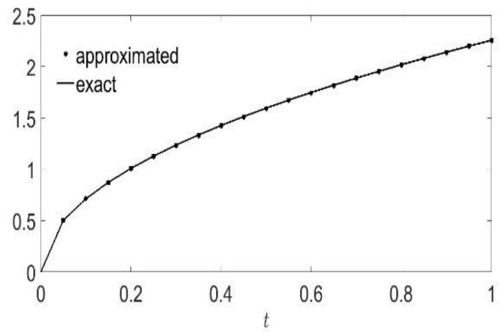

Figure 1.Graph of ${}_0^tD^{3/2}t^2$ (solid line) and its approximated form (stars) using equation (13) with $N = 2$.

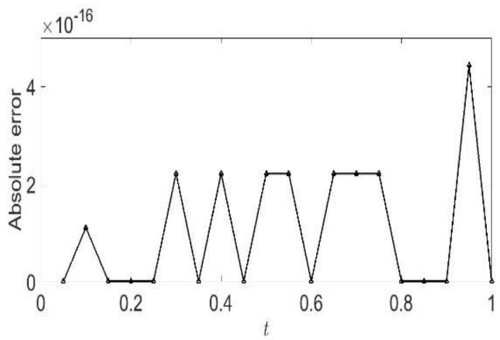

Figure 2. The absolute error of the two graphs in Figure 1.

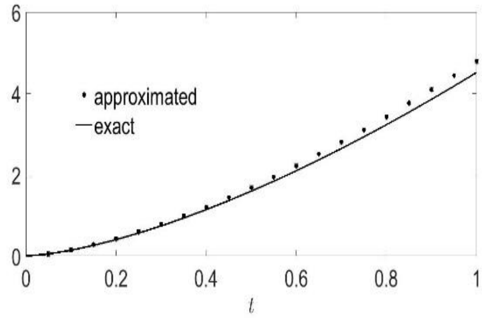

Figure 3.Graph of ${}_0^tD^{3/2}t^3$ (solid line) and its approximated form (stars) using equation (13) with $N = 2$.

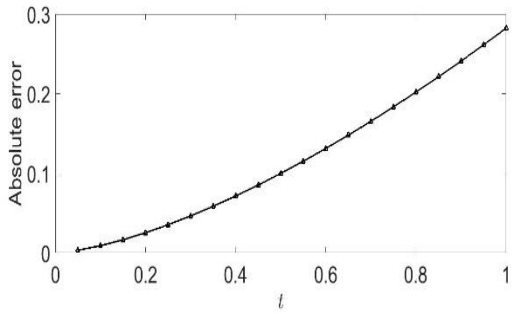

Figure 4. The absolute error between the two graphs in Figure 3.

For Figure 1 with $\lambda = 3$ and Figure 3 with $\lambda = 3$ we used property 2.1, equation (6), to compute and plot the exact graph of ${}_0^tD^{3/2}t^\lambda$. Also equation (13) is used to plot approximately ${}_0^tD^{3/2}t^\lambda$ with $N = 2$, three first terms of equation (13). In Figure 2 and Figure 4, the absolute error of the corresponding figures are shown. Figure 2 shows that equation (13) represents ${}_0^tD^{3/2}t^2$ almost exactly by its three first terms as the absolute error is of order $O(10^{-16})$, while in Figure 4 this difference is significant.

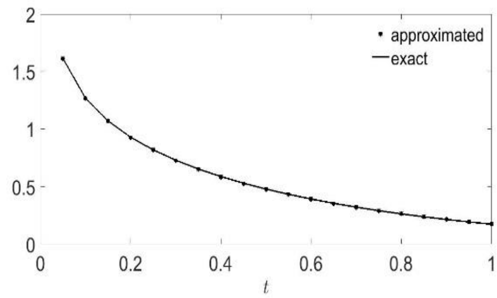

Figure 5. Graph of ${}_0^tD^{1/4}e^{-x}$ (solid line) and its approximated form (stars) using equation (13) with $N = 2$.

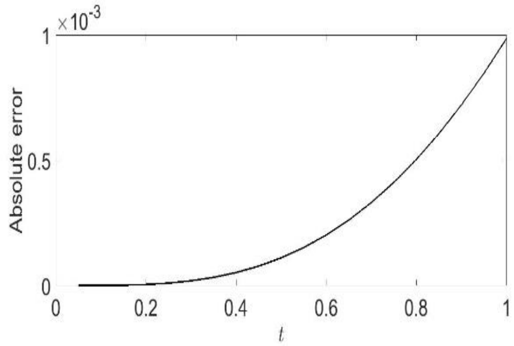

Figure 6. The absolute error of the two graphs in Figure 5.

In Figure 5 we used property 2.2, equation (9), to plot the graph of ${}_0^tD^{1/4}e^{-t}$ and we used equation (13) for approximately plotting ${}_0^tD^{1/4}e^{-t}$ again with $N=2$. For numerical computation of Mettag-Liffler function we calculated the first hundred terms, $k=0,1,\ldots,100$. In Figure 6 the absolute error shows a difference of order $O(10^{-3})$.

Now we continue our proposed scheme for solving Bagley-Torvik equation (1) with initial conditions (2-3). The equation (14) approximately can be written as

$$2\sqrt{\pi}t^{\frac{3}{2}}\frac{d^{3/2}y(t)}{dt^{3/2}} \cong -y(t)+3t\frac{dy(t)}{dt}+\frac{3}{2}t^2\frac{d^2y(t)}{dt^2}. \quad (2)$$

Substitution of equation (15) for $\frac{d^{3/2}y}{dt^{3/2}}$ into equation (1) implies

$$\frac{d^2y(t)}{dt^2}+A(t)\frac{dy(t)}{dt}+B(t)\,y(t)\cong G(t). \quad (16)$$

Where

$$A(t)=\frac{3a_2}{\sqrt{t}}\,\mu(t), \quad (17)$$

$$B(t)=\left(3a_3\sqrt{\pi}-a_2t^{-3/2}\right)\mu(t), \quad (18)$$

$$G(t)=2\sqrt{\pi}\,g(t)\,\mu(t), \quad (19)$$

and

$$\mu(t)=\left(2a_1\sqrt{\pi}+3/2\,a_2\sqrt{t}\right)^{-1}. \quad (20)$$

Equation (16) is an approximated form for the Bagley-Torvik equation (1) after we used the infinite series of integer-order derivative expansion given in equation (10). We converted the second-order ordinary differential equation, equation (1), with fractional order and constant coefficients into equation (16). Equation (16) is a second-order ODE with all integer-order derivatives but with variable coefficients. It is shown in (Gladkina, Shchedrin, Khawaja, & Carr, 2017) that this approximation on differential equation with constant coefficients works more accurate than those with variable coefficients.

The numerical solution to the equation (16), can be found in different ways. We follow the fourth order Runge-Kutta method. This will be done by rewriting equation (16) in the form of a system of two first order ODEs using the following change of variables

$$u(t)=y(t),\;\; v(t)=\frac{dy(t)}{dt}. \quad (21)$$

The corresponding system of equation (16) by using the change of variables (21) takes the form

$$\frac{du(t)}{dt}=v(t),$$
$$\frac{dv(t)}{dt}=G(t)-A(t)v(t)-B(t)u(t)\,. \quad (22)$$

Where the variable coefficients $A(t), B(t)$ and $G(t)$ are defined by equations (17-19), respectively. From initial conditions (2-3), system (22) is supplemented with the following homogeneous initial conditions

$$u|_{t=0}=0, \quad (23)$$

and

$$v|_{t=0}=0. \quad (24)$$

This method can be generalised by increasing the number of the chosen first terms, say M>N, which results a system of M first order ODEs and M initial homogeneous conditions.

4. NUMERICAL RESULTS

In this section we choose two examples with known analytical solutions and we apply our numerical method to make a comparison. For the following example we choose the constant coefficients of equation (1) as $a_1=a_2=a_3=1$. We use the numerical solver command "ode45" which is available in MATLAB to solve the system (22-24).

4.1 Example 1: For the first example we consider the following Bagley-Torvik equation

$$\frac{d^2y(t)}{dt^2}+\frac{d^{3/2}y(t)}{dt^{3/2}}+y(t)=t\left(t^2+8\sqrt{\frac{x}{\pi}}+6\right). \quad (25)$$

With $y(t)$ satisfies the homogeneous initial conditions (2-3). The exact solution of equation (25) which satisfies initial conditions (2-3) is

$$y(t)=t^3. \quad (26)$$

Approximating the term with fractional derivative in equation (25), using only the first three terms of the infinite series (12), we arrive at the system (22-24).

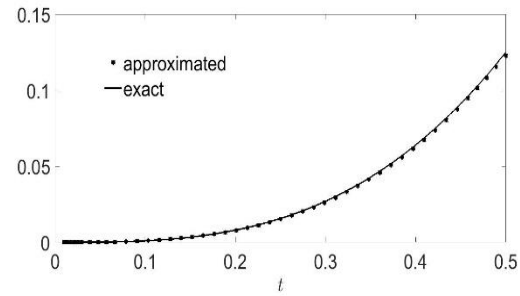

Figure 7. A comparison between the exact solution (solid line), equation (26), and the approximated numerical solution of the equation (25).

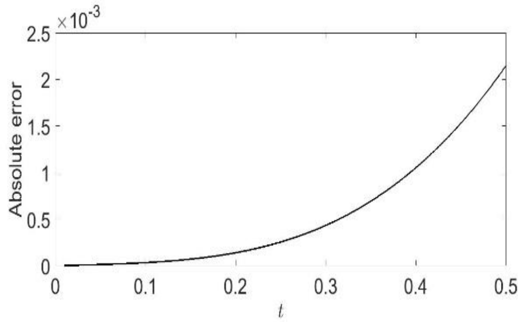

Figure 8. The absolute error of the two graphs in Figure 7.

Figure 7 shows the exact solution of equation (25) and its numerical solution after it has been approximated. Their absolute error is plotted in Figure 8. It is shown that absolute error is of order $O(10^{-3})$. This confirms that the first three terms which represent the fractional derivative term are almost dominant in equation (25). Results for both numerical and analytical solutions with their absolute error are also shown in Table 1.

Table 1. Comparison of results for the solution of the first example.

t	Approximated	Exact	Absolute error
0.11	0.00149424	0.0013	0.000041988615692
0.16	0.00435591	0.0041	0.000086231348493
0.21	0.00955164	0.0093	0.000164549047641
0.26	0.01777239	0.0176	0.000291891919603
0.31	0.02970800	0.0298	0.000484303260347
0.36	0.04604743	0.0467	0.000758710698154
0.40	0.06747894	0.0640	0.001132765351892
0.45	0.09469016	0.0911	0.001624712612298
0.50	0.12285095	0.1250	0.002149044622339

4.2 Example 2: Our second example is the following Bagley-Torvik equation

$$\frac{d^2y(t)}{dt^2}+\frac{d^{3/2}y(t)}{dt^{3/2}}+y(t)=2+4\sqrt{x/\pi}+t^2. \quad (27)$$

The exact analytical solution of equation (27) is

$$y(t)=t^2, \quad (28)$$

which satisfies the homogeneous initial conditions (2-3). In Figure 9 we plot the exact solution of equation (27) and its numerical solution after an approximated form for equation (27) is transformed into a system of ODEs. To show the accuracy of our numerical method the absolute error is plotted in Figure 10. It is noticed that absolute error is of order $O(10^{-8})$. Therefore, again, we have confirmed that by remaining only the first three terms of the infinite series of integer-order expansion, equation (12), which represents the term with fractional derivative in equation (27) we are keeping almost all the dominant terms and it produces an error in our numerical solution which is of order $O(10^{-8})$.
This example is also numerically solved by Ghorbani and Alavi, (Ghorbani, & Alavi, 2008), and the absolute error that they found between the numerical and analytical solution is of order $O(10^{-5})$. A comparison is made between our numerical and analytical results in Table 2.

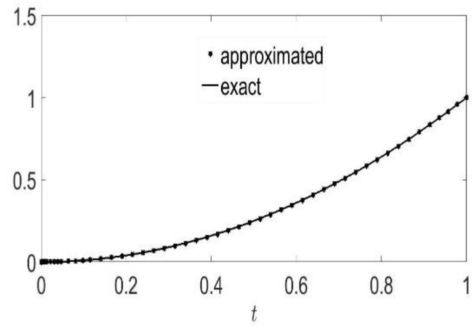

Figure 9. A comparison between the exact solution (solid line), equation (28), and the approximated numerical solution of the equation (27).

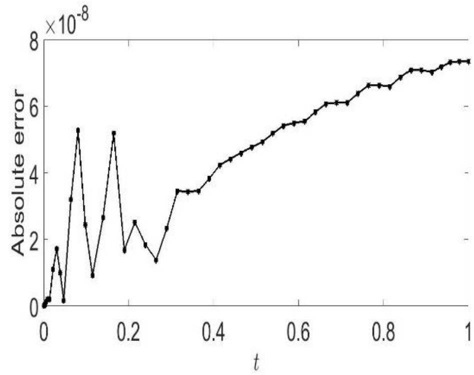

Figure 10. The absolute error of the two graphs in Figure 7.

Table 2. Comparison of results for the solution of the second example.

t	Approximated	Exact	Absolute error
0.1	0.010000001723771	0.01	0.1828e-07
0.2	0.040000046369413	0.04	0.0637e-07
0.3	0.090000089366190	0.09	0.2937e-07
0.4	0.160000120342059	0.16	0.4034e-07
0.5	0.250000148387098	0.25	0.4839e-07
0.6	0.360000175093239	0.36	0.5509e-07
0.7	0.490000200907060	0.49	0.6091e-07
0.8	0.640000225992179	0.64	0.6599e-07
0.9	0.810000250421551	0.81	0.7042e-07
1.0	1.000000273512726	1.00	0.7351e-07

5. CONCLUSION

The Bagley-Torvik equation with constant coefficients is approximated using the infinite series representation of integer-order derivatives. The derived system solved numerically by the fourth order Runge-Kutta method. The plots and their absolute error are presented by figures and tables. By providing two examples we illustrated the accuracy and efficiency of our proposed scheme for numerically solving the fractional differential equations. By considering the Bagley-Torvik equation which contains a term with fractional derivative, we showed through two examples that the scheme presented in this work is producing a close solution comparing to its exact analytical solution.

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