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ON γ-CLEAN RINGS

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ABSTRACT:

In this paper we introduce the concept of $\boldsymbol{\gamma}$-clean ring and we discuss some relations between $\boldsymbol{\gamma}$- clean ring and other rings with explaining by some examples. Also, we give some basic properties of it.

KEYWORDS: γ-clean rings, r-clean rings, clean rings and γ-regular rings, regular rings.

1. INTRODUCTION

The concept of Von Neumann regular rings was first introduced by Von Neumann (1936), called R (briefly) regular (resp. strongly regular) if for every $r \in R$ there exists $b \in R$ such that $r = rbr$
(resp. $r = r^2 b$). Mohammad and Salih (2006) called R is γ-regular (resp. strongly γ-regular) ring if $a = ax^n a$ (resp. $a = a^2 x^n$) for some $x \in R$ and $n \neq 1$ positive integer. Clearly every γ-regular rings is regular rings. Esa (2010), investigates some concept of γ-regular called an element a has a γ-reflexive inverse if there exists $0 \neq x \in R$ and $n \neq 1$ positive integer such that $x^n = x^n a x^n$, and she proved that every γ-regular element has a γ-reflexive inverse. An element x of a ring R is called clean if $x = u + e$, where u is a unit in R and e is an idempotent element in R. A ring R is called clean if each of its elements is clean. Clean rings were firstly introduced by Nicholson (1977). Moreover, Asharafi and Nasibi (2013), had extended the definition of clean rings by using definition of regular element and defined that an element x in R is called r-clean if $x = r + e$, where r is a regular element and e is an idempotent, if every element of R is r-clean then R is called r-clean. Esa (2015), combined between the definition of γ-regular ring and local rings, called γ-Von Neumann local ring, (briefly γVNL-rings) defined that, R is γVNL-ring if a or $1 - a$ is γ-regular.

In this paper we study the concept of r-clean by using the definition of γ-regular element and defined that, an element x in R is called γ-clean if $x = a + e$, where a is a γ-regular element and e is an idempotent, if every element of R is γ-clean then R is called γ-clean. We show that, if R is an abelain γ-clean ring. Then R is γ-regular ring and how we discuss the necessary conditions for clean rings to be γ-clean rings. Finally, we get some important properties of γ-clean rings.

Throughout this paper, all rings are associative with identity and we denote $Id(R)$ for the set of all idempotents in R, $U(R)$ is the group of units and $J(R)$ is the Jacobson radical. A ring R is called abelain if every idempotent is central. A ring R is called reduced if it has no non-zero nilpotent element. A ring R is called directly finite (Esa, 2015), if for any elements $a, b \in R$, $ab = 1$ implies $ba = 1$.

2. PRELIMINARIES

In this section we view some basic definitions and relations on γ-regular rings.

Theorem 2.1. A commutative regular ring is a clean ring (Anderson and Camillo, 2002).

Lemma 2.2. Let R be an abelian ring. Let $a \in R$ be a clean element in R and let $e \in Id(R)$ (Asharafi and Nasibi, 2013). Then:

(i) The element ae is clean.

(ii) If $-a$ is clean, then $a + e$ is also clean.

Definition 2.3. A ring R is said to be a quasi-commutative if for every $a, b \in R$**,** when $1 \neq a$, there exists positive integer m such that $ab = b^m a$. (Mohammad and Salih, 2006)

Remark 2.4. Let R be a ring, then:

(i) For every $a, b \in R$ when $1 \neq a$ there exists $m > 1$ positive integer such that $ab = b^m a$, For $a = 1$ the above condition does not satisfy.

(ii) For $1 \cdot b = b^m \cdot 1$ then $b = b^m$ and this is a trivial case where $m = 1$.

Lemma 2.5. If R be a reduced ring. Then R is an abelian ring (Esa, 2015).

3. γ-CLEAN RINGS

In this section we introduce the definition of γ-clean ring and we discuss some relations of γ-clean rings with other rings such as γ-regular rings, r-clean rings, γ-Von Neumann local rings and clean rings. Also, we illustrate these relations by examples.

Definition 3.1. An element x in a ring R is called γ-clean if x can be written as $x = a + e$, where a is a γ-regular element and e is an idempotent.

Some Examples:

1. $M_2(Z_2)$ is γ-clean ring, because, for all $X \in M_2(Z_2)$, we can write X as A + I, where the entries of $A_{2\times 2}$'s are γ-regular element and $I \in Id(M_2(Z_2))$.

2. Boolean rings are γ- clean rings

3. Every field is a γ-clean ring as well as Z_p where p is a prime.

4. It is clear that every γ-regular rings which are clean rings are γ-clean rings.

Remark 3.2. Clearly every γ-regular ring is γ-clean ring, but the converse is not true, for example the ring $M_2(Z_2)$ is γ-clean ring but not γ-regular ring because at least $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \in M_2(Z_2)$, where $\gamma = 2$ is not γ-regular element.

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Theorem 3.3. Let R be an abelain ring. If R is γ-clean, then R is γ-regular ring.

Proof: let R be a γ-clean ring, then for every $x \in R$, we can write $x = a + e$, where a is γ-regular element and e is an idempotent. Assume $e = ab^n = b^n a$, where $n \neq 1$ positive integer then $e^2 = e$, so $ea = ae = a$. Now, $x = a + e$, if $a = 0$ then $x = e = ab^n = b^n a$. Since a has a γ-reflexive inverse then $x = ab^n ab^n = ab^n b^n a = ab^{2n} a$. Assume $y^m = b^{2n}$, so $x = ay^m a$, hence a is γ-regular. And if $a \neq 0$ then $x = a + e = ab^n a + ab^n = ab^n(a + 1) = e(a + 1)$. Since $a = ae$ then $a - ae = 0$, so $a(1 - e) = 0$. But $a \neq 0$ then $(1 - e) = 0$ thus $e = 1$, hence $x = e(a + 1) = a + 1$. By Theorem 4.7, $1 - x$ is also γ-clean then $x = -a = -(ab^n a) = (-a)(-b^n)(-a)$. Therefore x is γ-regular, so R is γ-regular.■

Remark 3.4. Every γ-clean ring is a r-clean ring, since every γ-regular ring is regular ring; But the converse is not true, for example Z_4 is r-clean (Asharafi and Nasibi, 2013), but not γ-clean since $\bar{2} \in Z_4$ is not γ-regular so Z_4 is not γ-clean.

Theorem 3.5. Let R be a quasi-commutative ring as defined in Remark 2.4.(i). If R is r-clean, then R is γ-clean.

Proof: let R be r-clean ring, then for every $x \in R$, we can write $x = r + e$, where r is a regular element and e is an idempotent. Now, $x = r + e = rbr + e$, for some $b \in R$. Since R is a quasi-commutative ring, then for every pair $a, b \in R$, $ab^n = ba$, where $n \neq 1$ a positive integer. Then $x = r^2 b^n + e$ so $x = rb^n r + e$. Thus x is γ-clean, hence R is γ-clean. ■

Remark 3.6. Every γVNL-ring is γ-clean ring, but the converse is not true, for example we can use the trivial and nontrivial idempotent in γ-clean ring, such as $M_2(Z_2)$, but in the definition of γVNL-ring we must use only the trivial idempotent to obtain the definition of γVNL-ring (Esa, 2015).

Theorem 3.7. Let R be a ring such that 0 and 1 are the only idempotents in R. Then R is γ-clean ring if and only if it is γVNL-ring.

Proof: Let R be a γ-clean ring and assume that 0 and 1 are the only idempotents in R. Then for any $x \in R$, we have $x = a + e$, where a is γ-regular element and $e \in Id(R)$. Now, if $x = a + 0 = a$, then x is γ-regular. And if $x = a + 1$, then $1 - x = -a$, since $-a$ is γ-regular. So, $1 - x$ is γ-regular. Hence R is γVNL-ring.

Conversely, let R be γVNL-ring. Then, for any $a \in R$, either a or $1 - a$ is γ-regular. Now, if a is γ-regular, by Remark 3.2., then a is γ-clean. And if $1 - a$ is γ-regular, also by Remark 3.2., then $1 - a$ is γ-clean. Hence, a is γ -clean. ■

Now, we discuss the relation between γ-clean rings and clean rings by giving some conditions on γ-clean rings to be clean rings.

Theorem 3.8. Let R be a quasi-commutative ring under the Remark 2.4.(i). If R is directly finite, then R is clean ring γ-clean ring.

Proof: Let R γ-clean, then every $x \in R$, $x = a + e$, where a is γ-regular element and $e \in Id(R)$.

Now, if $a = 0$ then $x = 0 + e = e = (2e - 1) + (1 - e)$. Since $(2e - 1) \in U(R)$ and $(1 - e) \in Id(R)$, hence x is clean. And if $a \neq 0$, then there exists $b \in R$ and $n \neq 1$ positive integer such that $a = ab^n a$, but R is quasi-commutative then $ab = b^n a$ so $a = a^2 b = aba$, thus $ab \in Id(R)$. By hypothesis either $ab = 0$ or $ab = 1$. So, if $ab = 0$, then $a = aba = 0$, which is contradiction.

Therefore, $ab = 1$, so R is directly finite. Thus, $ab = ba = 1$ then $a \in U(R)$. So, x is clean and hence R is clean. ■

Theorem 3.9. Let R be a commutative γ-clean ring and each pair of idempotent in R is orthogonal. Then R is clean.

Proof: If R is a commutative ring then every γ-regular ring is regular and so by Theorem 2.1. R is clean ring. Now, for any $x \in R$ we can write $x = e_1 + e_2 + u$ where $e_1, e_2 \in Id(R)$ and $u \in U(R)$, since $e_1 + e_2$ are orthogonal then $e_1 + e_2 = e \in Id(R)$. Hence, $x = e + u$ is clean, so R is clean. ■

Theorem 3.10. Let R be an abelian ring. If R is γ-clean, then R is clean.

Proof: Let R be a γ-clean, then for every $x \in R$, we have $x = a + e$, where a is γ-regular and $e \in Id(R)$. So, $x = ab^n a + e$, for some $b \in R$ and $n \neq 1$, positive integer. Assume $e = ab^\gamma = b^\gamma a$, then $\big(ae + (1 - e)\big)\big(b^\gamma a + (1 - e)\big) = 1$. Since R is abelian so for every $a \in R$, $ae = ea$ and $eb^\gamma = b^\gamma e$. Then $\big(ae + (1 - e)\big)\big(b^\gamma e + (1 - e)\big) = aeb^\gamma e + ae(1 - e) + (1 - e)b^\gamma e + (1 - e)^2$

$$= ab^\gamma e + ae - ae^2 + b^\gamma e - eb^\gamma e + (1 - e)^2$$

$$= e^2 + ae - ae + b^\gamma e - b^\gamma e + 1 - 2e + e^2 = 1.$$

So, $u = ae + (1 - e)$ is a unit, furthermore $a = ue$ and set $f = (1 - e)$ an idempotent. Then $ue + f$ is a unit, also $-(ue + f)$ is a unit. Since f is an idempotent, so $-a = f + (-(eu + f))$, since a is regular

Then by Lemma 2.2.(ii), we get x is clean. ■

Corollary 3.11. Let R be a reduced ring. If R is γ-clean, then R is clean.

Proof: By Lemma 2.5., the proof is complete.

Theorem 3.12. Let R be a ring without zero divisor. If R is γ-clean, then R is clean.

Proof: Let R be a γ-clean, then for every $x \in R$, we can write x as $x = a + e$, where a is γ-regular and $e \in Id(R)$. If $a = 0$, then $x = 0 + e = e = (2e - 1) + (1 - e)$. So, $(2e - 1) \in U(R)$ and $(1 - e) \in Id(R)$, then x is clean. Now, if $a \neq 0$, then $x = ab^n a + e$. Since R has no zero divisor, so inverse must be exist. So, $ab^n = b^n a = 1$, and $a \in U(R)$. Hence x is clean, so R is clean ring. ■

4. OTHER RESULTS ON γ-CLEAN RINGS

In this section, some other results of γ-clean rings will be given.

Theorem 4.1. Let R be a γ-clean ring and I an ideal of a ring R. Then R/I is a γ-clean ring.

Proof: let R be a γ-clean ring and I an ideal of a ring R and let $\acute{x} = x + I \in R/I$. Since R is a γ-clean, so for any $x \in R$, $x = a + e$, where a is γ-regular and $e \in Id(R)$. Thus, $\acute{x} = \acute{a} + \acute{e}$ there exists $b \in R$ and $n \neq 1$ positive integer such that $a = ab^n a$. Therefore, $(ab^n a)' = \acute{a}$. So $\acute{a}$ is γ-regular element and $\acute{e} \in Id(R)$. It follows that R/I is γ-clean. ■

Remark 4.2. In general, the converse of Theorem 4.1. is not true, for example; if p is a prime number then $\mathbb{Z}/p\mathbb{Z}$ and its clearly that $\mathbb{Z}/p\mathbb{Z} \cong Z_p$ which is a field, so Z_p is γ-clean, but $\mathbb{Z}$ is not γ-clean.

Now, we discuss the center of γ-clean ring by using the center γ-regular ring as follows:

Lemma 4.3.[1, Theorem 2.4], The center of γ-regular ring is also γ-regular.

Theorem 4.4. Let R be a γ-clean ring with only idempotents 0 and 1. Then the center of R is also γ-clean.

Proof: let R be γ-clean ring and let $x \in Z(R)$, then $x = a + e$, where a is γ-regular element and $e \in Id(R)$. By hypothesis, either $x = a$ or $x = a + 1$. Now, if $x = a$ then $x = ab^n a$, for some $b \in R$ and $1 \neq n$ positive integer. By Lemma 4.3., a is γ-regular in $Z(R)$, hence $x = a + 0$ is γ-clean in $Z(R)$

But if $x = a + 1$, then $a = x - 1$. Since a is γ-regular then $(x - 1) = (x - 1)b^n(x - 1)$ for some $b \in R$ and $1 \neq n$ positive integer. By Lemma 4.3., $(x - 1)$ is γ-regular in $Z(R)$

and so $(x-1)$ is γ-clean in $Z(R)$. Therefore $Z(R)$ is γ-clean. ■

Theorem 4.5. If e is an idempotent element of R and eRe and $(1-e)R(1-e)$ are both γ-clean. Then R is also γ-clean.

Proof: let R be a ring and let $e \in Id(R)$. By using Pierce decomposition for R;

$R = eRe \oplus eR(1-e) \oplus (1-e)Re \oplus (1-e)R(1-e)$. Since e is central in R, so

$$R = eRe \oplus (1-e)R(1-e) \cong \begin{pmatrix} eRe & 0 \\ 0 & (1-e)R(1-e) \end{pmatrix}.$$

For each $X \in R$, give $X = \begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix}$, where $x \in eRe$ and $y \in (1-e)R(1-e)$. By hypothesis x, y are γ-clean element. Thus $x = a_1 + e_1$ and $y = a_2 + e_2$, where a_1, a_2 are γ-regular element and $e_1, e_2 \in Id(R)$. So $X = \begin{pmatrix} x & 0 \\ 0 & y \end{pmatrix} = \begin{pmatrix} a_1 + e_1 & 0 \\ 0 & a_2 + e_2 \end{pmatrix} = \begin{pmatrix} a_1 & 0 \\ 0 & a_2 \end{pmatrix} + \begin{pmatrix} e_1 & 0 \\ 0 & e_2 \end{pmatrix}$

But there exists $b_1, b_2 \in R$ and $n \neq 1$ positive integer such that $a_1 = a_1 b_1^n a_1$, $a_2 = a_2 b_2^n a_2$

Therefore,

$$\begin{pmatrix} a_1 b_1^n a_1 & 0 \\ 0 & a_2 b_2^n a_2 \end{pmatrix} + \begin{pmatrix} e_1 & 0 \\ 0 & e_2 \end{pmatrix} = \begin{pmatrix} a_1 & 0 \\ 0 & a_2 \end{pmatrix} \begin{pmatrix} b_1^n & 0 \\ 0 & b_2^n \end{pmatrix} \begin{pmatrix} a_1 & 0 \\ 0 & a_2 \end{pmatrix} + \begin{pmatrix} e_1 & 0 \\ 0 & e_2 \end{pmatrix}$$

$$= \begin{pmatrix} a_1 & 0 \\ 0 & a_2 \end{pmatrix} \begin{pmatrix} b_1 & 0 \\ 0 & b_1 \end{pmatrix}^n \begin{pmatrix} a_1 & 0 \\ 0 & a_2 \end{pmatrix} + \begin{pmatrix} e_1 & 0 \\ 0 & e_2 \end{pmatrix}$$

$$= \begin{pmatrix} a_1 & 0 \\ 0 & a_2 \end{pmatrix} + \begin{pmatrix} e_1 & 0 \\ 0 & e_2 \end{pmatrix}$$

So $\begin{pmatrix} a_1 & 0 \\ 0 & a_2 \end{pmatrix}$ is γ-regular element and $\begin{pmatrix} e_1 & 0 \\ 0 & e_2 \end{pmatrix} \in Id(R)$. Hence R is γ-clean ring. ■

Theorem 4.6. Let $e_1, e_2, \dots, e_n$ be orthogonal central idempotent with $e_1 + e_2 + \cdots + e_n = 1$. Then $e_i R e_i$ is γ-clean for each i if and only if R is γ-clean.

Proof: By using Theorem 4.5. and by induction. Then $e_i R e_i$ is a γ-clean ring.

On the other hand, Let R be a γ-clean ring and $e_1, e_2, \dots, e_n$ be orthogonal central idempotent with $e_1 + e_2 + \cdots + e_n = 1$. Since $R = e_1 R e_1 \oplus \dots . \oplus e_n R e_n$.

Then by Theorem 4.3., $e_i R e_i$ is γ-clean for each i. ■

Theorem 4.7. Let R be a ring. Then R is γ-clean if and only if for every $x \in R$ can be written as $x = a - e$, where a is γ-regular element and $e \in Id(R)$.

Proof: Let R be a γ-clean, then for every $x \in R$, then $x = a + e$, where a is γ-regular element and $e \in Id(R)$. So $-x \in R$, we can write $-x = a + e$. Hence $x = (-a) - e$, where $-a$ is γ-regular and $e \in Id(R)$.

Conversely, suppose that for every $x \in R$ we can write $x = a - e$ where a is γ-regular and $e \in Id(R)$. So, for $-x \in R$, we can write $-x = a - e$. Hence $x = (-a) + e$, where $-a$ is γ-regular and $e \in Id(R)$. ■

Theorem 4.8. Let R be a ring. Then $x \in R$ is γ-clean element if and only if $1 - x$ is γ-clean.

Proof: Let $x \in R$ be a γ-clean then $x = e + a$ where a is γ-regular element and $e \in Id(R)$. Thus $1 - x = 1 - (e + a) = 1 - e - a = (1 - e) + (-a)$

But there exists $b \in R$ and $n \neq 1$ positive integer such that $a = ab^n a$

Hence $(-a)(-b)^n(-a) = -(ab^n a) = -a$

So, $-a$ is γ-regular and $(1-e) \in Id(R)$

Therefore, $1 - x$ is γ-clean. ■

Corollary 4.9. Let R be a ring and $x \in J(R)$. Then x is γ-clean.

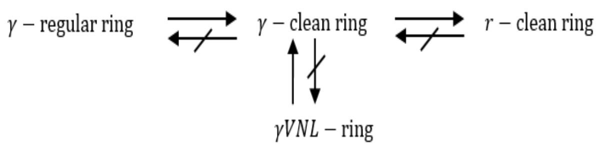
Proof: let $x \in J(R)$ then $1 - x \in U(R)$ so $1 - x$ is regular. Hence $1 - x$ is γ-clean

Therefore, by Theorem 4.8., x is γ-clean. ■

5. CONCLUSION

After all these relations and properties, we conclude that:

i. Every γ-regular ring is γ-clean and the converse is not true, see Theorem 3.3.
ii. Every γ-clean ring is r-clean and the converse is not true, see Theorem 3.5.
iii. Every γVNL-ring is γ-clean and the converse is not true, see Theorem 3.7.
iv. There is no relation between γ-clean ring and clean, we can get this diagram:



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