ON THE NULLITY OF GENERALIZED ROOTED T-TUPLE WITH B-BRIDGE COALESCENCE GRAPHS

Nechirvan Badal Ibrahim
Dept. of Mathematics, College of Science University of Duhok, Kurdistan Region-Iraq.

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ABSTRACT:

In this paper, we determine the nullity of generalized rooted t-tuple coalescence graphs and the nullity of generalized rooted t-tuple with b-bridge coalescence graphs. Finally, the nullity of generalized rooted t-tuple with b-path coalescence graph and (b-bridge)-tuple coalescence graph and nullity of (b-path)-tuple coalescence graph is obtained for some special types of graphs.

KEYWORDS: *Nullity, t-tuple coalescence graph, b-bridge coalescence graph, b-path coalescence graph.*

1- INTRODUCTION

The nullity (degree of singularity) of a graph G is the algebraic multiplicity of the number zero in the spectrum of G. It is denoted by η(G) and was studied by (Cvetkovic, Doob and Sachs, (1979)., n.d.). Let $\{(G_1, v_1), (G_2, v_2), \ldots, (G_t, v_t)\}$ be a family of not necessary distinct connected graphs with roots $v_1, v_2, \ldots, v_t$, respectively. A connected graph $G= G_1 \circ G_2 \circ \ldots \circ G_t$ is called the multiple coalescence of $G_1, G_2, \ldots, G_t$ provided that the vertices $v_1, v_2, \ldots, v_t$ are identified to reform the coalescence vertex v. The t-tuple coalescence graph is denoted by $G^{|t|}$ is the multiple coalescence of t isomorphic copies of a graph G (Sharaf & Ali, 2014).

2- Nullity of Generalized Rooted t-Tuple Coalescence Graphs

In this part, we introduce some results about the nullity and now the nullity of generalized rooted t-tuple coalescence graphs can be determined.

Proposition 2.1. (Cvetkovic, Doob and Sachs, (1979)., n.d.)

i) $\eta(P_n)=\begin{cases} 1, & \text{if n is odd,} \\ 0, & \text{if n is even.} \end{cases}$

ii) $\eta(C_n)=\begin{cases} 2, & \text{if } n \equiv 0 \pmod 4, \\ 0, & \text{otherwise.} \end{cases}$

iii) $\eta(K_{m,n})=m+n-2$, for all m, n.

iv) $\eta(K_n)=\begin{cases} 1, & \text{if } n = 1, \\ 0, & \text{if } n > 1. \end{cases}$

Corollary 2.2.(Cvetkovic, Doob and Sachs, (1979)., n.d.) **(End Vertex Corollary (E.V.C.))** If G is a bipartite graph with an end vertex, and if H is an induced subgraph of G obtained by deleting this vertex together with the vertex adjacent to it, then η(G)=η(H).

Theorem 2.3. (Gong & Xu, 2012) Let v be a cut-vertex of a graph G of order n and $G_1, G_2, ..., G_t$ be all components of G-v. If there exists a component, say G_1, among $G_1, G_2, ..., G_t$ such that $\eta(G_1)=\eta(G_1+v)+1$, then $\eta(G)=\eta(G-v) - 1 = \sum_{i=1}^{t} \eta(G_i) - 1$.

Theorem 2.4. (Gong & Xu, 2012) Let v be a cut-vertex of a graph G of order n and G_1 be a component of G-v. If $\eta(G_1)=\eta(G_1 + v)-1$, then $\eta(G)=\eta(G_1)+\eta(G-G_1)$.

Lemma 2.5. (Sharaf & Ali, 2014) **(Coneighbor Lemma (C.L.))** For any pair of coneighbor vertices u and v in a graph G, η(G)=η(G-u)+1=η(G-v)+1.

Lemma 2.6: (Ibrahim, 2013) **(Generalized Coneighbor Lemma(G.C.L.))** If $v_1, v_2, \ldots, v_t$ are pairwise coneighbor vertices of a graph G, then $\eta(G)=\eta(G-(S-\{v_j\}))+t-1$, $1 \le j \le t$, in which $S=\{v_1, v_2, \ldots, v_t\}$.

Lemma 2.7: (Ibrahim, 2013) **(Semi-Coneighbor Lemma (S.C.L.))** If u and v are semi-coneighbor vertices of a graph G, then

η(G) ≤ η(G-u)=η(G-v).

In the following, we define a new concept of t-tuple coalescence graphs and determined their nullities.

Definition 2.8. Let G be a graph consisting of n vertices and $G^*= \{G_1^{|t|}, G_2^{|t|}, \ldots, G_n^{|t|}\}$ be a family of rooted t-tuple coalescence graphs with rooted vertices $v_1, v_2, \ldots, v_n$, respectively. Then, the graph formed by identifying the rooted of t-tuple coalescence graph G_k^* to the k^{th} $(1 \le k \le n)$ vertex of G is called the **generalized rooted t-tuple coalescence graph** and is denoted by $G(G^*)$. G itself is called the **core** of $G(G^*)$. If each member of G^* is isomorphic to the rooted graph G_k^*, then the graph $G(G^*)$ is denoted by $G(G_k^{|t|})$.

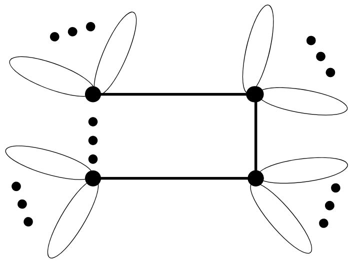

Fig. (1) Generalized rooted t-tuple coalescence graphs $G(G_k^{|t|})$.

The nullity of generalized rooted t-tuple coalescence for cycle graph C_n is determined in the following proposition.

Proposition 2.9.

1) $\eta\left(C_n\left(P_{2m+1}^{|t|}\right)\right) = \begin{cases} 2, \text{if } n = 0(\text{mod}4), \\ 0, \ \text{otherwise.} \end{cases}$

2) $\eta\left(C_n\left(P_{2m}^{|t|}\right)\right) = n(t-1)$.

3) $\eta\left(C_n\left(C_m^{|t|}\right)\right) = \begin{cases} nt + 2, \text{if } n, m = 0(\text{mod}4), \\ nt, \text{if } n \neq 0(\text{mod } 4)\text{and } m = 0(\text{mod}4), \\ 0, \text{if } m = 3. \end{cases}$

4) $\eta\left(C_n\left(K_{m,m}^{|t|}\right)\right) = \begin{cases} nt(2m-3) + 2, \text{if } n = 0(\text{mod}4), \\ nt(2m-3), \ \text{otherwise.} \end{cases}$

5) $\eta\left(C_n\left(K_m^{|t|}\right)\right) = 0, \ \text{if } m \geq 3$.

Proof.

1) Applying E.V.C. (nmt times), we get: $\eta\left(C_n\left(P_{2m+1}^{|t|}\right)\right) = \eta(C_n)$, and by Proposition 2.1(ii) , we get the result.

2) Applying E.V.C. (nm times) in each tuple graph, we get: $\eta\left(C_n\left(P_{2m}^{|t|}\right)\right) = t - 1 + t - 1 + \cdots + t - 1 = n(t-1)$.

3) If n, m=0(mod 4), then by using Theorem 2.4 (n times), we have: $\eta\left(C_n\left(C_m^{|t|}\right)\right) = t + t + \cdots + \eta(C_n) = nt + 2$.

If n≠0(mod 4) and m=0(mod 4), using Theorem 2.4 (n times), we have: $\eta\left(C_n\left(C_m^{|t|}\right)\right) = t + t + \cdots + \eta(C_n)$, and by Proposition 2.1(ii), we get: $\eta\left(C_n\left(C_m^{|t|}\right)\right) = nt$.

If m=3, using S.C.L. with E.V.C., we get the result.

4) Applying Theorem 2.4 (n times), we get: $\eta\left(C_n\left(K_{m,m}^{|t|}\right)\right) = t(2m-3) + t(2m-3) + \cdots + \eta(C_n)$ $= nt(2m-3) + \eta(C_n)$, then by proposition 2.1(ii), we have: $\eta\left(C_n\left(K_{m,m}^{|t|}\right)\right) = \begin{cases} nt(2m-3) + 2, \text{if } n = 0(\text{mod}4), \\ nt(2m-3), \ \text{otherwise.} \end{cases}$

5) The case m=2 is proved by (2). If m≥3, using S.C.L. with E.V.C. (n times), we get the result.

The nullity of generalized rooted t-tuple coalescence for complete bipartite graph $K_{n,n}$ is determined in the following proposition.

Proposition 2.10.

1) $\eta\left(K_{n,n}\left(P_m^{|t|}\right)\right) = \begin{cases} 2n(t-1), \text{if m is even}, \\ 2n-2, \text{if m is odd}. \end{cases}$

2) $\eta\left(K_{n,n}\left(C_m^{|t|}\right)\right) = \begin{cases} 2nt+2n-2, \text{if } m = 0 \text{ (mod 4)}, \\ 0 \text{ , if } m = 3 \text{ .} \end{cases}$

3) $\eta\left(K_{n,n}\left(K_{m,m}^{|t|}\right)\right) = 2nt(2m-3)+2n-2.$

4) $\eta\left(K_{n,n}\left(K_m^{|t|}\right)\right) = \begin{cases} 2n(t-1), \text{if } m = 2 \text{ and } t > 1, \\ 0 \text{ , if } m \geq 3 \text{ .} \end{cases}$.

Proof.

1) Applying E.V.C. (2n times), we get the result.

2) By applying Theorem 2.4 (2n times), we have: $\eta\left(K_{n,n}\left(C_m^{|t|}\right)\right) = t+t+\cdots+\eta(K_{n,n})$, then by Proposition 2.1(iii), we get: $\eta\left(K_{n,n}\left(C_m^{|t|}\right)\right) = 2nt+2n-2.$

If m=3, using S.C.L. with E.V.C. (2n times), we get the result.

3) By applying Theorem 2.4 (2n times) we have: $\eta\left(K_{n,n}\left(K_{m,m}^{|t|}\right)\right) = t(2m-3)+t(2m-3)+\cdots+\eta(K_{n,n})$, then by proposition 2.1(iii), we get: $\eta\left(K_{n,n}\left(C_m^{|t|}\right)\right) = 2n(t(2m-3)+2n-2.$

4) If m=2, is the same proof of case (1), if $m \geq 3$, is the same proof of case 2.

The nullity of generalized rooted t-tuple coalescence for complete graph K_n is determined in the following proposition.

Proposition 2.11.

1) $\eta\left(K_n\left(P_m^{|t|}\right)\right) = \begin{cases} n(t-1), \text{if m is even}, \\ 0, \text{if m is odd}. \end{cases}$

2) $\eta\left(K_n\left(C_m^{|t|}\right)\right) = \begin{cases} nt, \text{if } m = 0 \text{(mod 4)}, \\ 0, \text{if } m = 3. \end{cases}$

3) $\eta\left(K_n\left(K_{m,m}^{|t|}\right)\right) = nt(2m-3).$

4) $\eta\left(K_n\left(K_m^{|t|}\right)\right) = \begin{cases} n(t-1), \text{if } m = 2, \\ 0, \text{if } m > 2. \end{cases}$

Proof.

The proof is similar to the proof of the Proposition 2.10.

3- Nullity of Generalized Rooted t-Tuple with b-Bridges Coalescence Graphs

In this part, we define a new t-tuple coalescence graph having bridges and study the nullity of such composite tuple graphs.

Definition 3.1. Let G be a graph on n vertices and each vertex in G is a rooted vertex and $G_i^{|t|}$={$G_1^{|t|}$, $G_2^{|t|}$, …, $G_n^{|t|}$} be a family of rooted t-tuple coalescence graphs. Then, the graph formed by introducing the rooted of t-tuple coalescence graph $G_i^{|t|}$to the rooted vertex in G by an edge called bridge is called **generalized rooted t-tuple with b-bridges coalescence graph** and denoted by G(b) $G_i^{|t|}$, as illustrated in Figure 2.

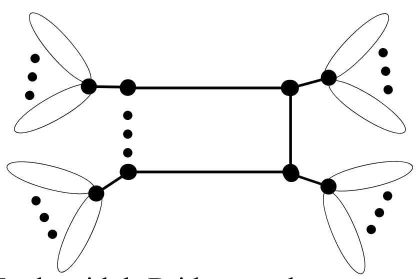

Fig. (2) Generalized rooted t-Tuple with b-Bridges coalescence graphs G(b) $G_i^{|t|}$.

Proposition 3.2.

1) $\eta\left(P_n(b)P_m^{|t|}\right) = \begin{cases} n(t-1), \text{if n and m is even}, \\ n(t-1)+1, \text{if m is even and n is odd}, \\ 0, \ \text{if m is odd}. \end{cases}$

2) $\eta\left(P_n(b)C_m^{|t|}\right) = \begin{cases} 0, \text{if } m = 3 \text{ and n is even}, \\ 1, \text{if } m = 3 \text{ and n is odd}, \\ nt, \text{if n } m = 0(\text{mod } 4). \end{cases}$

3) $\eta\left(P_n\ (b)K_{m,m}^{|t|}\right) = nt(2m-3).$

4) $\eta\left(P_n(b)K_m^{|t|}\right) = \begin{cases} n(t-1)+1, \text{if } m = 2 \text{ and n is odd}, \\ n(t-1), \text{if } m = 2 \text{ and n is even}, \\ 0, \text{if } \ m \geq 3. \end{cases}$

Proof.

1) If n and m is even, applying E.V.C. (n(m/2) times) in each tuple graph and by Proposition 2.1(i), we get the result.

If m is odd, applying E.V.C. (m-1/2 times) in each tuple graph and after them using E.V.C., we get the result.

2) If m=3, using S.C.L. with E.V.C. (n times), we get the result, if m=0(mod 4), applying Theorem 2.3(n times), we get: $\eta\left(P_n(b)C_m^{|t|}\right) = \eta\left(C_m^{|t|}\right) - 1 + \eta\left(C_m^{|t|}\right) - 1 + \cdots + \eta\left(C_m^{|t|}\right) - 1$

$$= n(t+1) - n = nt.$$

3) Applying Theorem 2.3 (n times), we get:

$$\eta\left(P_n(b)K_{m,m}^{|t|}\right) = t(2m-3)+1-1+t(2m-3)+1-1+\cdots+t(2m-3)+1-1$$
$$= nt(2m-3).$$

4) If m=2, applying E.V.C. (n times), we get:

$\eta\left(P_n(b)K_m^{|t|}\right) = t-1+t-1+\cdots+\eta(P_n)$, and by Proposition 2.1(i), we get the result. And if m >2 using S.C.L. with E.V.C., we get the result.

Proposition 3.3.

1) $\eta\left(C_n(b)P_{2m+1}^{|t|}\right) = 0.$

2) $\eta\left(C_n(b)P_{2m}^{|t|}\right) = \begin{cases} n(t-1)+2, \text{if } n = 0(\text{mod}4), \\ n(t-1), \ \text{otherwise}. \end{cases}$

3) $\eta\left(C_n(b)C_m^{|t|}\right) = \begin{cases} 2, \text{if } m = 3 \text{ and } n = 0\ (\text{mod } 4), \\ 0, \text{if } m = 3 \text{ and } n \neq 0(\text{mod } 4), \\ nt, \text{if } \ m = 0(\text{mod } 4). \end{cases}$

4) $\eta\left(C_n(b)K_{m,m}^{|t|}\right) = nt(2m-3).$

5) $\eta(C_n(b)K_m^{|t|}) = \begin{cases} n(t-1)+2, \text{if } m = 2 \text{ and } n = 0\ (\text{mod } 4), \\ n(t-1), \text{if } m = 2 \text{ and } n \neq 0(\text{mod } 4), \\ 0, \text{if } \ m \geq 3. \end{cases}$

Proof.

1) By applying E.V.C. ((t(2m-1/2)) times) in each tuple graph we get the result.

2) By applying E.V.C. (m times) in each tuple graph we get the result.

3) If m=3, by using S.C.L. with E.V.C. (n times) we get the result, if m=3 and n=0(mod 4) the proof is easy.

If m=0(mod 4), applying Theorem 2.3(n times), we get:

$$\eta\left(C_n(b)C_m^{|t|}\right) = \eta\left(C_m^{|t|}\right) - 1 + \eta\left(C_m^{|t|}\right) - 1 + \cdots + \eta\left(C_m^{|t|}\right) - 1 = n(t+1) - n = nt.$$

4) Applying Theorem 2.3 (n times), we get:

$$\eta\left(C_n(b)K_{m,m}^{|t|}\right) = t(2m-3)+1-1+t(2m-3)+1-1+\cdots+t(2m-3)+1-1$$
$$= nt(2m-3).$$

5) If m=2, using S.C.L with E.V.C. (n times), we get:

$\eta\left(C_n(b)K_m^{|t|}\right) = t-1+t-1+\cdots+\eta(C_n)$, and by Proposition 2.1(ii), we get the result.

If m ≥ 3, using S.C.L. with E.V.C., we get the result.

Proposition 3.4.

1) $\eta\left(K_{n,n}(b)P_m^{|t|}\right) = \begin{cases} 2(nt-1), \text{if m is even,} \\ 0, \text{ if m is odd.} \end{cases}$

2) $\eta\left(K_{n,n}(b)C_m^{|t|}\right) = \begin{cases} 2n-2, \text{if } m=3 \text{ and n is even,} \\ 2nt, \text{if and } m=0(\text{mod } 4). \end{cases}$

3) $\eta\left(K_{n,n}(b)K_{m,m}^{|t|}\right) = 2nt(2m-3).$

4) $\eta\left(K_{n,n}(b)K_m^{|t|}\right) = \begin{cases} 2n(t-1)+2n-2, \text{if } m=2 \text{ and n is odd,} \\ 0, \text{if } m \geq 3. \end{cases}$

Proof.

The proof is similar to the proof of the Proposition 3.2.

Proposition 3.5.

1) $\eta\left(K_n(b)P_m^{|t|}\right) = \begin{cases} n(t-1), \text{if m is even,} \\ 0, \text{ if m is odd.} \end{cases}$

2) $\eta\left(K_n(b)C_m^{|t|}\right) = \begin{cases} n(t-1), \text{if } m=3, \\ nt, \text{if } m=0(\text{mod } 4). \end{cases}$

3) $\eta\left(K_n(b)K_{m,m}^{|t|}\right) = nt(2m-3).$

4) $\eta\left(K_n(b)K_m^{|t|}\right) = \begin{cases} n(t-1), \text{if } m=2, \\ 0, \text{if } m \geq 3. \end{cases}$

Proof.

The proof is similar to the proof of the Proposition 3.2.

4- The Nullity of Generalized Rooted t-Tuple with b-path Coalescence Graphs

If the b-bridge in definition 3.1 is replaced by a b-path graph of odd order, then we call such a t-tuple with b-bridge coalescence graph by a t-tuple with b-path coalescence graph and symbolized it by G(b-path) $G_i^{|t|}$, as shown in Figure 3.

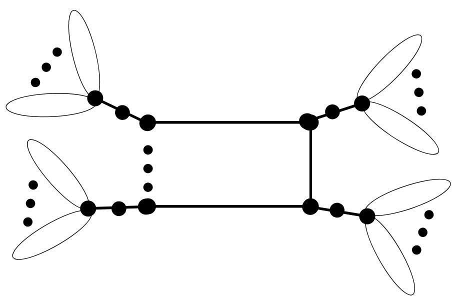

Fig. (3) Generalized Rooted t-Tuple with b-path Coalescence Graph G(b-path) $G_i^{|t|}$.

In the following, we obtain the nullity of generalized rooted t-tuple with b-path coalescence graph of some special graphs**.**

Proposition 4.1.

1) If n is odd, then

$\eta\left(P_n(b-\text{path})P_m^{|t|}\right) = \begin{cases} n(t-1), \text{if m is even,} \\ 1, \text{ if m is odd.} \end{cases}$

2) If n is even, then

$\eta\left(P_n(b-\text{path})P_m^{|t|}\right) = \begin{cases} n(t-1), \text{if m is even,} \\ 0, \text{ if m is odd.} \end{cases}$

3) $\eta\left(P_n(b-\text{path})C_m^{|t|}\right) = \begin{cases} 0, \text{if } m=3 \text{ and n is even,} \\ 1, \text{if } m=3 \text{ and n is odd,} \\ nt+1, \text{if n is odd and } m=0(\text{mod } 4). \\ nt, \text{if n is even and } m=0(\text{mod } 4). \end{cases}$

4) $\eta\left(P_n\ (b-\text{path})K_{m,m}^{|t|}\right) = \begin{cases} n(t(2m-3)+1, \text{if n is odd,} \\ n(t(2m-3), \text{ if n is even.} \end{cases}$.

5) $\eta\left(P_n(b-\text{path})K_m^{|t|}\right)=\begin{cases} n(t-1), \text{if } m=2, \\ 0, \text{if } m\geq 3. \end{cases}$

Proof.

1) If n is odd and m is even, applying E.V.C. (m/2 times) in each tuple graph, we get: $\eta\left(P_n(b-\text{path})P_m^{|t|}\right)=t-1+t-1+\cdots+\eta\left(P_n(b-\text{path})P_{m-1}^{|1|}\right)$, also using E.V.C., we get: $\eta\left(P_n(b-\text{path})P_m^{|t|}\right)=n(t-1)+\eta(P_{even})$, and by Proposition 2.1(i), we have: $\eta\left(P_n(b-\text{path})P_m^{|t|}\right)=n(t-1)$.

If n and m is odd, applying E.V.C., we get the result.

2) The proof is similar to part 1.

3) If m=3, using S.C.L. with E.V.C., we get the result.

If m 3, applying Theorem 2.3 (n times), we get: $\eta\left(P_n(b-\text{path})C_m^{|t|}\right)=t+1-1+t+1-1+\cdots+\eta(P_n)$, and by Proposition 2.1(i) , we get the result.

4) Applying Theorem 2.3 (n times) with E.V.C., we get: $\eta\left(P_n\ (b-\text{path})K_{m,m}^{|t|}\right)=n\big(t(2m-3+1)\big)-n+\eta(P_n)$, and by Proposition 2.1(i) , we get the result.

5) If m=2 is the special case of 1. And if m ≥3, using S.C.L. with E.V.C., we get the result.

Proposition 4.2.

1) $\eta\left(C_n(b-\text{path})P_m^{|t|}\right)=\begin{cases} 2, \text{if } n=0(\text{mod}4) \text{and m is odd}, \\ n(t-1), \text{if } n=0(\text{mod}4) \text{and m is even}, \\ 0, \text{if } n\neq 0(\text{mod}4) \text{and m is odd}, \\ n(t-1), \ \text{if } n\neq 0(\text{mod}4) \text{and m is odd}. \end{cases}$

2) $\eta\left(C_n(b-\text{path})C_{4m}^{|t|}\right)=\begin{cases} nt+2, \text{if } n=0\ (\text{mod } 4), \\ nt, \text{if } n\neq 0(\text{mod } 4), \end{cases}$

3) $\eta\left(C_n\ (b-\text{path})K_{m,m}^{|t|}\right)=\begin{cases} nt(2m-3)+2, \text{if } n=0\ (\text{mod } 4), \\ nt(2m-3), \text{if } n\neq 0(\text{mod } 4), \end{cases}$

4) $\eta\left(C_n(b-\text{path})K_m^{|t|}\right)=\begin{cases} n(t-1), \text{if } m=2, \\ 0, \text{if } m\geq 3. \end{cases}$

Proof.

1) For all cases using E.V.C., we can get the result.

2) Applying Theorem 2.3 (n times), we get: $\eta\left(C_n(b-\text{path})C_{4m}^{|t|}\right)=t+1-1+t+1-1+\cdots+\eta(C_n)$, and by Proposition 2.1(ii) , we get the result.

3) Applying Theorem 2.3 (n times), we get: $\eta\left(C_n\ (b-\text{path})K_{m,m}^{|t|}\right)=t(2m-3)+t(2n-3)+\cdots+\eta(C_n)$, and by Proposition 2.1(ii), we get the result.

4) The proof is similar to the proof of Proposition 4.1.

Proposition 4.3.

1) $\eta\left(K_{n,n}(b-\text{path})P_m^{|t|}\right)=\begin{cases} 2n(t-1), \text{if m is even}, \\ 2n-2, \ \text{if m is odd}. \end{cases}$

2) $\eta\left(K_{n,n}(b-\text{path})C_m^{|t|}\right)=\begin{cases} 0, \text{if } m=3, \\ 2(nt+n-1), \text{if n and } m=0(\text{mod } 4). \end{cases}$

3) $\eta\left(K_{n,n}\ (b-\text{path})K_{m,m}^{|t|}\right)=2nt(2m-3)+2n-2.$

4) $\eta\left(K_{n,n}(b-\text{path})K_m^{|t|}\right)=\begin{cases} 2n(t-1)+, \text{if } m=2, \\ 0, \text{if } m\geq 3. \end{cases}$

Proof.

The proof is similar to the proof of the Proposition 4.2.

Proposition 4.4: If n>2. Then

1) $\eta\left(K_n(b-\text{path})P_m^{|t|}\right)=\begin{cases} n(t-1), \text{if m is even,} \\ 0, \ \text{if m is odd.}\end{cases}$

2) $\eta\left(K_n(b-\text{path})C_m^{|t|}\right)=\begin{cases} 0, \text{if } m=3\ , \\ nt, \text{if } \ m=0(\text{mod } 4).\end{cases}$

3) $\eta\left(K_n(b-\text{path})K_{m,m}^{|t|}\right)=nt(2m-3).$

4) $\eta\left(K_n(b-\text{path})K_m^{|t|}\right)=0, \text{if } m>2.$

Proof.

The proof is similar to the proof of the Proposition 4.2.

5- The Nullity of Generalized Rooted b-Bridge Tuple Coalescence Graphs

In this part, we introduce the generalized rooted b-bridge tuple coalescence graph, defined as follows:

Definition 5.1. Let G be a graph of order n with vertex set V(G)={ $v_1, v_2, \ldots, v_n$} and each vertex in G is a rooted vertex. Let $G\left(K_1^{b(G_i)}\right)$ be the graph obtained from G by identifying each rooted vertex in G with b-bridge tuple coalescence graph $K_1^{b(G_i)}$. As illustrated in Figure 4.

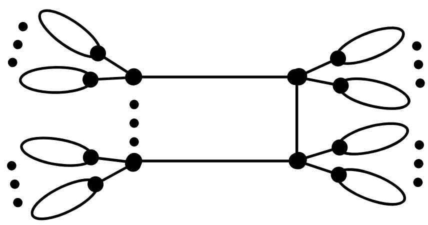

Fig. (4) Generalized Rooted b-bridge tuple coalescence graph $G\left(K_1^{b(G_i)}\right)$.

The nullity of generalized rooted b-bridge tuple coalescence for cycle graph C_n is evaluated in the next proposition.

Proposition 5.2.

1) $\eta\left(C_n\left(K_1^{b(P_m)}\right)\right)=\begin{cases} b-1, \text{if m is odd,} \\ 2, \text{if m is even.}\end{cases}$

2) $\eta\left(C_n\left(K_1^{b(C_m)}\right)\right)=\begin{cases} n(2b-1), \text{if } m=0(\text{mod } 4), \\ 0, \text{if } m\neq 0(\text{mod } 4).\end{cases}$

3) $\eta\left(C_n\left(K_1^{b(K_{m,m})}\right)\right)=nb(2m-2)-n.$

4) $\eta\left(C_n\left(K_1^{b(K_m)}\right)\right)=0.$

Proof.

1) Applying E.V.C., we can get the result.

2) For case (i), using Theorem 2.3 (n times), we get: $\eta\left(C_n\left(K_1^{b(C_m)}\right)\right)=2b+2b+\cdots+2b-n=2nb-n.$

For case (ii), using S.C.L., we get the result.

3) Applying Theorem 2.3 (n times), we get: $\eta\left(C_n\left(K_1^{b(K_{m,m})}\right)\right)=b(2m-2)+b(2m-2)+\cdots+b(2m-2)=nb(2m-2)-n.$

4) Using S.C.L., we get the result.

The nullity of generalized rooted b-bridge tuple coalescence for complete bipartite graph $K_{n,n}$ is evaluated in the next proposition.

Proposition 5.3.

1) $\eta\left(K_{n,n}\left(K_1^{b(P_m)}\right)\right)=\begin{cases}2n(b-1), \text{if m is odd,}\\ 2n-2, \text{if m is even.}\end{cases}$

2) $\eta\left(K_{n,n}\left(K_1^{b(C_m)}\right)\right)=\begin{cases}2n(2b-1), \text{if } m=0(\text{mod } 4),\\ 0, \text{O.W.}\end{cases}$

3) $\eta\left(K_{n,n}\left(K_1^{b(K_{m,m})}\right)\right)=2n(b(2m-2)-1).$

4) $\eta\left(K_{n,n}\left(K_1^{b(K_m)}\right)\right)=0.$

Proof.

The proof is similar to the proof of the Proposition 5.2.

The nullity of generalized rooted b-bridge tuple coalescence for complete graph K_n is evaluated in the next proposition.

Proposition 5.4.

1) $\eta\left(K_n\left(K_1^{b(P_m)}\right)\right)=\begin{cases}n(b-1), \text{if m is odd,}\\ 0, \text{if m is even.}\end{cases}$

2) $\eta\left(K_n\left(K_1^{b(C_m)}\right)\right)=\begin{cases}n(2b-1), \text{if } m=0(\text{mod } 4),\\ 0, \text{O.W.}\end{cases}$

3) $\eta\left(K_n\left(K_1^{b(K_{m,m})}\right)\right)=n(b(2m-2)-1).$

4) $\eta\left(K_n\left(K_1^{b(K_m)}\right)\right)=0.$

Proof.

The proof is similar to the proof of the Proposition 5.2.

Definition 5.5: If the b-bridge tuple graph in Definition 3.1 is replaced by a b-path P_b, then we call such a b- tuple graph by a **b-path tuple graph** and symbolized it by $G\left(K_1^{b-\text{path}(G_i)}\right)$. That is $G\left(K_1^{b-\text{path}(G_i)}\right)$ obtained from G by identifying each rooted vertex in G with b-path tuple graph $K_1^{b-\text{path}(G_i)}$.

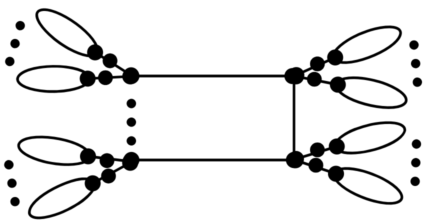

Fig. (5) Generalized rooted b-path tuple graph $G\left(K_1^{b-\text{path}(G_i)}\right)$.

Proposition 5.6.

1) $\eta\left(C_n\left(K_1^{b-\text{path}(P_{2m+1})}\right)\right)=\begin{cases}2, \text{ if } n=0(\text{mod } 4),\\ 0, \text{otherwise.}\end{cases}$

2) $\eta\left(C_n\left(K_1^{b-\text{path}(P_{2m})}\right)\right)=n(b-1), \text{if } n=0(\text{mod } 4).$

3) $\eta\left(C_n\left(K_1^{b-\text{path}(C_{4m})}\right)\right)=\begin{cases}nb+2, \text{ if } n=0(\text{mod } 4),\\ nb, \text{otherwise.}\end{cases}$

4) $\eta\left(C_n\left(K_1^{b-\text{path}(K_{m,m})}\right)\right)=\begin{cases}nmb+2, \text{ if } n=0(\text{mod } 4),\\ nmb, \text{otherwise.}\end{cases}$

5) $\eta\left(C_n\left(K_1^{b-\text{path}(K_m)}\right)\right)=\begin{cases}2, \text{if } n=0(\text{mod } 4) \text{and } m=1,\\ nb, \text{if } m=2,\\ 0, \text{if } m>2.\end{cases}$

Proof.

1) Using E.V.C., we get the result.
2) Using E.V.C., we get the result.
3) Applying Theorem 2.4 (n times), we get:
$\eta\left(C_n\left(K_1^{b-path(C_{4m})}\right)\right) = b + b + \cdots + \eta(C_n)$, and by Proposition 2.1(ii), we get the result.
4) Applying Theorem 2.4(n times), we get:
$\eta\left(C_n\left(K_1^{b-path(K_{m,m})}\right)\right) = b + b + \cdots + \eta(C_n)$, and by Proposition 2.1(ii), we get the result.
5) By using S.C.L., we get the result.

Proposition 5.7.

1) $\eta\left(K_{n,n}\left(K_1^{b-path(P_m)}\right)\right) = \begin{cases} 2n-2, \text{if m is odd,} \\ 2n(b-1), \text{if m is even.} \end{cases}$

2) $\eta\left(K_{n,n}\left(K_1^{b-path(C_m)}\right)\right) = \begin{cases} 2n(b+1)-2, \text{if } m = 0(\text{mod } 4), \\ 0, \text{if } m = 3. \end{cases}$

3) $\eta\left(K_{n,n}\left(K_1^{b-path(K_{m,m})}\right)\right) = 2n(mb+1)-2.$

4) $\eta\left(K_{n,n}\left(K_1^{b-path(K_m)}\right)\right) = \begin{cases} 2n-2, \text{if } m = 1, \\ 2n(b-1), \text{if } m = 2, \\ 0, \text{if } m > 2. \end{cases}$

Proof.

1) If m is odd, applying E.V.C., we get the result.
If m is even, applying E.V.C., (2n times) we get:

$$\eta\left(K_{n,n}\left(K_1^{b-path(P_m)}\right)\right) = b-1+b-1+\cdots+b-1 = 2n(b-1).$$

2) If m=3, apply S.C.L., we get the result.
If m=0 (mod 4), applying Theorem 2.4(2n times), we get:

$$\eta\left(K_{n,n}\left(K_1^{b-path(C_m)}\right)\right) = b+b+\cdots+\eta(K_{n,n}) = 2nb+2n-2 = 2n(b+1)-2.$$

3) Applying Theorem 2.4 (2n times), we have:

$$\eta\left(K_{n,n}\left(K_1^{b-path(K_{m,m})}\right)\right) = mb+mb+\cdots+\eta(K_{n,n}) = 2mnb+2n-2.$$

4) For m=1 is a special case of case 1 part (1). For n=2, applying E.V.C, we get the result. For case n > 2 using S.C.L. with E.V.C., we get the result.

The nullity of generalized rooted b-path tuple coalescence for complete graph is evaluated in the next proposition.

Proposition 5.8.

1) $\eta\left(K_n\left(K_1^{b-path(P_m)}\right)\right) = \begin{cases} n(b-1), \text{if n is ven,} \\ 0, \text{if n is odd.} \end{cases}$

2) $\eta\left(K_n\left(K_1^{b-path(C_{4m})}\right)\right) = nb, \text{if } n > 2.$

3) $\eta\left(K_n\left(K_1^{b-path(K_{m,m})}\right)\right) = nmb.$

4) $\eta\left(K_n\left(K_1^{b-path(K_m)}\right)\right) = 0, \text{if } m \geq 3.$

Proof.

1) The proof is similar to the proof of case (1) of the Proposition 5.7.
2) The proof is similar to the proof of case (3) of the Proposition 5.6.
3) The proof is similar to the proof of case (3) of the Proposition 5.6.
4) The proof is similar to the proof of case (5) of the Proposition 5.6.

Definition 5.9. In Definition 5.1, we introduced each rooted vertex in G by a bridge with $K_1^{b(G_i)}$, and we denote it by $(G\,(b)K_1^{b(G_i)})$. As illustrated in Figure 6.

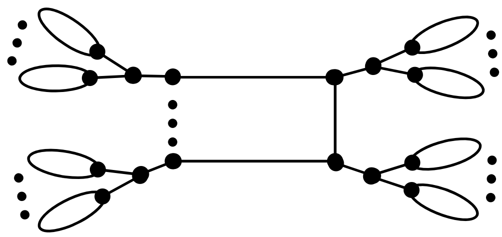

Fig. (6) A graph $G\ (b)K_1^{b(G_i)}$.

Proposition 5.10.

1) $\eta\left(C_n\ (b)K_1^{b(P_m)}\right) = \begin{cases} n(b-1)+2, \text{if m is even and } n = 0(\text{mod } 4), \\ n(b-1), \text{if m is even and } n \neq 0(\text{mod}4), \\ 0, \text{if m is odd.} \end{cases}$

2) $\eta\left(C_n\ (b)K_1^{b(C_m)}\right) = \begin{cases} n(2b-1)+2, \text{if } n, m = 0(\text{mod } 4), \\ n(2b-1), \text{if } m = 0(\text{mod } 4) \text{ and } n \neq 0(\text{mod } 4). \end{cases}$

3) $\eta\left(C_n\ (b)K_1^{b(K_{m,m})}\right) = \begin{cases} 2nmb - 2nb - n + 2, \text{if } n = 0(\text{mod } 4), \\ 2nmb - 2nb - n, \text{if } n \neq 0(\text{mod}4), \end{cases}$

4) $\eta\left(C_n\ (b)K_1^{b(K_m)}\right) = 0, \text{if } m \geq 3.$

Proof.

The proof is similar to the proof of the Proposition 5.8.

Proposition 5.11.

1) $\eta\left(P_n\ (b)K_1^{b(P_m)}\right) = \begin{cases} n(b-1)+1, \text{if n, m is odd}, \\ n(b-1), \text{if n is even}, \\ 0, \text{O.W.} \end{cases}$

2) $\eta\left(P_n\ (b)K_1^{b(C_m)}\right) = \begin{cases} n(2b-1)+1, \text{if n is odd and} m = 0(\text{mod } 4), \\ n(2b-1), \text{if n is even and } m = 0(\text{mod } 4). \\ 0, \text{if } m = 3. \end{cases}$

3) $\eta\left(P_n\ (b)K_1^{b(K_{m,m})}\right) = \begin{cases} 2nmb - 2nb - n + 1, \text{if nis odd}, \\ 2nmb - 2nb - n, \text{if n is even.} \end{cases}$

4) $\eta\left(P_n\ (b)K_1^{b(K_m)}\right) = \begin{cases} n(b-1)+1, \text{ifn is odd and } m = 1, \\ n(b-1), \text{if n is even and } m = 1, \\ 0, \text{ if } n \geq 2. \end{cases}$

Proof.

1) Applying E.V.C., we get the result.

2) For cases (if n is odd and$m = 0(\text{mod } 4)$, andif n is even and $m = 0(\text{mod } 4)$), applying Theorem 2.4 (n times), we get:

$\eta\left(P_n\ (b)K_1^{b(C_m)}\right) = 2b - 1 + 2b - 1 + \cdots + \eta(P_n)$, and by Proposition 2.1(i), we have $\eta\left(P_n\ (b)K_1^{b(C_m)}\right) = \begin{cases} n(2b-1)+1, \text{if n is odd and} m = 0(\text{mod } 4), \\ n(2b-1), \text{if n is even and } m = 0(\text{mod } 4). \end{cases}$

And for case if m=3, apply S.C.L. with E.V.C., we get the result.

3) The proof is similar to the proof of case 2.

4) For cases (if n is odd and m=1) and (if n is even and m=1), using E.V.C., we get the result. And for case (if n $\geq$ 2), using S.C.L. with E.V.C., we get the result.

Proposition 5.12.

1) $\eta\left(K_{n,n}(b)K_1^{b(P_m)}\right) = \begin{cases} 2n(b-1) + 2m - 2, \text{if m is odd}, \\ 0, \text{if m is even.} \end{cases}$

2) $\eta\left(K_{n,n}(b)K_1^{b(C_m)}\right) = \begin{cases} 4nb - 2, \text{if } m = 0(\text{mod } 4). \\ 0, \text{if } m = 3. \end{cases}$

3) $\eta\left(K_{n,n}(b)K_1^{b(K_{m,m})}\right) = 4nbm - 4nb - 2.$

4) $\eta\left(K_{n,n}(b)K_1^{b(K_m)}\right) = 0.$

Proof.
The proof is similar to the proof of the Proposition 5.11.

Proposition 5.13.

1) $\eta\left(K_n(b)K_1^{b(P_m)}\right)=\begin{cases} n(b-1), \text{if m is odd} ,\\ 0, \text{if n is even} ,\end{cases}$

2) $\eta\left(K_n(b)K_1^{b(C_m)}\right)=\begin{cases} n(b-1), \text{if } m=0(\text{mod } 4),\\ 0, \text{if } m=3.\end{cases}$

3) $\eta\left(K_n(b)K_1^{b(K_{m,m})}\right)=n(2b(m-1)-1).$

4) $\eta\left(K_n(b)K_1^{b(K_m)}\right)=0.$

Proof.
The proof is similar to the proof of the Proposition 5.11.

Definition 5.14. In Definition 6.1, we introduced each rooted vertex in G by a b-path with $K_1^{b(G_i)}$, and we denote it by $(G\ (b-\text{path})\ K_1^{b(G_i)})$.

Proposition 5.15.

1) $\eta\left(C_n\ (b-\text{path})K_1^{b(P_m)}\right)=\begin{cases} 2, \text{if m is even and } n=0(\text{mod } 4),\\ 0, \text{if m is even and } n\neq 0(\text{mod}4),\\ n(b-1), \text{if m is odd}\end{cases}$

2) $\eta\left(C_n\ (b-\text{path})K_1^{b(C_m)}\right)=\begin{cases} n(b-1), \text{if } m=0(\text{mod } 4),\\ 2,\ \ \text{if } m\ \neq 3 \text{ and } n=0(\text{mod } 4),\\ 0, \text{if } m=3 \text{ and } n\neq 0(\text{mod } 4).\end{cases}$

3) $\eta\left(C_n\ (b-\text{path})K_1^{b(K_{m,m})}\right)=2nb(m-1).$

4) $\eta\left(C_n\ (b-\text{path})K_1^{b(K_m)}\right)=0.$

Proof.
1) Applying E.V.C., we get the result.
2) For case m= 0 (mod 4), applying Theorem 2.4 (n times), we get the result. For case m=3, using S.C.L. with E.V.C., we get the result. .
3) The proof is similar to the proof of case 2.
4) By using S.C.L. with E.V.C., we get the result.

Proposition 5.16.

1) $\eta\left(P_{2n}\ (b-\text{path})K_1^{b(P_m)}\right)=\begin{cases} 0, \text{if m is even},\\ n(b-1), \text{if m is odd}\end{cases}$

2) $\eta\left(P_{2n+1}\ (b-\text{path})K_1^{b(P_m)}\right)=\begin{cases} 1, \text{if m is even},\\ n(b-1), \text{if m is odd}\end{cases}$

3) $\eta\left(P_n\ (b-\text{path})K_1^{b(C_m)}\right)=\begin{cases} n(b-1), \text{if } m=0\ (\text{mod } 4)\ ,\\ 0,\ \ \text{if } m=3 \text{ and n is even},\\ 1, \text{if } m=3 \text{ and nis odd}.\end{cases}$

4) $\eta\left(P_n\ (b-\text{path})K_1^{b(K_{m,m})}\right)=2nb(m-1).$

5) $\eta\left(P_n\ (b-\text{path})K_1^{b(K_m)}\right)=0.$

Proof.
The proof is similar to the proof of the Proposition 5.15.

Proposition 5.17.

1) $\eta\left(K_{n,n}\ (b-\text{path})K_1^{b(P_m)}\right)=\begin{cases} 2n, \text{if m is odd},\\ 2n-2, \text{if m is even}.\end{cases}$

2) $\eta\left(K_{n,n}\ (b-\text{path})K_1^{b(C_m)}\right)=\begin{cases} 4nb-2, \text{if } m=0(\text{mod } 4),\\ 0, \text{if } m=3.\end{cases}$

3) $\eta\left(K_{n,n}\ (b-\text{path})K_1^{b(K_{m,m})}\right)$

$$=4nmb-4nb-2.$$

4) $\eta\left(K_{n,n}\ (b-\text{path})K_1^{b(K_m)}\right) = 2n-2.$

Proof.

The proof is similar to the proof of the Proposition 5.15.

Proposition 5.18.

1) $\eta\left(K_n\ (b-\text{path})K_1^{b(P_m)}\right) = \begin{cases} n(b-1), \text{if m is odd,} \\ 0, \text{if m is even.} \end{cases}$

2) $\eta\left(K_n\ (b-\text{path})K_1^{b(C_m)}\right) = \begin{cases} 2nb, \text{if } m = 0(\text{mod } 4), \\ 0, \text{if } m \neq 0(\text{mod } 4). \end{cases}$

3) $\eta\left(K_n(b-\text{path})K_1^{b(K_{m,m})}\right) = nb(2m-2).$

4) $\eta\left(K_n(b-\text{path})K_1^{b(K_m)}\right) = 0$

Proof.

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The proof is similar to the proof of the Proposition 5.15.

ل دور پلا ناباو یا گرافین یەك گرتی ژکومەلا رووتین جیٚکری ژ شیوازی(t-tuple) دگەل شیوازی (b-bridge)

پوختە:

د ڤی ڤەکولینیدا پلا ناباو یا گرافین یەك گرتی ژکومەلا رووتین جیٚکری ژ شیوازی(t-tuple) و پلا ناباو یا گرافین یەك گرتی ژکومەلا رووتین جیٚکری ژ شیوازی(t-tuple) دگەل شیوازی (b-bridge) هاتن هەژمارتن . د دوماهی دا پلا ناباو یا گرافین یەك گرتی ژکومەلا رووتین جیٚکری ژ شیوازی(t-tuple) د گەل شیوازی(b-path) و دگەل شیوازی (b-bridge) وپلا ناباو یا گرافین یەك گرتی ژکومەلا رووتین جیٚکری ژ شیوازی(b-path) هاتن هە ژمارتن.

حول البطلان للبيانات المتحدة للجذورالمعممة من النمط (t-tuple) مع النمط (b-bridge)

الملخص:

في هذا البحث تم احتساب البطلان للبيانات المتحدة للجذورالمعممة من النمط (t-tuple) و البطلان للبيانات المتحدة للجذورالمعممة من النمط (t-tuple) مع النمط (b-bridge). اخيراَ تم ايجاد البطلان للبيانات المتحدة للجذورالمعممة من النمط (t-tuple) مع النمط (b-path) و مع النمط (b-bridge) والبطلان للبيانات المتحدة للجذورالمعممة من النمط (b-path).