Operation Approaches on P_s-Open Sets and its Separation Axioms

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Abstract:
The purpose of this paper is to introduce the concept of an operation on a class of P_S-open subsets of topological spaces. Using this operation, we define the concept of P_S^{γ} -open sets, and study some of their related topological properties. Furthermore, some separation axioms by utilizing the operation γ on $P_SO(X)$ and the set P_S^{γ} -open have been investigated.

KEYWORDS: P_S^{γ} -open sets, P_S^{γ} - separation axioms

1. Introductions and Preliminaries

Throughout this paper, the space (X, τ) (or simply X) always means a topological space on which no separation axioms are assumed unless explicitly stated. The concept of preopen sets and semiopen sets was introduced respectively by Mashhour, Abd El-Monsef and El-Deeb (1982) and Levine (1963). On the other hand, Kasahara (1979) defined the concept of an operation on τ and introduced the concept of α-closed graphs of functions. The operation α had been renamed as γ operation on τ by Ogata (1991). Recently, An, Cuong and Maki (2008) defined and investigated the concept of the mapping γ_p on the collection of all preopen $PO(X)$ subsets of (X, τ), and introduced the notion of pre γ_p-open sets and studied some of their properties.

Khalaf and Asaad (2009) defined the concept of P_S-open sets in a space X which is denoted by $P_SO(X)$. The aim of this paper is to introduce the concept of an operation γ on $P_SO(X)$ and to define the notion of P_S^{γ} -open sets of (X, τ) by using this operation. Section 3, contains some topological properties of P_S^{γ} -open sets and some relationships with other concepts. In the last section, some separation axioms by utilizing the operation γ on $P_SO(X)$ and the set P_S^{γ} -open are investigated.

For a subset A of a space X, the closure and the interior of A are denoted by $Cl(A)$ and $Int(A)$, respectively.

Definition 1.1: A subset A of a space X is said to be:

1. preopen if $A \subseteq Int(Cl(A))$ (Mashhour et al., 1982).
2. semiopen if $A \subseteq Cl(Int(A))$ (Levine, 1963).

The complement of a semiopen set is said to be semiclosed (Crossley and Hildebrand, 1971). A preopen subset A of a topological space (X, τ) is called P_S-open (Khalaf and Asaad, 2009) if for each $x \in A$, there exists a semiclosed set F such that $x \in F \subseteq A$. The complement of a P_S-open set is P_S-closed (Khalaf and Asaad, 2009), and the P_S-closure of a subset A of X is defined as the intersection of all P_S-closed sets containing A and it is denoted by $P_SCl(A)$ (Khalaf and Asaad, 2009). The P_S-interior of a subset A of X is defined as the union of all P_S-open sets contained in A and it is denoted by $P_SInt(A)$ (Khalaf and Asaad, 2009). An operation γ on the topology τ on X is a mapping $\gamma : \tau \rightarrow P(X)$ such that $U \subseteq U^{\gamma}$ for each $U \in \tau$, where $P(X)$ is the power set of X and U^{γ} denotes the value of γ at U (Ogata, 1991). A nonempty subset A of a topological space (X, τ) with an operation γ on τ is said to be γ-open if for each $x \in A$, there exists an open set U containing x such that $U^{\gamma} \subseteq A$ (Ogata, 1991).

Definition 1.2 (An et al., 2008)**:** An operation γ_p on $PO(X)$ is a mapping $\gamma_p : PO(X) \rightarrow P(X)$ such that $U \subseteq U^{\gamma p}$ for each $U \in PO(X)$, where $U^{\gamma p}$ denotes the value of γ_p at U. A nonempty set A of X is said to be pre γ_p-open if for each $x \in A$, there exists a preopen set U such that $x \in U$ and $U^{\gamma p} \subseteq A$.

Recall that a topological space (X, τ) is locally indiscrete (Dontchev, 1998) if every open subset of X is closed, or if every closed subset of X is open. Also, a topological space (X, τ) is called semi-T_1 (Maheshwari and Prasad, 1975) if for each pair of distinct points x, y in X, there exist two semiopen sets U and V such that $x \in U$ but $y \notin U$ and $y \in V$ but $x \notin V$.

Theorem 1.3 (Khalaf and Asaad, 2009)**:** Let (X, τ) be a topological space. Then the following statements are true:

1. If X is locally indiscrete, then $P_SO(X) = \tau$.
2. If X is semi-T_1, then $P_SO(X) = PO(X)$.

2. P_S^γ-OPEN SETS

Definition 2.1: An operation γ on $P_SO(X)$ is a mapping $\gamma : P_SO(X) \to P(X)$ such that $G \subseteq G^\gamma$ for every $G \in P_SO(X)$, where $P(X)$ is the power set of X and G^γ is the value of γ at G.

From this definition, we can easy to find $X^\gamma = X$ for any operation $\gamma : P_SO(X) \to P(X)$.

Definition 2.2: Let (X, τ) be a topological space and $\gamma : P_SO(X) \to P(X)$ be an operation on $P_SO(X)$. A nonempty set A of X is said to be P_S^γ-open if for each $x \in A$, there exists a P_S-open set G such that $x \in G$ and $G^\gamma \subseteq A$.

The complement of a P_S^γ-open set of X is P_S^γ-closed. Assume that the empty set φ is also P_S^γ-open set for any operation $\gamma : P_SO(X) \to P(X)$. The class of all P_S^γ-open subsets of a space (X, τ) is denoted by $P_S^\gamma O(X)$. For each $x \in X$, the class of all P_S^γ-open sets of (X, τ) containing a point x is denoted by $P_S^\gamma O(X, x)$.

It is obvious that every P_S^γ-open set is P_S-open (that is, $P_S^\gamma O(X) \subseteq P_SO(X)$), but the converse is not true as shown from the following example.

Example 2.3: Let $X = \{a, b, c\}$ and $\tau = \{\varphi, X, \{a\}, \{b, c\}\} = P_SO(X)$. Define an operation $\gamma : P_SO(X) \to P(X)$ as follows: For every $A \in P_SO(X)$

$$A^\gamma = \begin{cases} A & if \quad a \in A \\ X & if \quad a \notin A \end{cases}$$

Clearly, $P_S^\gamma O(X) = \{\varphi, X, \{a\}\}$. Then the set $\{b, c\}$ is P_S-open, but $\{b, c\} \notin P_S^\gamma O(X)$. Therefore, $P_SO(X) \not\subseteq P_S^\gamma O(X)$.

We can say that a subset A is P_S^{id}-open of X if and only if A is P_S-open in X. The identity operation id on $P_SO(X)$ is a mapping $id : P_SO(X) \to P(X)$ such that $H^{id} = H$ for every $H \in P_SO(X)$. Then $P_S^{id}O(X) = P_SO(X)$.

Theorem 2.4: Let (X, τ) be a topological space and $\gamma : P_SO(X) \to P(X)$ be an operation on $P_SO(X)$. Then the following are true:

1. The union of any class of P_S^γ-open sets in X is also a P_S^γ-open.
2. The intersection of any class of P_S^γ-closed sets in X is also a P_S^γ-closed.

Proof: (1) Let $x \in \bigcup_{\lambda \in \Lambda} \{A_\lambda\}$, where $A_\lambda \in P_S^\gamma O(X)$ for all λ. Then $x \in A_\lambda$ for some $\lambda \in \Lambda$. Since A_λ is P_S^γ-open set in X, then there exists a P_S-open set G such that $x \in G \subseteq G^\gamma \subseteq A_\lambda \subseteq \bigcup_{\lambda \in \Lambda} \{A_\lambda\}$. Therefore, $\bigcup_{\lambda \in \Lambda} \{A_\lambda\}$ is P_S^γ-open set in X.

(2) Similar to part (1) using complement.

Generally, the intersection (resp. the union) of any two P_S^γ-open (resp. P_S^γ-closed) sets in (X, τ) may not be a P_S^γ-open (resp. P_S^γ-closed) set as demonstrated in the following example.

Example 2.5: Consider the space $X = \{a, b, c\}$ and $\tau = P(X) = P_SO(X)$. Let $\gamma : P_SO(X) \to P(X)$ be an operation on $P_SO(X)$ defined as follows: For every $A \in P_SO(X)$

$$A^\gamma = \begin{cases} A & if \quad A \neq \{c\} \\ \{b, c\} & if \quad A = \{c\} \end{cases}$$

Obviously, $P_S^\gamma O(X) = P(X) - \{c\}$. Then the sets $\{a, c\}$ and $\{b, c\}$ are P_S^γ-open, but $\{a, c\} \bigcap \{b, c\} = \{c\}$ is not a P_S^γ-open set since $\{c\} \notin P_S^\gamma O(X)$. Also, $\{a\}$ and $\{b\}$ are P_S^γ-closed sets, but $\{a\} \bigcup \{b\} = \{a, b\}$ is not a P_S^γ-closed set.

Therefore, the class of all P_S^γ-open sets of any topological space (X, τ) need not be a topology on X in general.

Theorem 2.6: The following properties are true for any topological space (X, τ):

1. If (X, τ) is locally indiscrete, then the operations γ on $P_SO(X)$ and γ on τ are equivalent and hence the concept of P_S^γ-open set and γ-open set are identical.

2. If (X, τ) is semi-T_1, then the operations γ on $P_SO(X)$ and γ_p on $PO(X)$ are equivalent and hence the concept of P_S^γ-open set and pre γ_p-open set are identical.

Proof: (1) Follows from their definitions and Theorem 1.3 (1).
(2) Follows from their definitions and Theorem 1.3 (2).

Definition 2.7: Let (X, τ) be any topological space. An operation γ on $P_SO(X)$ is said to be P_S-regular if for each $x \in X$ and for every pair of P_S-open sets G_1 and G_2 such that both containing x, there exists a P_S-open set H containing x such that $H^\gamma \subseteq G_1^\gamma \cap G_2^\gamma$.

For example, the mapping $\gamma : P_SO(X) \to P(X)$ defined in Example 2.3 is P_S-regular operation.

Theorem 2.8: Let a mapping γ be P_S-regular operation on $P_SO(X)$. If the subsets A and B are P_S^γ-open in a topological space (X, τ), then $A \cap B$ is also P_S^γ-open set in (X, τ).

Proof: Suppose $x \in A \cap B$ for any P_S^γ-open subsets A and B in (X, τ) both containing x. Then there exist P_S-open sets G_1 and G_2 such that $x \in G_1 \subseteq A$ and $x \in G_2 \subseteq B$. Since γ is a P_S-regular operation on $P_SO(X)$, then there exists a P_S-open set H containing x such that $H^\gamma \subseteq G_1^\gamma \cap G_2^\gamma \subseteq A \cap B$. Therefore, $A \cap B$ is P_S^γ-open set in (X, τ).

By applying Theorem 2.8, it is easy to show that $P_S^\gamma O(X)$ forms a topology on X for any P_S-regular operation γ on $P_SO(X)$.

Definition 2.9: A topological space (X, τ) with an operation γ on $P_SO(X)$ is said to be P_S^γ-regular if for each $x \in X$ and for each P_S-open set G containing x, there exists a P_S-open set H containing x such that $H^\gamma \subseteq G$.

Theorem 2.10: Let (X, τ) be a topological space and $\gamma : P_SO(X) \to P(X)$ be an operation on $P_SO(X)$. Then the following conditions are equivalent:

1. $P_SO(X) = P_S^\gamma O(X)$.
2. (X, τ) is a P_S^γ-regular space.
3. For every $x \in X$ and for every P_S-open set G of (X, τ) containing x, there exists a P_S^γ-open set H of (X, τ) containing x such that $H \subseteq G$.

Proof: (1) ⇒ (2) Let $x \in X$ and G be a P_S-open set in X such that $x \in G$. It follows from assumption that G is a P_S^γ-open set. This implies that there exists a P_S-open set H such that $x \in H$ and $H^\gamma \subseteq G$. Therefore, the space (X, τ) is P_S^γ-regular.

(2) ⇒ (3) Let $x \in X$ and G be a P_S-open set in (X, τ) containing x. Then by (2), there is a P_S-open set H such that $x \in H \subseteq H^\gamma \subseteq G$. Again, by using (2) for the set H, it is shown that H is P_S^γ-open. Hence H is a P_S^γ-open set containing x such that $H \subseteq G$.

(3) ⇒ (1) By applying the part (3) and Theorem 2.4 (1), it follows that every P_S-open set of X is P_S^γ-open in X. That is, $P_SO(X) \subseteq P_S^\gamma O(X)$. Since every P_S^γ-open set is P_S-open (that is, $P_S^\gamma O(X) \subseteq P_SO(X)$ in general). Hence $P_SO(X) = P_S^\gamma O(X)$.

3. TOPOLOGICAL PROPERTIES OF P_S^γ-OPEN SETS

We begin with the definition of P_S^γ-limit point of a set A.

Definition 3.1: Let A be any subset of a topological space (X, τ) and γ be an operation on $P_SO(X)$. A point $x \in X$ is said to be P_S^γ-limit point of A if for every P_S^γ-open set G containing x, $G \cap (A - \{x\}) \neq \varphi$. The set of all P_S^γ-limit points of A is called a P_S^γ-derived set of A and it is denoted by $P_S^\gamma D(A)$.

Some basic properties of P_S^γ-derived set are mentioned in the following theorem.

Theorem 3.2: The following properties hold for any sets A and B in a topological space (X, τ) with an operation γ on $P_SO(X)$.

1. $P_S^\gamma D(\varphi) = \varphi$.
2. If $A \subseteq B$, then $P_S^\gamma D(A) \subseteq P_S^\gamma D(B)$.
3. $P_S^\gamma D(A \cap B) \subseteq P_S^\gamma D(A) \cap P_S^\gamma D(B)$.
4. $P_S^\gamma D(A \cup B) \supseteq P_S^\gamma D(A) \cup P_S^\gamma D(B)$.
5. $P_S^\gamma D(P_S^\gamma D(A)) - A \subseteq P_S^\gamma D(A)$.
6. $P_S^\gamma D(A \cup P_S^\gamma D(A)) \subseteq A \cup P_S^\gamma D(A)$.

Proof: It is enough to proof the parts (5) and (6) since the proofs of the other parts are obvious and hence they are omitted.

(5) Let $x \in P_S^\gamma D(P_S^\gamma D(A)) - A$ and G is a P_S^γ-open set containing x. Then $G \cap (P_S^\gamma D(A)) - \{x\} \neq \varphi$. Let $y \in G \cap (P_S^\gamma D(A)) - \{x\})$. Then, since $y \in P_S^\gamma D(A)$ and $y \in G$, $G \cap (A - \{y\}) \neq \varphi$. Let $z \in G \cap (A - \{y\})$. Hence $z \neq x$ for $z \in A$ and $x \notin A$. So $G \cap (A - \{x\}) \neq \varphi$. Therefore, $x \in P_S^\gamma D(A)$.

(6) Let $x \in P_S^\gamma D(A \cup P_S^\gamma D(A))$. If $x \in A$, the result is clear. So, let $x \in P_S^\gamma D(A \cup P_S^\gamma D(A)) - A$. Then, for P_S^γ-open set G containing x, $G \cap (A \cup P_S^\gamma D(A)) - \{x\} \neq \varphi$. Thus, $G \cap (A - \{x\}) \neq \varphi$ or $G \cap (P_S^\gamma D(A) - \{x\}) \neq \varphi$. Now, it follows similarly from the part (5) that $G \cap (A - \{x\}) \neq \varphi$. Hence, $x \in P_S^\gamma D(A)$. Therefore, in any case, $P_S^\gamma D(A \cup P_S^\gamma D(A)) \subseteq A \cup P_S^\gamma D(A)$.

Definition 3.3: Let A be any subset of a topological space (X, τ) and γ be an operation on $P_SO(X)$. The P_S^γ-closure of A is defined as the intersection of all P_S^γ-closed sets of X containing A and it is denoted by $P_S^\gamma Cl(A)$. That is, $P_S^\gamma Cl(A) = \cap \{F : A \subseteq F, X - F \text{ is } P_S^\gamma\text{-open set in } X\}$.

Some important properties of P_S^γ-closure operator will be given in the next lemma.

Lemma 3.4: The following statements are true for any subsets A and B of a topological space (X, τ) with an operation γ on $P_SO(X)$.

1. $P_S^\gamma Cl(\varphi) = \varphi$ and $P_S^\gamma Cl(X) = X$.
2. $P_S^\gamma Cl(A)$ is the smallest P_S^γ-closed set containing A.
3. $P_S^\gamma Cl(A)$ is P_S^γ-closed set in X.
4. $A \subseteq P_SCl(A) \subseteq P_S^\gamma Cl(A)$.
5. A is P_S^γ-closed if and only if $P_S^\gamma Cl(A) = A$.
6. If $P_S^\gamma Cl(A) \cap P_S^\gamma Cl(B) = \varphi$, then $A \cap B = \varphi$.
7. If $A \subseteq B$, then $P_S^\gamma Cl(A) \subseteq P_S^\gamma Cl(B)$.
8. $P_S^\gamma Cl(A \cap B) \subseteq P_S^\gamma Cl(A) \cap P_S^\gamma Cl(B)$.
9. $P_S^\gamma Cl(A) \cup P_S^\gamma Cl(B) \subseteq P_S^\gamma Cl(A \cup B)$.
10. $P_S^\gamma Cl(P_S^\gamma Cl(A)) = P_S^\gamma Cl(A)$.

Proof: Straightforward.

Theorem 3.5: Let A be any subset of a topological space (X, τ) and γ be an operation on $P_SO(X)$. Then $x \in P_S^\gamma Cl(A)$ if and only if $A \cap G \neq \varphi$ for every P_S^γ-open set G of X containing x.

Proof: Let $x \in P_S^\gamma Cl(A)$ and let $A \cap G = \varphi$ for some P_S^γ-open set G of X containing x. Then $A \subseteq X - G$ and $X - G$ is P_S^γ-closed set in X. So $P_S^\gamma Cl(A) \subseteq X - G$. Thus, $x \in X - G$. This is a contradiction. Hence $A \cap G \neq \varphi$ for every P_S^γ-open set G of X containing x.

Conversely, suppose that $x \notin P_S^\gamma Cl(A)$. So there exists a P_S^γ-closed set F such that $A \subseteq F$ and $x \notin F$. Then $X - F$ is a P_S^γ-open set such that $x \in X - F$ and $A \cap (X - F) = \varphi$. Contradiction of hypothesis. Therefore, $x \in P_S^\gamma Cl(A)$.

From Definition 3.1 and Theorem 3.5, we have the following corollary.

Corollary 3.6: Let A be any subset of a topological space (X, τ) and γ be an operation on $P_SO(X)$. Then $P_S^\gamma D(A) \subseteq P_S^\gamma Cl(A)$.

Theorem 3.7: Let A be any subset of a topological space (X, τ) and γ be an operation on $P_SO(X)$. Then A is P_S^γ-closed if and only if A contains the set of its P_S^γ-limit points.

Proof: Suppose that A is P_S^γ-closed subset of a space (X, τ) and let $x \notin A$, then $x \in X - A$ and $X - A$ is P_S^γ-open set in X such that $A \cap X - A = \varphi$. This means that $x \notin P_S^\gamma D(A)$. Hence $P_S^\gamma D(A) \subseteq A$.

Conversely, assume that A contains the set of its P_S^γ-limit points (That is, $P_S^\gamma D(A) \subseteq A$). To show that A is P_S^γ-closed (or $X - A$ is P_S^γ-open) set in X. Let $x \in X - A$, then $x \notin A$. By assumption that there exists a P_S^{γ}-open set U_x of X containing x such that $A \bigcap U_x = \varphi$. That is, $U_x \subseteq X - A$ and hence $X - A = \bigcup_{x \in (X - A)} U_x$. So $X - A$ is a union of P_S^{γ}-open sets and by Theorem 2.4 (1), $X - A$ is P_S^{γ}-open. Consequently, A is P_S^{γ}-closed set in X.

Lemma 3.8: For any subset A of a topological space (X, τ) with an operation γ on $P_SO(X)$. The set $A \bigcup P_S^{\gamma}D(A)$ is P_S^{γ}-closed in X.

Proof: Let $x \notin A \bigcup P_S^{\gamma}D(A)$. Then $x \notin A$ and $x \notin P_S^{\gamma}D(A)$. Since $x \notin P_S^{\gamma}D(A)$, there exists a P_S^{γ}-open set $G_x \subseteq X$ containing x which contains no point of A other than x but $x \notin A$. So G_x contains no point of A, which implies $G_x \subseteq X - A$. Again, G_x is a P_S^{γ}-open set of each of its points. But as G_x does not contain any point of A, no point of G_x can be a P_S^{γ}-limit point of A. Therefore, no point of G_x can belong to $P_S^{\gamma}D(A)$. This implies that $G_x \subseteq X - P_S^{\gamma}D(A)$. Hence, it follows that $x \in G_x \subseteq X - A \bigcap X - P_S^{\gamma}D(A) = X - (A \bigcup P_S^{\gamma}D(A))$. Therefore, $A \bigcup P_S^{\gamma}D(A)$ is P_S^{γ}-closed.

P_S^{γ}-limit points provide us with an easy means to find the P_S^{γ}-closure of a set A.

Theorem 3.9: Let (X, τ) be a topological space and γ be an operation on $P_SO(X)$. Then $P_S^{\gamma}Cl(A) = A \bigcup P_S^{\gamma}D(A)$ for any subset A of a space X.

Proof: Since $A \subseteq A \bigcup P_S^{\gamma}D(A)$. Then by Lemma 3.8, $P_S^{\gamma}Cl(A) \subseteq A \bigcup P_S^{\gamma}D(A)$.

On the other hand, since $A \subseteq P_S^{\gamma}Cl(A)$ in general and by Corollary 3.6, $P_S^{\gamma}D(A) \subseteq P_S^{\gamma}Cl(A)$. So we have $A \bigcup P_S^{\gamma}D(A) \subseteq P_S^{\gamma}Cl(A)$. Therefore, in both cases, we obtain that $P_S^{\gamma}Cl(A) = A \bigcup P_S^{\gamma}D(A)$.

Theorem 3.10: For any subsets A, B of a topological space (X, τ). If γ is a P_S-regular operation on $P_SO(X)$, then $P_S^{\gamma}Cl(A) \bigcup P_S^{\gamma}Cl(B) = P_S^{\gamma}Cl(A \bigcup B)$.

Proof: It is enough to proof that $P_S^{\gamma}Cl(A \bigcup B) \subseteq P_S^{\gamma}Cl(A) \bigcup P_S^{\gamma}Cl(B)$ since the other part follows directly from Lemma 3.4 (9). Let $x \notin P_S^{\gamma}Cl(A) \bigcup P_S^{\gamma}Cl(B)$. Then there exist two P_S^{γ}-open sets G and H containing x such that $A \bigcap G = \varphi$ and $B \bigcap H = \varphi$. Since γ is a P_S-regular operation on $P_SO(X)$, then by Theorem 2.8, $G \bigcap H$ is P_S^{γ}-open in X such that $(G \bigcap H) \bigcap (A \bigcup B) = \varphi$. Therefore, we have $x \notin P_S^{\gamma}Cl(A \bigcup B)$ and hence $P_S^{\gamma}Cl(A \bigcup B) \subseteq P_S^{\gamma}Cl(A) \bigcup P_S^{\gamma}Cl(B)$.

Definition 3.11: A subset N of a topological space (X, τ) is called a P_S^{γ}-neighbourhood of a point $x \in X$, if there exists a P_S^{γ}-open set G in X such that $x \in G \subseteq N$.

The following is a relation between P_S^{γ}-open set and P_S^{γ}-neighbourhood of a point $x \in X$.

Theorem 3.12: Let $U \subseteq (X, \tau)$ be a P_S^{γ}-open if and only if it is a P_S^{γ}-neighbourhood of each of its points.

Proof: Let U be any P_S^{γ}-open set in (X, τ). Then by Definition 3.11, it is clear that U is a P_S^{γ}-neighbourhood of each of its points, since for every $x \in U$, $x \in U \subseteq U$ and $U \in P_S^{\gamma}O(X)$.

Conversely, suppose U is a P_S^{γ}-neighbourhood of each of its points. Then for each $x \in U$, there exists a P_S^{γ}-open set V_x containing x such that $V_x \subseteq U$. Then $U = \bigcup_{x \in U} V_x$. Since each V_x is P_S^{γ}-open. It follows from Theorem 2.4 (1) that U is P_S^{γ}-open set in X.

Definition 3.13: Let A be any subset of a topological space (X, τ) and γ be an operation on $P_SO(X)$. The P_S^{γ}-interior of A is defined as the union of all P_S^{γ}-open sets of X contained in A and it is denoted by $P_S^{\gamma}Int(A)$. That is, $P_S^{\gamma}Int(A) = \bigcup \{G : G$ is a P_S^{γ}-open set in X and $G \subseteq A\}$.

Some important properties of P_S^γ-interior operator will be given in Lemma 3.14.

Lemma 3.14: Let A and B be subset of a topological space (X, τ) and γ be an operation on $P_SO(X)$. Then the following conditions hold:

1. $P_S^\gamma Int(\varphi) = \varphi$ and $P_S^\gamma Int(X) = X$.
2. $P_S^\gamma Int(A)$ is the largest P_S^γ-open set contained in A.
3. $P_S^\gamma Int(A)$ is P_S^γ-open set in X.
4. $P_S^\gamma Int(A) \subseteq P_SInt(A) \subseteq A$.
5. A is P_S^γ-open if and only if $P_S^\gamma Int(A) = A$.
6. If $A \cap B = \varphi$, then $P_S^\gamma Int(A) \cap P_S^\gamma Int(B) = \varphi$.
7. If $A \subseteq B$, then $P_S^\gamma Int(A) \subseteq P_S^\gamma Int(B)$.
8. $P_S^\gamma Int(A \cap B) \subseteq P_S^\gamma Int(A) \cap P_S^\gamma Int(B)$.
9. $P_S^\gamma Int(A) \cup P_S^\gamma Int(B) \subseteq P_S^\gamma Int(A \cup B)$.
10. $P_S^\gamma Int(P_S^\gamma Int(A)) = P_S^\gamma Int(A)$.
11. $P_S^\gamma Int(X - A) = X - P_S^\gamma Cl(A)$.

Proof: Straightforward.

P_S^γ-limit points provide us to find the P_S^γ-interior of a set A.

Theorem 3.15: Let (X, τ) be a topological space and γ be an operation on $P_SO(X)$. Let A be a subset of a space X. Then $P_S^\gamma Int(A) = A - P_S^\gamma D(X - A)$.

Proof: Let $x \in A - P_S^\gamma D(X - A)$, then $x \notin P_S^\gamma D(X - A)$ and hence there exists a P_S^γ-open set G_x containing x such that $G_x \cap (X - A) = \varphi$. That is, $x \in G_x \subseteq A$ and hence $A = \bigcup_{x \in A} G_x$. So A is a union of P_S^γ-open sets and hence by Theorem 2.4 (1), A is P_S^γ-open set in X containing x. Then by Lemma 3.14 (5), $x \in P_S^\gamma Int(A)$. Thus, $A - P_S^\gamma D(X - A) \subseteq P_S^\gamma Int(A)$.

On the other hand, if $x \in P_S^\gamma Int(A) \subseteq A$, then $x \notin P_S^\gamma D(X - A)$ since $P_S^\gamma Int(A)$ is P_S^γ-open set and $P_S^\gamma Int(A) \cap (X - A) = \varphi$. So $x \in A - P_S^\gamma D(X - A)$. This implies that $P_S^\gamma Int(A) \subseteq A - P_S^\gamma D(X - A)$. Therefore, in both cases, we obtain that $P_S^\gamma Int(A) = A - P_S^\gamma D(X - A)$.

Theorem 3.16: If γ is a P_S-regular operation on $P_SO(X)$, then for any subsets A, B of a space X, we have $P_S^\gamma Int(A) \cap P_S^\gamma Int(B) = P_S^\gamma Int(A \cap B)$.

Proof: Follows directly from Theorem 3.10 and using Lemma 3.14 (11).

Lemma 3.17: Let (X, τ) be a topological space and γ be a P_S-regular operation on $P_SO(X)$. Then $P_S^\gamma Cl(A) \cap G \subseteq P_S^\gamma Cl(A \cap G)$ holds for every P_S^γ-open set G and every subset A of X.

Proof: Suppose that $x \in P_S^\gamma Cl(A) \cap G$ for every P_S^γ-open set G, then $x \in P_S^\gamma Cl(A)$ and $x \in G$. Let U be any P_S^γ-open set of X containing x. Since γ is P_S-regular on $P_SO(X)$. So by Theorem 2.8, $G \cap U$ is P_S^γ-open set containing x. Since $x \in P_S^\gamma Cl(A)$, then by Theorem 3.5, we have $A \cap (G \cap U) \neq \varphi$. This means that $(A \cap G) \cap U \neq \varphi$. Therefore, again by Theorem 3.5, we obtain that $x \in P_S^\gamma Cl(A \cap G)$. Thus, $P_S^\gamma Cl(A) \cap G \subseteq P_S^\gamma Cl(A \cap G)$.

The proof of the following lemma is similar to Lemma 3.17 and using Lemma 3.14 (11).

Lemma 3.18: Let (X, τ) be a topological space and γ be a P_S-regular operation on $P_SO(X)$. Then $P_S^\gamma Int(A \cup F) \subseteq P_S^\gamma Int(A) \cup F$ holds for every P_S^γ-closed set F and every subset A of X.

4. P_S^γ-T_n SPACES FOR $n \in \{0, 1, 2\}$ IN TERMS OF P_S^γ-OPEN SETS

In this section, we introduce some types of P_S^γ- separation axioms called P_S^γ-T_n for $n \in \{0, 1, 2\}$ using P_S^γ-open set. Some basic properties of these spaces are investigated.

Definition 4.1: A topological space (X, τ) with an operation γ on $P_SO(X)$ is called P_S^γ-T_0 if for each pair of distinct points x, y in X, there exists a P_S^γ-open set G containing one of the points but not the other.

Definition 4.2: A topological space (X, τ) with an operation γ on $P_SO(X)$ is called P_S^γ-T_1 if for each pair of distinct points x, y in X, there exist two P_S^γ-open sets G and H such that $x \in G$ but $y \notin G$ and $y \in H$ but $x \notin H$.

Definition 4.3: A topological space (X, τ) with an operation γ on $P_SO(X)$ is called P_S^γ-T_2 if for each pair of distinct points x, y in X, there exist disjoint P_S^γ-open sets G and H containing x and y respectively.

Lemma 4.4: If (X, τ) is P_S^γ-T_n space, then (X, τ) is P_S^γ-T_{n-1} for $n = 1, 2$.

Proof: Obvious.

Observe that the converse of Lemma 4.4 is not true as shown by the following example.

Example 4.5: Suppose $X = \{a, b, c\}$ and τ be the discrete topology on X. Define an operation γ on $P_SO(X)$ as follows:

1. For every $A \in P_S^\gamma O(X)$

$$A^\gamma = \begin{cases} A & if \quad c \in A \\ X & if \quad c \notin A \end{cases}$$

Then the space (X, τ) is P_S^γ-T_0, but it is not P_S^γ-T_1.

2. For every $B \in P_S^\gamma O(X)$

$$B^\gamma = \begin{cases} B & if \quad B = \{a, b\} \ or \ \{a, c\} \ or \ \{b, c\} \\ X & otherwise \end{cases}$$

Then (X, τ) is P_S^γ-T_1 space, but (X, τ) is not P_S^γ-T_2.

Theorem 4.6: Let (X, τ) be a topological space and γ be an operation on $P_SO(X)$. Then the following properties are equivalent:

1. X is P_S^γ-T_2.
2. If $x \in X$, then there exists a P_S^γ-open set G containing x such that $y \notin P_S^\gamma Cl(G)$ for each $y \in X$ such that $x \neq y$.
3. For each $x \in X$, $\bigcap\{P_S^\gamma Cl(G): G \in P_S^\gamma O(X, x)\} = \{x\}$.

Proof: (1) ⇒ (2) Let X be any P_S^γ-T_2 space. For each $x, y \in X$ with $x \neq y$, then there exist two P_S^γ-open sets G and H containing x and y respectively such that $G \cap H = \varphi$. This implies that $G \subseteq X - H$ and hence $P_S^\gamma Cl(\{G\}) \subseteq X - H$ since $X - H$ is P_S^γ-closed set in X and $y \notin X - H$. Therefore, $y \notin P_S^\gamma Cl(G)$.

(2) ⇒ (3) Obvious.

(3) ⇒ (1) Let $x, y \in X$ with $x \neq y$. By hypothesis there exists a P_S^γ-open set G containing x such that $y \notin G$ and hence $y \notin P_S^\gamma Cl(G)$. Then $y \in X - P_S^\gamma Cl(G)$ and $X - P_S^\gamma Cl(G)$ is P_S^γ-open set. So $G \cap X - P_S^\gamma Cl(G) = \varphi$. Therefore, X is P_S^γ-T_2 space.

Theorem 4.7: For a topological space (X, τ) with an operation γ on $P_SO(X)$. Then the following conditions are true:

1. X is P_S^γ-T_1 if and only if every singleton set in X is P_S^γ-closed.
2. X is P_S^γ-T_0 if and only if $P_S^\gamma Cl(\{x\}) \neq P_S^\gamma Cl(\{y\})$, for every pair of distinct points x, y of X.

Proof: (1) Necessary Part. Suppose the space X be P_S^γ-T_1. Let $x \in X$. Then for any point $y \in X$ such that $x \neq y$, there exists a P_S^γ-open set G such that $y \in G$ but $x \notin G$. Thus, $y \in G \subseteq X - \{x\}$. This implies that $X - \{x\} = \bigcup\{G: y \in X - \{x\}\}$. Since the union of P_S^γ-open sets is P_S^γ-open. Then $X - \{x\}$ is P_S^γ-open set in X. Hence $\{x\}$ is P_S^γ-closed set in X.

Sufficient Part. Suppose every singleton set in X is P_S^γ-closed. Let $x, y \in X$ such that $x \neq y$. This implies that $x \in X - \{y\}$. By hypothesis, we get $X - \{y\}$ is a P_S^γ-open set contains x but not y. Similarly $X - \{x\}$ is a P_S^γ-open set contains y but not x. Therefore, a space X is P_S^γ-T_1.

(2) Necessary Part. Let X be a P_S^γ-T_0 space and x, y be any two distinct points of X. Then there exists a P_S^γ-open set G containing x or y (say x, but not y). So $X - G$ is a P_S^γ-closed set, which does not contain x, but contains y. Since $P_S^\gamma Cl(\{y\})$ is the smallest P_S^γ-closed set containing y, $P_S^\gamma Cl(\{y\}) \subseteq X - G$, and so $x \notin$

$P_S^{\gamma}Cl(\{y\})$. Therefore, $P_S^{\gamma}Cl(\{x\}) \neq P_S^{\gamma}Cl(\{y\})$.

Sufficient Part. Suppose for any $x, y \in X$ with $x \neq y$, $P_S^{\gamma}Cl(\{x\}) \neq P_S^{\gamma}Cl(\{y\})$. Now, let $z \in X$ such that $z \in P_S^{\gamma}Cl(\{x\})$, but $z \notin P_S^{\gamma}Cl(\{y\})$. Now, we claim that $x \in P_S^{\gamma}Cl(\{y\})$. For, if $x \in P_S^{\gamma}Cl(\{y\})$, then $\{x\} \subseteq P_S^{\gamma}Cl(\{y\})$, which implies that $P_S^{\gamma}Cl(\{x\}) \subseteq P_S^{\gamma}Cl(\{y\})$. This is contradiction to the fact that $z \notin P_S^{\gamma}Cl(\{y\})$. Hence x belongs to the P_S^{γ}-open set $X - P_S^{\gamma}Cl(\{y\})$ to which y does not belong. It gives that X is P_S^{γ}-T_0 space.

See Definition 4.8 in (An et al., 2008) introduced the concept of pre γ_p-$T_n^{'}$ spaces, where $n \in \{0, 1, 2\}$ and $\gamma_p : PO(X) \rightarrow P(X)$ is an operation on $PO(X)$, using the notion of pre γ_p-open sets. So by applying Theorem 2.6 (2), we obtain the following lemma.

Lemma 4.8: Let (X, τ) be a semi-T_1 space. Then (X, τ) is P_S^{γ}-T_n if and only if it is pre γ_p-$T_n^{'}$, where $n \in \{0, 1, 2\}$, γ is an operation on $P_SO(X)$ and γ_p is an operation on $PO(X)$.

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کورتیا لێکولینێ:

ئارمانج ژ ڤێ ڤەکولینێ تێگەهێن کریار لسەر جورێ کومێن ڤەکری ژ جورێ P_S ل ڤالاهییێن توبولوجی دا کوما ئەم بدەنە نیاسین. ب کارئینانا ڤێ کریارێ، مە تێگەهێن ژکومێن ڤەکری ژ جورێ P_S^{γ} دانە نیاسین، ومە هندەك ژتایبەتمەندییێن توبولوجی یێ گرێدای پێڤە دانە خواندن. زێدەباری ڤێ چەندێ هندەك بەلگەنەڤیتێن ژێکڤەکری ب کارئینانا کریار γ لسەر $P_SO(X)$ وکومێن ڤەکری ژ جورێ P_S^{γ} هاتنە ڤەکولینکرن.

الملخص:

الهدف من هذا البحث هو تقديم مفهوم العملية على صنف من المجوعات الجزئية المفتوحة من النمط P_S في الفضاءات التبولوجية. باستخدام هذه العملية، عرفنا مفهوم المجوعات المفتوحة من النمط P_S^{γ} ، ودرسنا بعض الخصائص التبولوجية المتعلقة بها. إضافة الى ذلك تم التحقق من بعض بديهيات الفصل باستخدام العملية γ على $P_SO(X)$ والمجموعة المفتوحة من النمط P_S^{γ} .