APPROXIMATE SOLUTIONS FOR A MODEL OF REACTION-DIFFUSION SYSTEM WITH SLOW REACTION AND FAST DIFFUSION

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(Accepted for publication: May 25, 2016)

Abstract:

In this paper, perturbation and finite difference methods are used to solve a reaction diffusion system. This system is modeled for describing the interaction between species in ecology. The interaction is interpreted as traveling wave solutions for both species under three types of initial conditions which describe some ecological cases. Types of traveling wave solutions are found and studied using numerical and approximate methods when there exists a small parameter $\lambda \ll 1$appear in one of the equation. The solutions of the two methods are compared and show a good agreement.

Keyword: *Reaction-Diffusion system, Perturbation method , Finite difference, Traveling wave solutions.*

Introduction

Reaction-diffusion models for the interaction of the species have been studied widely, the simplest of which is the Lotka-Volterra model(AL-Omari, and Gourley(2003), Gopalsamy (1982), Hosono (2003), Li (2008)), in which the interaction of the two species is to advantage for all, this interaction is called Mutualism (see, for example, Dean (1983), Priyanga (2004), Billingham(2001), Murray (2002), Hardler and Rothe (1975), Britton (1990), Gourley (2005), Graves et al. (2006)). Mutualism is defined as an interaction between species that is beneficial for both species. It plays the crucial role in promoting and even maintaining such species: plant and seed dispersal is one example. In this paper, we study a reaction-diffusion model for a system of two species which exhibits mutualistic population interactions, provided that the population is sufficiently small. The model we will study here is

$$\frac{\partial u}{\partial t} = D_u \frac{\partial^2 u}{\partial x^2} + k_u u(1 - m_u u^2 + n_u W),$$

$$\frac{\partial W}{\partial t} = D_w \frac{\partial^2 W}{\partial x^2} + k_w w(1 - m_w w^2 + n_w u), \qquad \dots(1)$$

where D_u and D_w are diffusion coefficients, $k_u u(1 - m_u u^2)$and $k_w w(1 - m_w w^2)$ are generalized logistic growth rates for the species u and w, the inter specific cooperation by $n_u w$ and $n_w u$. All parameters are positive. This is a natural extension of the Lotka-Volterra model, and although it is the simplest model of this type with a nonlinear growth rate ($(1 - m_u u^2)$ instead of$\left(1 - \frac{u}{k}\right)$). Note that when$n_u = n_w = 0$, (1) decouple, and each is equivalent to a generalized Fisher equation model studied in (8).

We define dimensionless variables

$$u = U\bar{u}, w = W\bar{w}, \qquad x = (\frac{D_u}{k_u})^{0.5}\bar{x}, \ \ t = \frac{\bar{t}}{k_u},$$

in terms of which (1) becomes

$$\frac{\partial \bar{u}}{\partial \bar{t}} = \frac{\partial^2 \bar{u}}{\partial \bar{x}^2} + \bar{u}(1 - \alpha_1 \bar{u}^2 + \gamma_1 \bar{w}),$$

$$\frac{\partial \bar{W}}{\partial \bar{t}} = \frac{D}{\lambda}\frac{\partial^2 \bar{w}}{\partial \bar{x}^2} + \lambda \bar{w}(1 - \alpha_2 \bar{w}^2 + \gamma_2 \bar{u}), \qquad \dots(2)$$

Here U and W are the unique single species equilibrium states given by the positive solutions of

$$1 - m_u U^2 = 0, \;\; 1 - m_w W^2 = 0$$

The dimensionless parameters are

$$\alpha_1 = \mathrm{m_u U}, \qquad \gamma_1 = \mathrm{n_u W}, \qquad \lambda = \frac{\mathrm{k_w}}{\mathrm{k_u}}, \qquad \mathrm{D} = \frac{\mathrm{D_w}}{\mathrm{D_u}}, \qquad \alpha_2 = \mathrm{m_w W}, \qquad \gamma_2 = \mathrm{n_w U}.$$

For notational convenience we will omit the overbars in what follows.

A similar system of equations was studied in Billingham (2004). In particular, Billingham (2004) examines a system that has the same evolution equation for u, but a simpler, linear equation for w. In spite of this, the dynamics of the system studied in Billingham (2004) are significantly more complex than those exhibited by (2) in the limit $\lambda \ll 1$. We shall see in this paper that there are many different types of traveling wave solution of (2), the propagation of these waves is steady. In this paper, we will study (2) in the same limit, $\lambda \ll 1$, and study the structure of the possible equilibrium states and traveling wave solutions that connect them.

For simplicity, we will consider initial conditions that are symmetric about the origin, so we need only consider the problem for $x \geq 0$ and $t \geq 0$, with

$$u(x,0) = u_0(x), \qquad w(x,0) = w_0(x),$$

and boundary conditions

$$\frac{\partial u(0,t)}{\partial x} = 0, \qquad \frac{\partial w(0,t)}{\partial x} = 0$$

We will consider initial conditions which represent ecological situation, namely

$$A) \qquad u_0(x) = \begin{cases} 1, & x \leq L_0 \\ 0, & x > L_0 \end{cases}$$
$$w_0(x) = 1,$$

where L_0 is the width of the step function. The far field boundary conditions are therefore $u \to 0$ and $w \to 1$ as $x \to \infty$. Species w is native, and the species u is introduced locally.

$$B) \qquad w_0(x) = \begin{cases} 1, & x \leq L_0 \\ 0, & x > L_0 \end{cases}$$
$$u_0(x) = 1,$$

The far field boundary conditions are therefore $u \to 1$ and w$\to 0$ as as $x \to \infty$. Species u is native, and the species w is introduced locally.

$$C) \qquad u_0(x) = \begin{cases} 1, & x \leq L_0 \\ 0, & x > L_0 \end{cases}$$
$$w_0(x) = \begin{cases} 1, & x \leq L_0 \\ 0, & x > L_0 \end{cases}$$

The far field boundary conditions are therefore $u \to 0$ and w$\to 0$ as $x \to \infty$. Both species are introduced locally.

We begin by studying the stability of spatially uniform solutions in section 2. In section 3 we solve some typical initial value problems numerically and determine which traveling waves develop. We study traveling wave solutions when $\lambda \ll 1$ in section 4 and conclude in section 5.

Spatially uniform solutions

Spatially uniform solutions of(2) satisfy

$$\frac{\partial u}{\partial t} = u(1 - \alpha_1 u^2 + \gamma_1 w),$$

$$\frac{\partial w}{\partial t} = \lambda w(1 - \alpha_2 w^2 + \gamma_2 u), \qquad \ldots.(3)$$

We focus on the coexistence equilibrium solution in (1,1), so in what follows we assume $\gamma_1 = \alpha_1 - 1$ and $\gamma_2 = \alpha_2 - 1$. There are three obvious equilibrium states of (3): $(0,0)$,$(u_0, 0)$ and $(0, w_0)$.

In order to study the stability of the equilibrium points, we linearize (3) at the equilibrium points, which can be done through finding the Jacobean matrix. So if we let

$f(u, w) = u(1 - \alpha_1 u^2 + \gamma_1 w)$ and$g(u, w) = \lambda w(1 - \alpha_2 w^2 + \gamma_2 u)$,then

$$J = \begin{pmatrix} \frac{\partial f(\hat{u},\hat{w})}{\partial u} & \frac{\partial f(\hat{u},\hat{w})}{\partial w} \\ \frac{\partial g(\hat{u},\hat{w})}{\partial u} & \frac{\partial g(\hat{u},\hat{w})}{\partial w} \end{pmatrix},$$

where $\hat{u}$and $\hat{w}$are equilibrium points. From the determinant of the Jacobean we can find the characteristic equation which is

$$K^2 + trJ + \det J = 0 \ldots..(4),$$

where tr and det denote the trace and determinant of Jacobean J, so that

$$trJ = \frac{\partial f(\hat{u}, \hat{w})}{\partial u} + \frac{\partial g(\hat{u}, \hat{w})}{\partial w}, \det J = \frac{\partial f(\hat{u}, \hat{w})}{\partial u}\frac{\partial g(\hat{u}, \hat{w})}{\partial w} - \frac{\partial f(\hat{u}, \hat{w})}{\partial w}\frac{\partial g(\hat{u}, \hat{w})}{\partial u}$$

The Eigenvalues can be found from the quadratic equation (4)

$$K_{1,2} = \frac{trJ \mp \sqrt{(trJ)^2 - 4detJ}}{2}$$

The Jacobeanhere is,

$$J = \begin{pmatrix} 1 - 3\alpha_1\hat{u}^2 + (\alpha_1 - 1)\hat{w} & (\alpha_1 - 1)\hat{u} \\ \lambda(\alpha_2 - 1)\hat{w} & \lambda(1 - 3\alpha_2 w^2 + (\alpha_2 - 1)\hat{u}) \end{pmatrix}$$

Also

$trJ = 1 - 3\alpha_1\hat{u}^2 + (\alpha_1 - 1)\hat{w} + \lambda(1 - 3\alpha_2\hat{w}^2 + (\alpha_2 - 1)\hat{u}), \det J = 1 - 3\alpha_1\hat{u}^2 + (\alpha_1 - 1)\hat{w} * \lambda(1 - 3\alpha_2\hat{w}^2 + (\alpha_2 - 1)\hat{u}) - (\alpha_1 - 1)\hat{u}\lambda(\alpha_2 - 1)\hat{w})$.

A local analysis at one of these equilibrium states, say the coexistence steady state(1,1), shows that the eigenvalues are

$$K_{12} = -\alpha_1 - \lambda\alpha_2 \pm \sqrt{\alpha_1{}^2 - \alpha_1\alpha_2\lambda - \alpha_1\lambda + \alpha_1{}^2\lambda^2 - \alpha_2\lambda + \lambda}$$

A simple plot of K_1andK_2, show that the equilibrium state (1,1), is always stable which can be seen in Figure(1) and Figure(2), and could be complex for small values of α_1. The extinction steady state(0,0), has the eigenvalues $K_1 = 1$ and $K_2 = \lambda$, and therefore is unstable (since λ is positive). The single species state $(u_0, 0)$is a saddle point which can be deduced from the following eigenvalues and the plot shown in Figure (3) and (4),

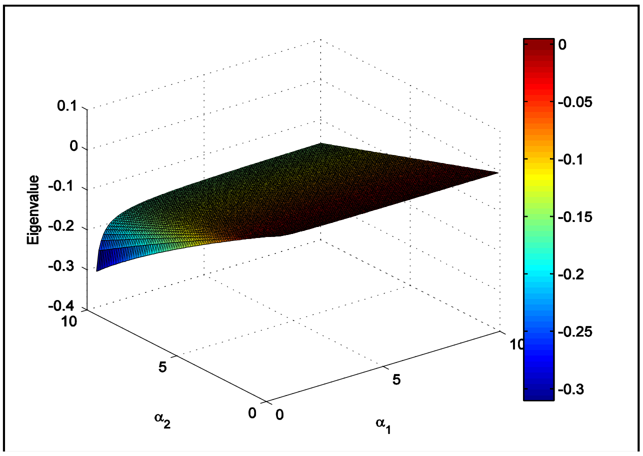

Figure 1: $\boldsymbol{Eigenvalues\ k_1\ of\ (1,1)}$

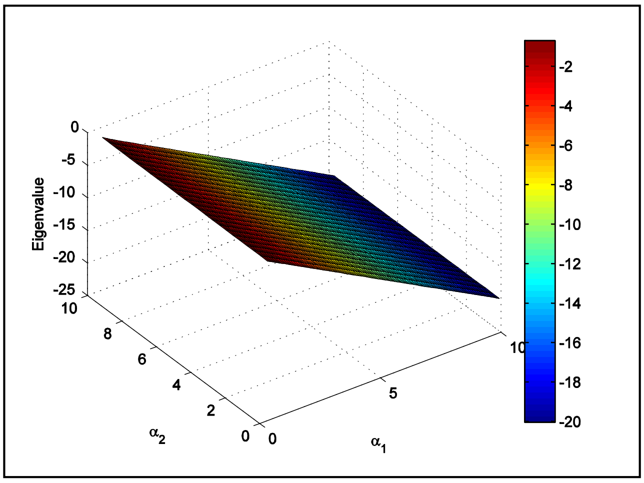

Figure 2: $\boldsymbol{Eigenvaluesk_2of(1,1)}$

$K_1 = -1 - 3\alpha_1 {u_0}^2,$

$K_2 = \lambda(1 - u_0 + \alpha_2 u_0).$

The single steady state $(0, w_0)$ is also a saddle point and similar to the other single species $(u_0, 0)$, this is clear from the eigenvalues

$K_1 = -\lambda - 3\lambda\alpha_2 {w_0}^2,$

$K_2 = 1 - w_0 + \alpha_1 w_0.$

The summery of the above analysis is shown in the following table,

Equilibrium state	Eigen values	Stability
(1,1)	$\alpha_1 - \lambda\alpha_2 \pm \sqrt{{\alpha_1}^2 - \alpha_1\alpha_2\lambda - \alpha_1\lambda + {\alpha_1}^2\lambda^2 - \alpha_2\lambda + \lambda}$	Stable node
(0,0)	$1, \lambda$	Unstable node
$(u_0, 0)$	$-1 - 3\alpha_1 {u_0}^2,\ \lambda(1 - u_0 + \alpha_2 u_0)$	Saddle point
$(0, w_0)$	$\lambda - 3\lambda\alpha_2 {w_0}^2, 1 - w_0 + \alpha_1 w_0$	Saddle point

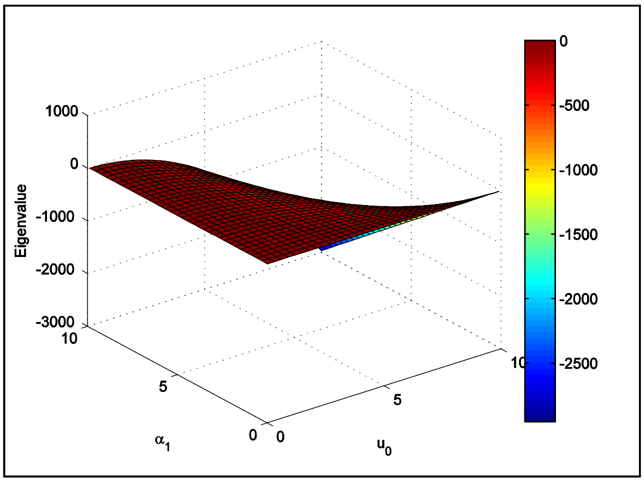

Figure 3: $\boldsymbol{Eigenvaluesk_1of(u_0,0)}$

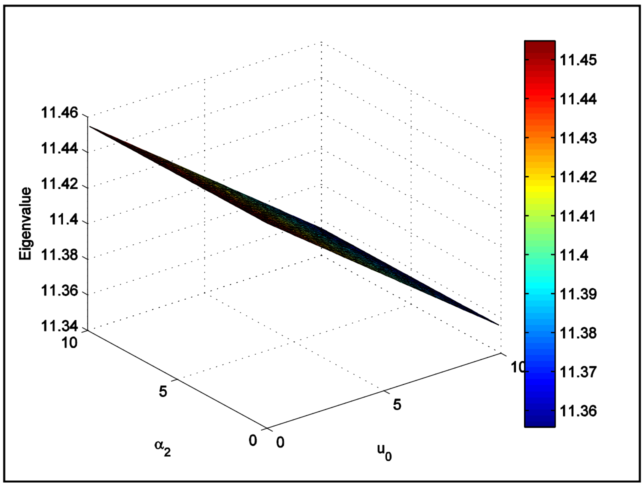

Figure 4: $\boldsymbol{Eigenvaluesk_2of(u_0,0)}$

Traveling wave solutions

Typical solutions of reaction-diffusion systems are traveling waves that connect equilibrium states. The three types of traveling wave that we study are:

Type (I_a), The traveling wave connects $(1,1)$ to $(u_0, 0)$.

Type (I_b), The traveling wave connects $(1,1)$ to $(0, w_0)$.

Type(I_c), The traveling wave connects $(1,1)$ to $(0,0)$.

We also add a subscript (II_a) or (II_b) to denote when there are two traveling waves connect each other.

These traveling wave solutions must connect a stable equilibrium solution to another equilibrium solution. Waves of type I can exist if the state$(1,1)$, is stable. In this paper we see only waves that exist when $(1,1)$is stable, because only these can be generated from the initial conditions that we study. An unstable uniform state $(1,1)$ will never exist when $t = 0$in any realistic initial value problem. Waves of type (II_a) and (II_b) require three steady states ($(1,1)$, $(u_0, 0)$ and $(0,0)$in type (II_a) and $(1,1)$, $(0, w_0)$ and $(0,0)$ in type II_b.

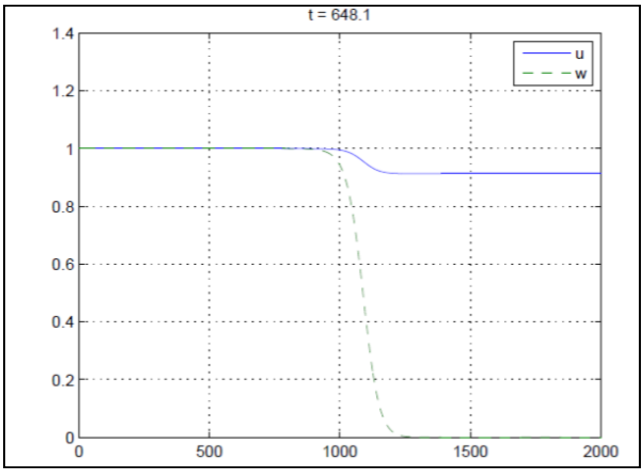

Figure 5: An overview of the types of traveling wave of type I_a.

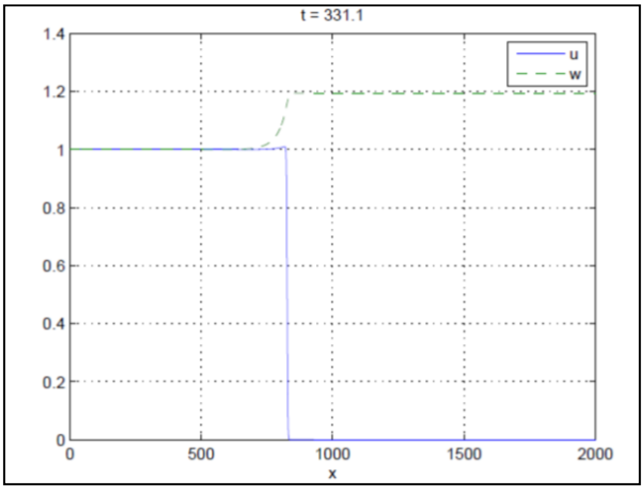

Figure 6: An overview of the types of traveling wave of type I_a.

Numerical solutions of the initial value problem

In order to get some idea of the dynamics of the system, we solve (2) numerically using a semi-implicit finite difference method (diffusion terms implicit, reaction terms explicit). The domain of solution is truncated to $0 \leq x \leq l$ with l large enough that the far field conditions are realized. We apply a Neumann boundary conditions on u and w. The domain is discretized using constant grid spacing, typically 0.1. When $\lambda \ll 1$, as we shall see, the solution sometimes develops over an $O(1)$ inner lengthscale at the wavefront and an $O(\lambda)^{-1}$ lengthscale elsewhere. Although it may well be more efficient to develop an adaptive method to capture these features of the solution, this is not the focus of this paper, so we have adopted an unsophisticated approach and accepted that our computation times may be quite long.

The numerical solutions show that at least one and sometimes two traveling waves are generated depending upon the initial conditions and choice of parameters.

Initial condition A

$$u_0(x) = \begin{cases} 1, & x \leq L_0 \\ 0, & x > L_0 \end{cases}$$

$w_0(x) = 1.$

We find that there are three qualitatively different types of behavior. For any positive value of the parameters, coexistence equilibrium state is stable node or stable focus, a simple traveling wave is generated, which connects (1,1) to $(u_0, 0)$, as shown in Figure (5). The value of parameters, $\alpha_1 = 1.2$, $\alpha_2 = 0.7$, $D = 1$ and $\lambda = 0.01$.

Initial condition B

$$w_0(x) = \begin{cases} 1, & x \le L_0 \\ 0, & x > L_0 \end{cases}$$

$u_0(x) = 1.$

With initial condition B, a simple traveling wave is generated, which connects $(1,1)$to$(0, w_0)$, as shown in Figure (5). The value of parameters,$\alpha_1 = 1.2,\ \alpha_2 = 0.7,\ D = 1.$

Notice that in all of these typical solutions we have takenand $\lambda = 0.05$. As we shall see in section 4, when $\lambda \ll 1$, traveling wave solutions develop on an $O(\lambda)^{-1}$lengthscale, with the exception of those that involve an equilibrium state with $u = 0$, in which there is an inner region at the wavefront where u changes on an $O(1)$lengthscale.

Initial condition C

$$u_0(x) = \begin{cases} 1, & x \le L_0 \\ 0, & x > L_0 \end{cases}$$

$$w_0(x) = \begin{cases} 1, & x \le L_0 \\ 0, & x > L_0 \end{cases}$$

In all cases, the state left behind the wave is determined by the initial conditions and the spatially-uniform system. The main difference from initial conditions A and B is that a traveling wave of type I_a and I_b is always generated, and propagates into the region where$u = w = 0$. For example, Figure (7), the parameters have the same values as in the waves of types I_a and I_b, but different initial condition. Waves of types$II_a$and $II_b$is also constructed when this type of initial condition exist. Figures (8-9), show the solution when the only difference is in the choice of D. In Figure (8), the speed of the wave that connects $(u_0, 0)$to $(0,0)$ is higher than that which connects $(0, w_0)$ to $(0,0)$, and vice versa for Figure(9).

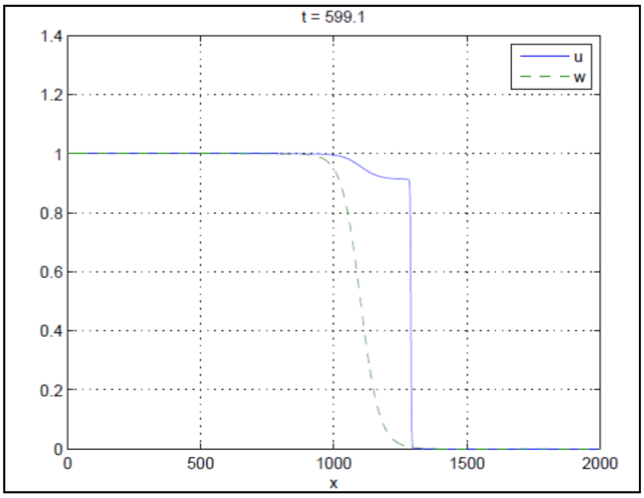

Figure 7: An overview of the types of traveling wave of type I_c.

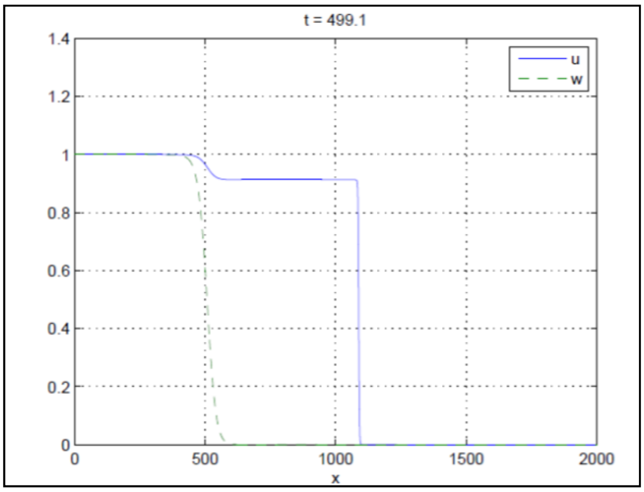

Figure 8: An overview of the types of traveling wave of type II_a.

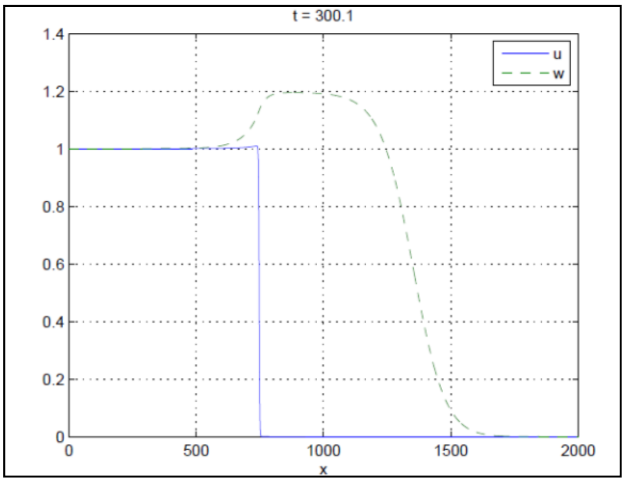

Figure 9: An overview of the types of traveling wave of type II_b.

Traveling wave solutions for $\lambda \ll 1$

In this section, we study the variety of traveling waves develop as solutions of the initial value problem, so we will study their structure, focusing on the analytically tractable case, $\lambda \ll 1$ (the second species diffuses faster and reproduces slower than the first species).

We define $z = x - ct$, and seek permanent form traveling wave solutions $u = \hat{u}z$ and $w = \hat{w}z$ with wave speed $c > 0$, so that (2) becomes

$$\frac{d^2\hat{u}}{dz^2} + c\,\frac{d\hat{u}}{dz} + \hat{u}(1 - \alpha_1\hat{u}^2 + \gamma_1\hat{w}) = 0,$$

$$\frac{D}{\lambda}\frac{d^2\hat{w}}{dz^2} + c\,\frac{d\hat{w}}{dz} + \lambda\hat{w}(1 - \alpha_2\hat{w}^2 + \gamma_2\hat{u}) = 0. \quad .....(5)$$

The appropriate boundary conditions depend upon which equilibrium states are connected by traveling wave solution, and we shall return to this question later.

This is a fourth order system of ordinary differential equations, which is difficult to study analytically. A limit where we can make some progress is$\lambda \ll 1$. The system is similar to that studied in Billingham (2004), where it was shown that the asymptotic structure of the solution consists of an inner region with lengthscale of $O(1)$at the wavefront, which we can place without loss of generality in the neighbourhood of $z = 0$, with outer solutions ahead of and behind the wavefront with lengthscale of $O(\lambda)^{-1}$. The inner region is only needed when one of the equilibrium states associated with the traveling wave has u=0, so, in contrast to the system studied in Billingham (2004), some traveling wave solutions can be described without the need to resort to the method of matched asymptotic expansions. We therefore begin by defining scaled outer variables as $Z = \lambda z$, $\hat{u} = U(z)$, $\hat{w} = W(z)$ with U, W, zof $O(1)$as $\lambda \to 0$

In terms of these new variables, (5) becomes

$$\lambda^2 \frac{d^2U}{dZ^2} + c\lambda \frac{dU}{dZ} + U(1 - \alpha_1 U^2 + \gamma_1 W) = 0,$$

$$D\frac{d^2W}{dz^2} + c\ \frac{dW}{dz} + W(1 - \alpha_2 W^2 + \gamma_2 U) = 0. \qquad ....(6)$$

Regular perturbation solutions

At leading order, provided that $U \nrightarrow 0$as $Z \to \mp\infty$, this is a regular perturbation problem, with the leading order equations

$$D\frac{d^2W}{dz^2} + c\ \frac{dW}{dz} + W(1 - \alpha_2 W^2 + \gamma_2 U) = 0,$$

$$W = \frac{\alpha_1 U^2 - 1}{\gamma_1},$$

or equivalently

$$\frac{dW}{dZ} = V,$$

$$\frac{dV}{dZ} = \frac{-c}{D} V - \frac{1}{D} W(1 - \alpha_2 W^2 + \gamma_2 U), \qquad .....(7)$$

$$W = \frac{\alpha_1 U^2 - 1}{\gamma_1}.$$

We use ode45 in Matlab to find traveling wave solutions of (7), which connect two equilibrium points of the system. In the (W, V) phase plane, this system has equilibrium points at (0,0), which corresponds to the steady state $U = u_0$, $W = 0$, and (1,0)where 1 is such that (1,1) is a coexistence equilibrium state (an intersection of the curves $\gamma_1 W = \alpha_1 U^2 - 1$ and $\gamma_2 U = \alpha_2 W^2 - 1$, as discussed in section 2). Possible traveling wave solutions with this structure therefore connect these two equilibria. We will focus on traveling wave solutions that satisfy $(W, V) \to (1,0)$ as $Z \to -\infty$ and $(W, V) \to (0,0)$as $Z \to \infty$.

By linearizing about $(1,0)$, we find that this is a saddle point and in each case we find that the stable coexistence equilibrium point corresponds to a saddle point in (7). If a traveling wave solution exists it is therefore represented by the unstable separatrix of $(1,0)$those pointsinto $Z < 0$.

Since $\gamma_2 > 0$,(0,0) is a stable node provided that$c^2 > 4D\ (1 + \gamma_2 U_0)$, and a stable focus for $c^2 < 4D\ (1 + \gamma_2 U_0)$. Since we require $W > 0$, this provides a lower bound, $c \geq c_{lb} \equiv 2\sqrt{D(1 - \gamma_2)}$, on the wavespeed. In this case we would expect a spectrum of wave speeds to exist, bounded below by some $c_{min} \geq c_{lb}$.

A comparison between numerical and asymptotic solutions are shown a good agreement as it can be seen in figures(11-12) for the value of parameters, $\alpha_1 = 1.2$, $\alpha_2 = 0.7$, $D = 1$and $\lambda = 0.01$.

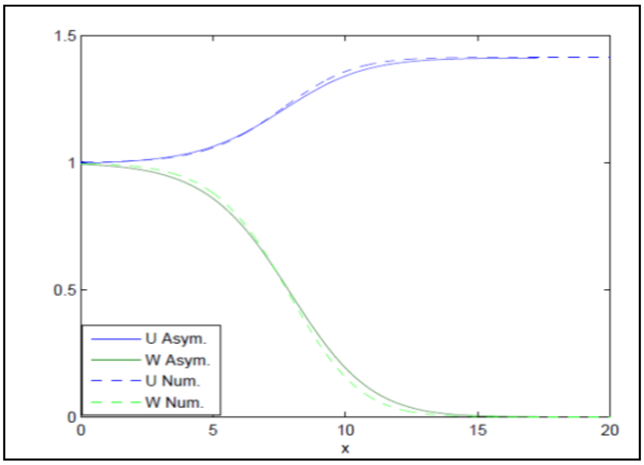

Figure10: Comparisonbetweennumericalandasymptoticsolutionsforaregularperturbationproblem.

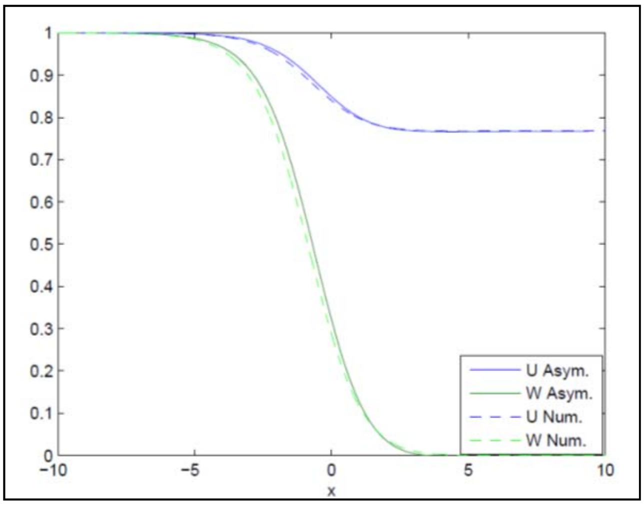

Figure11: **Comparisonbetweennumericalandasymptoticsolutionsforaregularper urbationproblem**.

Singular perturbation solutions

When one of the equilibrium states connected by the traveling wave solution has$U = 0$, we must solve a singular perturbation problem similar to that described in Billingham (2004). This is because the leading order problem in the outer region has, from (6),

$$\mathrm{U}(1 - \alpha_1 \mathrm{U}^2 + \gamma_1 \mathrm{W}).$$

The solution must smoothly connect a state with $U = 0$ to one with $1 - \alpha_1 \mathrm{U}^2 + \gamma_1 \mathrm{W} = 0$, so an inner asymptotic region is required. For example, if $U \to 1$ as $Z \to -\infty$ and $U \to 0$ as$Z \to \infty$, then for $Z < 0$, the solution must satisfy (7), while for $Z > 0, U \equiv 0$, (W satisfies)

$$D\frac{d^2\mathrm{W}}{dz^2} + c\,\frac{d\mathrm{W}}{dz} + \mathrm{W}(1 - \alpha_2 \mathrm{W}^2) = 0 \text{ ……… (8)}$$

In all cases, we need to solve either (8) for $Z > 0$ and (7) for $Z < 0$, or vice versa, subject to appropriate boundary conditions as $Z \to \pm\infty$ and satisfy the connection conditions that W and $\frac{d\mathrm{W}}{dz}$ should be continuous at $Z = 0$. We have seen that the whole solution of W can be found from (7) for $Z < 0$, without need to solve (8) for $Z > 0$which has a trivial solutions in the most of the cases. We can solve each system of differential or differential-algebraic equations in MATLAB, shooting from close to the equilibrium points towards $Z = 0$and use Newton's method to adjust the initial conditions to satisfy the connection conditions at $Z = 0$.

Inner solution

In the inner region, $Z = O(1)$ and $\hat{w}$is constant at leading order, with value $\hat{w}_0$ determined by matching with the outer solution. At leading order, (5) is therefore reduced to an ordinary differential equation for $\hat{u}$, namely

$$\frac{d^2\hat{u}}{d\hat{z}^2} + c\,\frac{d\hat{u}}{d\hat{z}} + \hat{u}(\mathrm{L} - \alpha_1\hat{u}^2) = 0$$

where $\mathrm{L} = 1 + \gamma_1\hat{w}_0$ subject to appropriate matching conditions as $\hat{z} \to \pm\infty$(depending on the type of traveling wave).

This system can be studied in the same way as in the previous section. For a given value of$c$, we can solve the outer problem and determine $\hat{w}_0$, and then $L$and c_m. A typical result is shown in Figure (12).Since the inner and outer wave speeds must be the same, the point of intersection of the two curves gives the speed of the wave that we expect to be generated in an initial value problem, either the traveling wave of minimum speed or the unique traveling wave solution.Figure(13) shows a good agreement between the asymptotic and numerical solutions.

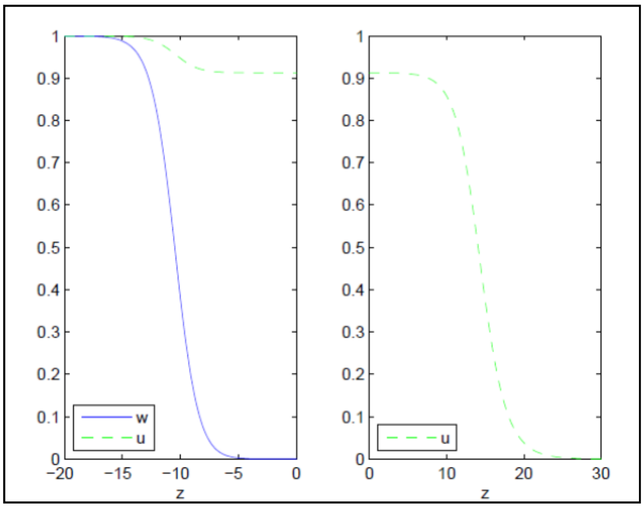

Figure 12: Traveling wave determined from the inner and outer solutions when $\alpha_1 = 1.2$, $\alpha_2 = 1.7$, $\lambda = 0.05$, and $\mathrm{D} = 1$.

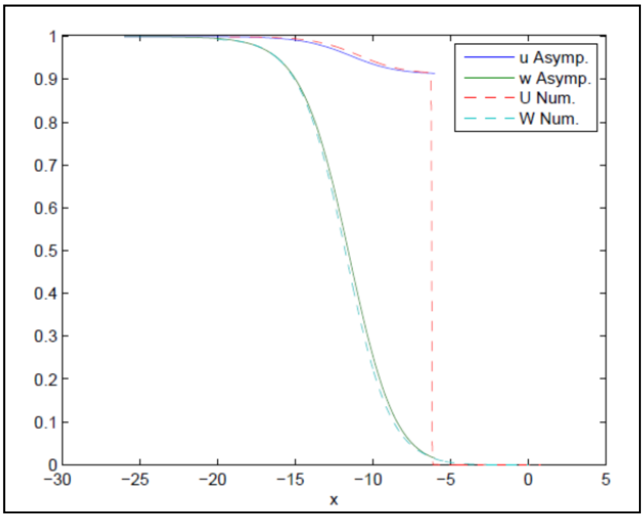

Figure 13: Comparison between numerical and asymptotic solutions for a singular perturbation problem.

Conclusions

In this paper we have studied the effect of cooperation on the types of equilibrium states and traveling waves that can exist in a two species reaction-diffusion system. We found that there can be anywhere from one coexistence equilibrium states in addition to the usual single species equilibria, and can always be stable. We also studied the dynamics of three ecologically-relevant initial value problems, and used asymptotic methods to study the traveling wave solutions that can emerge. We showed that, since the set of steady states has a richer structure than that of those in the Lotka-Volterra model, a wider range of traveling wave solutions is available, which in turn means that there is a wider range of possible outcomes in the wake of the final wavefront generated in an initial value problem. We also saw that, in the case of initial condition C, more than one traveling wave developed.

Future work could include a study of the stability of the traveling waves to lateral disturbances in two spatial dimensions, and of the existence and stability of spatially nonuniform states in finite domains.

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کورتی

د ڤێ ڤەکولینێدا رێکێن (نیزیك کرنا ب دوماهیك) هاتینە پێك ینان ژ بو شلوڤەکرنا سیستەمێ (Reaction diffusion) ئەوێن بێکڤە ژیانێ گرێددەت ل ئیك ژینکەهێ ئەڤ بێکڤە ژیانە دئیەتە پێکینان برەنگەکی شلوڤەکرنا ژ جورێ (گەشتکرن ب تایبەت بو باب و باپیرا) ل دیف سیبەرا سێ مەرجێن سەرەتایی ئەوێن هندەك دیاردا روندکەت ژ ژینگەهی. یێ هاتینە دیتن و بەراوردکرن دناڤ بەینا هەردووك رێکێن نەمرەیی لبن مەرجێ جودا جودا ژ پارامیتەرا.

الملخص

في هذا البحث تم استخدام طرق التقريبات المنتهية والمظطربة لحل منظومة معادلات(Reaction diffusion)والذي يمثل تعايش وتفاعل بين سلالتينن من بيئة واحدة. هذا التفاعل تترجم بشكل حلول راحلة للسلالتينفي ظل وجود ثلاثة انواع من الشروط الابتدائية والتي تفسر بعض الظواهر البيئية.تم ايجاد هذه الحلول ومقارنتها بين الطريفتين عند وجود معلمة صغبرة في هذه المنظومة.