On Some Types of Functions in Nonstandard Analysis

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Abstract:

In this paper, by using some nonstandard concepts given by Robinson and axiomatized by Nelson we study the behavior of functions defined on a discrete intervals, whose points are of infinitesimal distances. This study leads to introduce and define some new types of functions in nonstandard analysis and we get some nonstandard results for different nonstandard values (infinitesimals, infinitely close, unlimited …).

Keywords: *Nonstandard Analysis, Limited Function, Continuity, s-Continuity, g-Continuity.*

1-Introduction:

The following definitions and notations of nonstandard analysis will be needed in this paper, which can be found in (Hamad, 2008; Diener and Diener, 1995; Nelson, 1977; Davis, 1977).

Every set or elements defined in a classical mathematics is called **standard**. A real number x is called **unlimited** if $|x| > r$ for all standard positive real number r, otherwise it is called **limited**. A real number x is called **infinitesimal** if $|x| < r$ for all standard positive real numbers r. A real number x is called **appreciable** in case x is limited not infinitesimal. Two real numbers x and y are said to be **infinitely close** if $x - y$ is infinitesimal and is denoted by $x \simeq y$. If x is a real number, then the set of all numbers y which are infinitely close to x is called the **monad** of x, and is denoted by $m(x)$, that is $m(x) = \{y \in R: x \simeq y\}$, and $\alpha - m(x) = \left\{y \in R: \frac{y-x}{\alpha} \simeq 0\right\}$. If x is a real number, then the set of all numbers y such that $x - y$ is limited is called the **galaxy** of x and denoted by $gal(x)$, that is $gal(x) = \{y \in R: y - x \; limited\}$, and $\alpha - gal(x) = \left\{y \in R: \frac{y-x}{\alpha} \; limited\right\}$. If x is a limited real number in R^*, then it is infinitely close to a unique standard real number, this unique number is called the **standard part** of x and is denoted by $st(x)$. Any set or formula which does not involve a new predicates "standard, infinitesimals, limited, unlimited…etc" is called **internal**, otherwise is called **external**. A standard real valued function f is continuous at a standard point y if for all x, $x \simeq y$ then $f(x) \simeq f(y)$. A real valued function f is called s-continuous at y if for all $x, x \simeq y$ then $f(x) \simeq f(y)$. If $f: X \to Y$ is an internal function, then we say that f is a galaxy continuous at a point y if for all $x \in m(y)$ then $f(x) \in gal(f(y))$, and is denoted by g-continuous. A standard real valued function is **bounded** or **limited** if there exists a standard real number k such that $|f(x)| \leq k$.

Remark 1.1(Hamad, 2008):

1. From definitions of monad and galaxy, we have: $m(x) \subset gal(x)$, for all $x \in R$.

2. From definitions of continuous and s-continuous, we have for all $x, x \in m(y)$, then $f(x) \in m(f(y))$.

2. Some New Types of Nonstandard Functions

With nonstandard analysis the region of tangible elements is larger than that of standard analysis the problems that deal with unusual elements take its frame space in nonstandard analysis, so the study of the behavior of a function and its properties in nonstandard analysis give us a real phase and precision out comes, which can never be imagined classically.

According to this fact and the use of nonstandard tools, we introduce some new nonstandard types of functions.

Definition 2.1: Let $A \subseteq R$. A function $f: A \to R$ is said to be of type:

1. LC on A if $f(x)$ is linear and continuous for all $x \in A$, and denoted by $f \in$ LC.

2. LSC on A if $f(x)$ is linear and s-continuous for all $x \in A$, and denoted by $f \in$ LSC.

3. LGC on A if $f(x)$ is linear and g-continuous for all $x \in A$, and denoted by $f \in$ LGC.

4. SCLLP on A if $f(x)$ is s-continuous and limited at every limited point $x \in A$, and denoted by $f \in$ SCLLP.

5. GCLLP on A if $f(x)$ is g-continuous and limited at every limited point $x \in A$, and denoted by $f \in$ GCLLP.

Remark 2.2: In a classical mathematics, every linear function from usual matric space in to usual matric space is continuous while in nonstandard analysis, for s-continuous, this fact may not be true in general. For example:

Let $f(x) = \omega x + 1$ .Where ω is unlimited real number such that $\omega = \frac{1}{\varepsilon}$ for all $\varepsilon \simeq 0$. Take $x = 2 + \varepsilon$ and $y = 2$ its clear that $x \simeq y$.

But $f(x) = w(2+\varepsilon)+1$
$= 2\omega + \omega\varepsilon + 1$
$= 2\omega + 2 \not\simeq f(y) = 2\omega + 1.$
Hence f is not s-continuous.

Corollary 2.3 (David 2008)**:** If $x \simeq y$, and $u \simeq v$ then $x + u \simeq y + v$. If x and u are finite, then $xu \simeq yv$.

Theorem 2.4 (Robinson, 1970)**:** The standard real number c is the limit of $f(\mathrm{x})$ in R as x approachs to a standard point x_0, $\mathrm{a} < \mathrm{x}_0 < b$, it is necessary and sufficient that $f(\mathrm{x}) \simeq f(\mathrm{x}_0)$for all $\mathrm{x} \neq \mathrm{x}_0$ such that $\mathrm{x} \simeq \mathrm{x}_0$.

Theorem 2.5 (Robinson, 1970)**:** $f(x)$ is bounded at x_0 if and only if $f(x)$ is finite (limited) in the monad of x_0 in $\boldsymbol{R}$.

Corollary 2.6 (Hamad, 2008)**:** Every limited function is uniform g-continuous.

Lemma 2.7 (Robinson Lemma) (Robinson, 1970)**:** If $\{\mathrm{a_n}\}_{\mathrm{n}\in\mathrm{N}}$ is an internal sequence of real numbers such that $\mathrm{a_n} \simeq 0$ for $\mathrm{n} \in \mathrm{N}$,then there exists unlimited $\omega \in \mathrm{N}$ such that $\mathrm{a_n} \simeq 0\ \ \forall \mathrm{n} \leq \omega$.

Theorem 2.8 (Cauchy Principle Lemma) (Goldblatt, 1998)**:** If P is any internal property and if P(n) holds for all standard n, then there exists an unlimited $\omega \in \mathrm{N}$ such that $P(n)$ is holds for all $n \leq \omega$.

3. Main results:

Recall that the function f defined by $f\colon \mathrm{A} \to R$ for all $\mathrm{A} \subseteq R$.

Theorem 3.1: The following statements are direct consequence of the Definition 2.1.

1. Every standard linear function is of type LC.
2. Every infinitesimal function is of type SCLLP.
3. The identity function is of type LC, LSC, and SCLLP.
4. The constant function is of type LC, LSC, and SCLLP.
5. Every limited linear function is of type LC.
6. Any standard linear function defined on $[a,b]$ with the transformation $x \to \varepsilon x$ is of type LC.

Proof:

1. Let f be linear. We have to prove that f is continuous. Since every standard linear function is continuous at every standard real number, then by definition of continuity and Definition 2.1(1), we get that f is of type LC.
2. Let $f\colon \mathrm{A} \to R$ be infinitesimal function. Then f is limited and $f(x) \simeq 0$ for all $x \in \mathrm{A}$. Since every infinitesimal is continuous and s-continuous, then by Definition 2.1(4), we get that f is of type SCLLP.
3. Since every identity function is linear and both continuous and s-continuous function, then f is of types LC and LSC, and since every identity function is limited at every limited point, then f is of type SCLLP.
4. Put $x = k$, then the result can be obtained as a special case of (3).
5. Is obvious.
6. Suppose that f is standard linear function defines on $[a,b]$. Since every standard linear function is continuous, then f is continuous on $[a,b]$ and classically every continuous function defined on a closed interval is limited, then by (5) above, we get the result.

Corollary 3.2: Let f be a standard polynomial in R, then the transformation $x \to x\,\varepsilon$ changes f in to SCLLP type.

Proof: Let f be a standard polynomial in R then the transformation $x \to x\,\varepsilon$ for $\varepsilon \simeq 0$, changes f in to infinitesimal polynomial. Hence by Theorem 3.1(2), we get the result.

Lemma 3.3: Let f be a standard linear function. If $f \in$ SCLLP, then $f \in$ LSC and $f \in$ LC.

Proof: Let f be linear and of type SCLLP, then f is s-continuous and limited at every limited point. Since every s-continuous is continuous and by hypothesis, f is linear and hence f is of both types LSC and LC.

Remark 3.4:

1- Every standard linear function is continuous at every real number.
2- Every standard linear function is limited at every limited number.
3- Every limited function is uniformly g-continuous (Hamad, 2008).
4- In nonstandard analysis limited means bounded and classically every function has a limit at a point c means that f is bounded at c (Robinson, 1970).
5- Since infinitesimal is limited, then if f is s-continuous then f is g-continuous, but the convers is not true in general (Hamad, 2008) such as shown in the following example.

***Example*:**

Let $(x) = \frac{\mathrm{x}}{\varepsilon}$, where $\varepsilon \simeq 0$ .Take $x = \varepsilon$ and $y = 0$, then it is clear that $x \simeq y$ because $\varepsilon \simeq 0$ but $f(\mathrm{x}) \not\simeq f(\mathrm{y})$ because $f(x) = f(\varepsilon) = 1 \neq 0 = f(0)$, then $f(\mathrm{x}) - f(\mathrm{y}) = 1 - 0{=}1$, which is limited. Thus f is g-continuous, but not s-continuous.

6- The infinitesimal functions may be not linear, as shown in the following example.

***Example*:**

Let $f(x) = \varepsilon x^2 + \frac{\sin x}{\omega}$, where ω is unlimited and ε is infinitesimal, then $f(x) \simeq 0$ for all $x \in \mathrm{A}$, but $f(x)$ is not linear. Thus infinitesimal function is not of type LC and LSC. But by theorem 3.1(2) is of type SCLLP.

Lemma 3.5: If f is of type LC, then f is g-continuous.

Proof: The proof is obvious.

Remark 3.6: The converse of the above lemma is not true, for example:

Let $f(x) = \begin{cases} 3 & x \in Q \\ 0 & x \notin Q \end{cases}$

1- If $x \in Q$, then $x + \frac{\sqrt{2}}{x} \notin Q$ and for x is unlimited, we have $x \simeq x + \frac{\sqrt{2}}{x}$. Then $f(x) = 3 \neq 0 = f(x + \frac{\sqrt{2}}{x})$. That is $f(x) \not\simeq f(x + \frac{\sqrt{2}}{x})$.

2- If $x \notin Q$, then there exist $x_n \in Q$ such that $x_n \to x$, then $f(x_n) = 3 \nrightarrow f(x) = 0$.

Now, from 1 and 2 above, we get that f is not continuous, but it is g- continuous. Thus f is g-continuous. Hence f is not of type LC.

Theorem 3.7: Let f be a linear function and $f(x) \in \varepsilon - gal(f(y))$ whenever $x \simeq y$ for infinitesimal number ε. Then f is g-continuous if and only if $f \in$ LC.

Proof: Suppose that f is linear and g-continuous. Let $x \simeq y$. By definition of g-continuous, we have $f(x) \in gal(y)$. That is $f(x) - f(y)$ is limited. Since $x \simeq y$, then by hypothesis, we have $f(x) \in \varepsilon - gal(f(y))$ for $\varepsilon \simeq 0$. Then $\frac{f(x)-f(y)}{\varepsilon}$ is limited. That is $f(x) - f(y) = k\,\varepsilon$ for some standard k. Therefore, $f(x) - f(y) \simeq 0$. Hence $f(x) \simeq f(y)$ for standard f, x, y. Thus f is continuous and given f is linear, then f is of type LC.

Conversely; suppose that f is of type LC , then by Lemma 3.5, we get the result.

Theorem 3.8: Let f be a linear function with standard coefficients and $f(x) \in \varepsilon - gal(f(y))$ for all x, y such that $x \simeq y$ and ε is infinitesimal number .Then f is g-continuous if and only if $f \in$ SCLLP.

Proof: Suppose that f is a linear function. Let $x \simeq y$. Then by definition of g-continuous we have $f(x) \in gal(f(y))$. That is $f(x) - f(y)$ is limited. Since $f(x) \in \varepsilon - gal(f(y))$ for $x \simeq 0$, then $\frac{f(x)-f(y)}{\varepsilon}$ is limited. That is $f(x) - f(y) = k\,\varepsilon$ for some standard k. Thus $f(x) - f(y)$ is infinitesimal. Therefore, $(x) - f(y) \simeq 0$. Then $f(x) \simeq f(y)$ for all x, y. Hence f is s-continuous. Since the coefficient of f are standard, then f is limited at every limited point. Therefore, $f \in$ SCLLP.

The proof of the conversely part is obvious.

Theorem 3.9: Let f: [a ,b] $\to R$ be a standard continuous function and $h: R \to R$ be s-continuous function such that $h(x) \leq f(x)$ for all $x \in [a, b]$. Then h is of type SCLLP.

Proof: Suppose that f is a standard continuous function. Since every standard continuous function defined on a standard closed interval is limited, then f is limited for all $x \in [a, b]$. By hypothesis $h(x) \leq f(x)$ for all $x \in [a, b]$, then h is limited on $[a, b]$. Since h is s-continuous, then h is of type SCLLP.

Theorem 3.10: Let f: A $\to R$ for all A $\subseteq R$. Then f is of type SCLLP if and only if f is limited at every limited point and $f(x) \in \alpha - m(f(y))$ for all $x, y; x \simeq y$, where α is not infinitesimal.

Proof:

Let f be of type SCLLP and limited at every limited point. Then it is enough to prove that $f(x) \in \alpha - m(f(y))$ for non infinitesimal α. Now, let $x \simeq y$. We have to prove that $\frac{f(x)-f(y)}{\alpha} \simeq 0$. Since f is of type SCLLP, then f is s-continuous. Therefore, $f(x) - f(y)$ and $\frac{f(x)-f(y)}{\alpha}$ are infinitesimals, because α is not infinitesimal. Then either α is limited or unlimited and in both cases, we have $\frac{f(x)-f(y)}{\alpha} \simeq 0$ that is $f(x) \in \alpha - m(f(y))$.

Conversely; suppose that f is limited at every limited point and $f(x) \in \alpha - m(f(y))$ whenever $x \simeq y$ for all non infinitesimal α. To prove that f is of type SCLLP, it is enough to prove that f is s-continuous. Let $x \simeq y$, since $f(x) \in \alpha - m(f(y))$, therefore $\frac{f(x)-f(y)}{\alpha} \simeq 0$. Since α is not infinitesimal, then $f(x) - f(y) \simeq 0$, that is $f(x) \simeq f(y)$. Therefore, f is s-continuous. Hence f is of type SCLLP.

Theorem 3.11: If f is of type SCLLP and $f \simeq g$, then g is of type SCLLP.

Proof: Let f be of type SCLLP then f is s-continuous and limited at every limited point. Since f is s-continuous then $f(x) \simeq f(y)$ for all $x \simeq y$. Since $f \simeq g$, then for all $x, y \in$ R, $g(x) \simeq f(x) \simeq f(y) \simeq g(y)$. Thus for all $x, y \in$ R, $g(x) \simeq g(y)$. Hence g is s-continuous. To prove g is limited at every limited point. By contradiction if not then there exist a limited point β such that $g(\beta)$ is not limited. Since $f(x) \simeq g(x)$ for all x, then $f(\beta)$ is not limited which is contradiction since $f \in$ SCLLP.

Theorem 3.12: If f and g are of type SCLLP, then:

(1) $f + g$, **(2)** $f.g$ and **(3)** f^n, where n is limited positive integer, are of type SCLLP.

Proof:

(1) Suppose that f and g are of type SCLLP. Let $x \simeq y$. Since both f and g are s-continuous, then $f(x) \simeq f(y)$ and $g(x) \simeq g(y)$. By Corollary 2.3, we get that $f(x) + g(x) \simeq f(y) + g(y)$. That is $(f + g)(x) \simeq (f + g)(y)$. Hence $f + g$ is s-continuous. Now, since f and g are limited at every limited point then f and g are limited at a unique standard point, using Theorem 2.4, it is necessary

and sufficient for f and g that $f(x) \simeq f(x_0)$ and $g(x) \simeq g(x_0)$ for all $x \neq x_0$ such that $x \simeq x_0$.
Again by Corollary 2.7 we get $f(x) + g(x) \simeq f(x_0) + g(x_0)$ for all $x \neq x_0$ such that $x \simeq x_0$. That is $(f+g)(x) \simeq (f+g)(x_0)$ for all $x \neq x_0$ such that $x \simeq x_0$. Hence $f+g$ is limited at every limited point. Therefore, $f+g$ is s-continuous and limited at every limited point. Thus $f+g$ is of type SCLLP.

(2) Suppose that f and g are of type SCLLP, then f and g are s-continuous and limited at every limited point. By Theorem 2.5, f and g are finite. Since f and g are s-continuous, then $f(x) \simeq f(y)$ and $g(x) \simeq g(y)$. By Corollary 2.3, $f(x).g(x) \simeq f(y).g(y)$. That is $(f.g)(x) \simeq (f.g)(y)$. Hence $f.g$ is s-continuous. To prove that $f.g$ is limited at every limited point. Using Theorem 2.4 it is necessary and sufficient for f and g that $f(x) \simeq f(x_0)$ and $g(x) \simeq g(x_0)$ for all $x \neq x_0$ such that $x \simeq x_0$. Since f and g are finite by Corollary 2.3 we get $f(x).g(x) \simeq f(x_0).g(x_0)$ for all $x \neq x_0$ such that $x \simeq x_0$. That is $(f.g)(x) \simeq (f.g)(x_0)$ for all $x \neq x_0$ such that $x \simeq x_0$. Hence $f.g$ is limited at every limited point. Therefore, $f.g$ is s-continuous and limited at every limited point. Thus $f.g$ is of type SCLLP.

(3) Suppose that f is of type SCLLP, to prove that f^n is of type SCLLP. By mathematical induction:

Step 1: For $n = 1$. Since f is of type SCLLP, then the statement is true.

Step 2: Suppose the statement is true for $n-1$. That is f^{n-1} is of type SCLLP.

Step 3: To prove that the statement is true for all n ,we have $f^n = f^{n-1}.f$. From step (2) f^{n-1} is of type SCLLP and by hypothesis f is of type SCLLP, then by part (2) of this theorem, we get that f^n is of type SCLLP.

Remark 3.13: Let A and B be two nonempty subset of R. If f is of type LC, LSC, SCLLP on A and $B \subset A$, then f is of type LC, LSC, SCLLP on B.

Corollary 3.14: If f and g are of type LSC , then $f+g$ is of type LSC.

Proof: Since the sum of two linear functions is linear and by Corollary 2.3 the sum of two s-continuous functions is also s-continuous, then $f+g$ is linear and s-continuous. Thus $f+g$ is of type LSC.

Remark 3.15:

(1) If $\{f_n\}$ is a finite family of functions of type LSC, then $\sum_{n=1}^{k} f_n$ is of type LSC, where k is standard.

(2) On contrast of addition the multiplication operation is not closed for LSC type. That is f^n , $f.g$ and $\frac{f}{g}$ are not of type LSC, where $f, g \in$ LSC, and n is standard.

In general if $\{f_n\}$ is a finite family of functions of type LSC, then $\prod_{n=1}^{k} f_n$ is not of type LSC, where k is standard.

Lemma 3.16: Let $\{f_n\}$ be a finite family of functions of type LSC, then $\sum_{n=1}^{k} f_n$ for $k \leq \omega$, where ω is unlimited, is of type LSC.

Proof: For every standard $n \in N$, we have $\sum_{n=1}^{k} f_n$ is of type LSC. Then by Robinson Lemma there exist unlimited $\omega \in N$ such that $\sum_{n=1}^{\omega} f_n$ is of type LSC for all $n \leq \omega$.

Theorem 3.17: If f and g are of type LSC then $f o g$ is of type LSC.

Proof: Suppose that f and g are of type LSC. Since the composition of any two linear and s-continuous function is also linear and s-continuous function. Then $f o g$ is linear and s-continuous function. Hence $f o g$ is of type LSC.

Corollary 3.18: Let $\{f_n\}$ be a finite family of functions of type LSC then $O(f_n)$, where $O(f_n)$ is denoted to the multi composition between the functions f_n for $n \leq \omega$, ω is unlimited, is of type LSC.

Proof: The proof follows from Theorem 3.17 by using Theorem 2.8 (Cauchy Principle Lemma).

Corollary 3.19: Let $\{f_n\}$ be a finite family of functions of type LSC, then $O(f_n)$, where $O(f_n)$ is denoted to the multi composition between the functions f_n for any n is of type LSC.

Proof: The proof is follows from Theorem 3.17 and Corollary 3.18 by using mathematical induction.

Theorem 3.20: If $\{f_n\}$ is a finite family of functions of type SCLLP, then $\sum_{n=1}^{k} f_n$ and $\prod_{n=1}^{k} f_n$ are of type SCLLP.

Proof:

Since the finite sum and multiplication of s-continuous functions are s-continuous and the sum and multiplication of finite function which is limited at every limited point remains limited at every limited point, hence the results.

Corollary 3.21: Let $\{f_n\}$ be a finite family of functions of type SCLLP, then $\sum_{n=1}^{k} f_n$ and $\prod_{n=1}^{k} f_n$ for $n \leq \omega$, where ω is unlimited, are of type SCLLP.

Proof: The proof is similar to that of Lemma 3.16.

Theorem 3.22: If f and g are of type SCLLP, then $f o g$ is of type SCLLP.

Proof: Suppose that f and g are of type SCLLP. Since the composition of any two s-continuous function is also s-continuous function, then $f o g$ is s-continuous function. Again since the composition of two functions which are limited at every limited point is limited at every limited point, then $f o g$ is of type SCLLP.

The following statement are direct results of applying definitions of LSC and SCLLP on g-continuous.

Theorem 3.23:

(1) Every infinitesimal function is of type GCLLP.
(2) Every identity function is of type LGC.
(3) Every constant function is of type LGC.
(4) Let f be a standard polynomial in R. Then the transformation $x \to \varepsilon x$ changes f to be of type GCLLP.
(5) Every limited linear function is of type LGC.
(6) If f is of type LSC then f is of type LGC.

Proof:

(1) Let $f\colon A \to R$ be infinitesimal function. Since every s-continuous function is g-continuous, then by Theorem 3.1 part (2), we get that f is of type GCLLP.
(2) Obvious.
(3) Put $x = k$, then the result can be obtained as a special case of (2).
(4) Follows from corollary 3.2 and since every s-continuous is g-continuous. Hence f is of type GCLLP.
(5) Let f be limited linear function, to prove $f \in$ LGC. By Corollary 2.5 f is uniform g-continuous and since every uniform g-continuous is g-continuous, then f is g-continuous and given f is a linear function. Hence f is of type LGC.
(6) Obvious.

Lemma 3.24: Let f be a standard linear function .Then f is g-continuous if and only if f is of type GCLLP.

Proof: Suppose that f is g-continuous, since every standard linear function is limited at every limited point. Therefore, f is of type GCLLP.

Conversely; obvious.

Lemma 3.25: Let f:[a ,b] $\to R$ be a standard continuous function and $h\colon R \to R$ be s-continuous function such that $h(x) \le f(x)$ for all $x \in [a, b]$. Then h is of type GCLLP.

Proof: By Theorem 3.9 we have h is of type SCLLP and every s-continuous is g-continuous, hence h is of type GCLLP.

Lemma 3.26: If f is of type GCLLP and $f \simeq g$ then g is of type GCLLP.

Proof: Let f be of type GCLLP, then f is g-continuous and limited at every limited point. Since f is g-continuous, then for all $x \in R$, $f(x) \in gal(f(y))$. That is $f(x) - f(y)$ is limited

Since $f(x) \simeq g(x)$ for all $x \in R$. Therefore, $g(x) - g(y) \simeq f(x) - f(y)$, thence $g(x) - g(y)$ is limited for all $x, y \in R$, otherwise $f(x) \not\simeq f(y)$ which is contradiction. Then $g(x) \in gal\big(g(y)\big)$. Hence g is g-continuous.

To prove that g is limited at every limited point. By contradiction if not, then there exists a limited point t such that $g(t)$ is not limited. Since $f(x) \simeq g(x)$ for all x, then $f(t)$ is not limited which is contradiction to the fact that $f \in$ GCLLP.

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حول بعض انماط جديدة من الدوال في التحليل غير القياسي

الملخص

في هذا البحث ندرس سلوك دوال معرفة على فترات ذات محتوى متقطع حيث مسافة بين نقاطها غير متناهية الصغر. وتم التوصل الى ايجاد وتعريف بعض انواع جديدة من الدوال غير القياسية وحاولنا عرض بعض نتائج غير قياسية لهذه الدوال لقيم غير قياسية (قيم غير متناهية الصغر، غير متناهية القرب، غير متناهية الكبر،...)

لە بارەی هەندێك جۆری نوێ لە نەخشە لە شیكاری ناپێوانەیی

كورتیا لێكولینێ:

لەم تویژینەوەیەدا لیكۆلینەومان لە هەلسوكەوتی نەخشە كردوە كاتیك پیناسە كرابیت لەسەر ماوەیەك كە دانەكانی پچر پچر بن و بێ كۆتاش لیكنزیك بن. وتوانیمان هەندیك نەخشەی تازەی ناپێوانەیی دەستنیشان بكەین وپیناسەیان بكەین لەگەل پێشكەشكردنی هەندیك دەرئەنجامی ناپێوانەیی لەبارەی ئەم جۆرە نەخشانە بۆ نرخە ناپێوانەیەكانی(بێ كوتا بچوك، بێ كوتا گەورە، بێكوتا نزیك لە یەكتر،...)