EDGE DEGREE WEIGHT OF SEQUENTIAL JOIN OF GRAPHS

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ABSTRACT

Let the weight \( w \) of an edge \( e = uv = \{u,v\} \) of a graph \( G \) be defined by

\[
 w(e) = \text{deg}(u) + \text{deg}(v) - 2
\]

and the weight of \( G \) be defined by \( w(G) = \sum_{e \in E(G)} w(e) \), where \( E(G) \) is the edge set of \( G \). In this paper the weights of joins, sequential joins, unions, intersections, and products (Cartesian and Tensor) of sets of graphs are obtained. This leads to a variety of open questions and new studies.

Keywords: Graph Theory, Edge degree weights

1. Introduction

Let \( G=(V(G), E(G)) \) be a simple graph of order \( p \) and size \( q \). Degree of a vertex \( v \) in a graph \( G \) is the number of vertices incident with \( v \) and denoted by \( \text{deg}(v) \). Among many ways of assigning weights to the edges of a graph are those that are functions of the degrees of the vertices of a given edge \( e = \{u,v\} \). Such weightings are of interest since the vertex degrees of a graph are intrinsic to the structure of the graph. One of the most natural of such weightings was studied by [Kattimani 2008] and [DuCasse and etal. 2009] as:

\[
 w_0(e) = \text{deg}(u) + \text{deg}(v)
\]

And extended to the weight of a graph \( G=(V(G), E(G)) \) as:

\[
 w_0(G) = \sum_{e \in E(G)} w_0(e)
\]

Moreover, edge degree weight are generalizations by [DuCassee and Quintas 2011].

Here, we take one of the most natural types of vertex labeling, that of assigning the degree of a vertex as its label, and consider the induced labeling of the edge of a graph by assigning to an edge the sum of the degrees of the vertices that make up that edge minus two. This labeling possesses many properties and has non-trivial open problems associated with it. Our terminology are due to [Beineke and Wilson 1978] and [Buckely and Harary 1990].

Definition 1.1: The weight \( w(e) \) of an edge \( e = \{u,v\} \) is defined by:

\[
 w(e) = \text{deg}(u) + \text{deg}(v) - 2
\]

and the edge degree weight sum \( w(G) \) of \( G \) relative to this edge weighting is \( w(G) = \sum_{e \in E(G)} w(e) \).

The following Theorems are fundamental for the computation of \( w(G) \).

Theorem 1.2: For any graph \( G \),

\[
 w(G) = \sum_{v \in V(G)} ((\text{deg}(v))^2 - 2q(G)).
\]

Proof: By Definition 1.1, \( w(e) = \text{deg}(u) + \text{deg}(v) - 2 \) for edge \( e = \{u,v\} \in E(G) \). Thus, each vertex \( v \) in \( V(G) \) contributes the value \( \text{deg}(v) - 1 \) to the weights of \( \text{deg}(v) \) edges and thus contributes \( (\text{deg}(v) - 1) \text{deg}(v) = (\text{deg}(v))^2 - \text{deg}(v) \) to \( w(G) \). Hence

\[
 w(G) = \sum_{v \in V(G)} ((\text{deg}(v))^2 - \text{deg}(v))
 = \sum_{v \in V(G)} (\text{deg}(v))^2 - \sum_{v \in V(G)} \text{deg}(v)
\]

Since, the sum of the degrees of the vertices of a graph is twice the number of degree, thus

\[
 w(G) = \sum_{v \in V(G)} (\text{deg}(v))^2 - 2q(G).
\]

Theorem 1.3: If \( G \) is regular graph of degree \( r \), then \( w(e) = 2r - 2 \) for each edge \( e = \{u,v\} \in E(G) \) and \( w(G) = pr^2 - pr \).

Proof: By Definition 1.1, \( w(e) = 2r - 2 \). By Theorem 1.2,
\[ w(G) = \sum_{v \in V(G)} (\text{deg}(v))^2 - \sum_{v \in V(G)} \text{deg}(v) \]
\[ = \sum_{v \in V(G)} r^2 - \sum_{v \in V(G)} r = pr^2 - pr. \]

2. Remarks

The union \( G = \bigcup_{i=1}^{k} G_i \) of \( k \) graphs is defined by
\[ V(G) = \bigcup_{i=1}^{k} V(G_i) \quad \text{and} \quad E(G) = \bigcup_{i=1}^{k} E(G_i). \]

It is trivial to observe that if the \( G_i \) are vertex disjoint, then the order, size, and weight of \( G \) are simply the sums of the orders, sizes, and weights, respectively of the \( G_i \)'s. However, if the components are not vertex disjoint, then the order, size and weight of \( G \) cannot, in general, be expressed as functions of the orders, sizes and weights of the \( G_i \)'s, without first determining the graph \( G \).

Similarly, the order, size and weight of the intersection \( H = \bigcap_{i=1}^{k} H_i \) of \( k \) graphs \( H_i \), defined by
\[ V(H) = \bigcap_{i=1}^{k} V(H_i) \neq \emptyset, \quad \text{and} \]
\[ E(H) = \bigcap_{i=1}^{k} E(H_i) \]
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Similarly, the order, size and weight of the intersection \( H = \bigcap_{i=1}^{k} H_i \) of \( k \) graphs \( H_i \), defined by
\[ V(H) = \bigcap_{i=1}^{k} V(H_i) \neq \emptyset, \quad \text{and} \]
\[ E(H) = \bigcap_{i=1}^{k} E(H_i) \]

Cannot, in general be expressed as functions of the orders, sizes and weights of the components without first determining the graph \( H \).

3. Joins and Sequential Joins

The join \( G_1 + G_2 \) of two graphs \( G_1 \) and \( G_2 \) is a graph whose vertex set is \( V(G_1 + G_2) = V(G_1) \cup V(G_2) \) and \( E(G_1 + G_2) = E(G_1) \cup E(G_2) \cup \{uv : \forall u \in G_1 \quad \text{and} \quad \forall v \in G_2 \} \) with order \( p_1 p_2 \) and size \( q_1 + q_2 + p_1 p_2 \).

**Theorem 3.1:** Let \( G_1 \) and \( G_2 \) be two disjoint graphs of order \( p_1, p_2 \) and size \( q_1, q_2 \), respectively, then
\[ w(G_1 + G_2) = \begin{cases} w(G_1) + w(G_2) & \\
+ 4(q_1 p_2 + q_2 p_1) + \\
+ p_1(p_2)^2 + p_2(p_1)^2 - 2p_1 p_2 & 
\end{cases} \]

**Proof:** Let \( \text{deg}(u) \) and \( \text{deg}(v) \) be the degree of \( u \in V(G_1) \) and \( v \in V(G_2) \), respectively. Let \( \text{deg}^+(u) \) and \( \text{deg}^+(v) \), respectively, be the degree of \( u \) and \( v \) in the join \( G_1 + G_2 \). Then,
\[ \text{deg}^+(u) = \text{deg}(u) + p_2, \quad \text{deg}^+(v) = \text{deg}(v) + p_1, \]
and
\[ w(G_1 + G_2) = \begin{cases} \sum_{u \in V(G_1)} (\text{deg}(u) + p_2)^2 & \\
+ \sum_{v \in V(G_2)} (\text{deg}(v) + p_1)^2 & \\
-2q_1(G_1 + G_2) & 
\end{cases} \]

\[ \begin{align*}
&+ \sum_{u \in V(G_1)} (\text{deg}(u))^2 - 2q_1 \\
&+ \sum_{v \in V(G_2)} (\text{deg}(v))^2 - 2q_2 \\
&+ 2\text{deg}(u)p_2 \\
&+ \sum_{v \in V(G_2)} 2\text{deg}(v)p_1 + \sum_{u \in V(G_1)} (p_2)^2 \\
&+ \sum_{v \in V(G_2)} (p_1)^2 - 2p_1 p_2
\end{align*} \]

Noting that the sum of the degrees is equal to twice the size of the graph, it follows that
Corollary 3.2: Let \( G_1 \) and \( G_2 \) be two disjoint regular graphs of degree \( r_1 \) and \( r_2 \), respectively, then
\[
 w(G_1 + G_2) = \begin{cases}
 p_1(r_1 + p_2)^2 + p_2(r_2 + p_1)^2 \\
 -p_1r_1 + p_2r_2 + 2p_1p_2
\end{cases}
\]

Proof: By the above Theorem,
\[
 w(G_1 + G_2) = \begin{cases}
 w(G_1) + w(G_2) + \begin{cases}
 p_1(r_1 + p_2)^2 + 4q_1p_2 \\
 +p_2(p_1)^2 - 2p_1p_2
\end{cases}
\end{cases}
\]

Corollary 3.3: Let \( G_1 \) and \( G_2 \) be two disjoint regular graphs of the same degree \( r \) and the same order \( p \), then
\[
 w(G_1 + G_2) = 2p(r + p)^2 - 2p(r + p)
\]

Proof: Follows directly from the above Corollary.

The following Theorem concerns the sequential join of graphs. This graph operation is of interest in the study of distance problems in graphs. For concepts not explicitly defined here, see [Buckley and Harary 1990]. The sequential join \( G_1 + G_2 + \ldots + G_k \) of \( k \) graphs is the graph \((G_1 + G_2) \cup (G_3 + G_4) \cup \ldots \cup (G_{k-1} + G_k)\) where \( + \) and \( \cup \) are the join and union operations, respectively. See Fig.1.

Theorem 3.4: If \( G_i \) with \( 1 \leq i \leq k \) are \( k \) disjoint graphs with orders \( p_i \), sizes \( q_i \), and \( G = G_1 + G_2 + \ldots + G_k \) is their sequential join, then

1. The order of \( G \) is \( \sum_{i=1}^{k} p_i \),
2. The size of \( G \) is \( q(G) = \sum_{i=1}^{k} q_i + \sum_{i=1}^{k-1} p_i p_{i+1} \), and
3. The weight of \( G \) is
\[
 \begin{cases}
 \sum_{i=1}^{k} w(G_i) + 4q_1p_2 + q_k p_{k-1} \\
 + \sum_{i=2}^{k-1} q_i (p_{i-1} + p_{i+1}) + p_1(p_2)^2 + \\
 (p_{k-1})^2 p_k + \sum_{i=2}^{k-1} p_i (p_{i-2} + (p_{i-1}))^2 \\
 + \sum_{i=2}^{k-1} p_{i-1} p_{i+1} + 2 \sum_{i=1}^{k-1} p_i p_{i+1}
\end{cases}
\]

The graph shown in Fig. 1 can be used to illustrate Theorem 3.4. Note that, the weight of \( G \) obtained by Theorem 1.2 can verify the weight of \( G \) obtained by Theorem 3.4.
Figure 1: The sequential join graph
\[ G = P_3 + K_2 + K_3. \]

**Proof of Theorem 3.4:**
(1) The order of \( G \) is obviously the sum of the orders of the \( G_i \)s.
(2) The size of \( G \) is the sum of the sizes of the \( G_i \)s plus the number of edges joining these graphs.
(3) Let \( x_{ij} \) denote the \( J^{th} \) vertex in \( G_i \), \( \deg(x_{ij}) \) be the degree of \( x_{ij} \) in \( G_i \) and \( \deg^+(x_{ij}) \) be the degree of \( x_{ij} \) in \( G \). Then, \( \deg^+(x_{ij}) = p_{i-1} + \deg(x_{ij}) + p_{i+1}, \) where \( p_0 = 0 = p_{k+1}. \)

From this one obtains, by Theorem 1.2,
\[
W(G) = \begin{align*}
&\sum_{j=1}^{p_k} (\deg(x_{1j}))^2 + 2\sum_{j=1}^{p_k} \deg(x_{1j}) p_2 + (p_2)^2 \\
&- \sum_{j=1}^{p_k} (p_{k-1} + \deg(x_{k-1}))^2 \\
&+ \sum_{i=2}^{k-1} \sum_{j=1}^{p_k} (p_{i-1} + \deg(x_{ij}) + p_{i+1})^2 \\
&- 2q_1 - 2q_k - 2\sum_{i=2}^{k-1} q_i - 2\sum_{i=1}^{k-1} p_i p_{i+1}
\end{align*}
\]

Noting that the sum of the degrees is equal to twice the size of a graph, it follows that
\[
W(G) + 4q(p_1 p_2 + p_1 (p_2)^2) + \sum_{i=2}^{k-1} [W(G_i) + 4q_i (p_{i-1} + p_{i+1})] + p_i ((p_{i-1})^2 + (p_{i+1})^2) + 2p_{i-1} p_{i+1}
\]

\[
= \begin{align*}
&W(G_1) + 4q_1 p_2 + p_1 (p_2)^2 \\
&+ \sum_{i=2}^{k-1} [W(G_i) + 4q_i (p_{i-1} + p_{i+1})] + p_i ((p_{i-1})^2 + (p_{i+1})^2) + 2p_{i-1} p_{i+1}
\end{align*}
\]
The hypothesis of Theorem 3.4 contains two variables, namely, the order and the size of the $k$ graphs. By letting the order be a constant $p$, or the size be a constant $q$, or the both the order and the size be constant, or by restricting the $k$ graphs to be regular graphs, a variety of specialization of the general case covered by Theorem 3.4 can be obtained. These are given in the following table, where $G$ is the sequential join $G_1 + G_2 + \ldots + G_k$:

**Hypothesis 1:** Put $p_i = p$.

**Conclusion:**

1. **Order of $G$** is $kp$.
2. **Size of $G$** is $\sum_{i=1}^{k} q_i + (k-1)p^2$, and
3. **Weight of $G$** is
   
   \[
   \begin{align*}
   \sum_{i=1}^{k} w(G_i) + 4q(p_1 + q_k + 2\sum_{i=1}^{k-1} q_i) \\
   + 2(2k-3)p^3 - 2(k-1)p^2
   \end{align*}
   \]

**Hypothesis 2:** Put $q_i = q$.

**Conclusion:**

1. **Order of $G$** is $\sum_{i=1}^{k} p_i$,
2. **Size of $G$** is $kq + \sum_{i=1}^{k-1} p_ip_{i+1}$, and
3. **Weight of $G$** is
   
   \[
   \begin{align*}
   \sum_{i=1}^{k} w(G_i) + 4q(p_1 + p_k + 2\sum_{i=1}^{k-1} p_i) \\
   + p_1(p_2)^2 + (p_{k-1})^2 p_k \\
   + \sum_{i=2}^{k-1} p_i((p_{i-1})^2 + (p_{i+1})^2) \\
   + 2\sum_{i=2}^{k-1} p_{i-1}p_ip_{i+1} - 2\sum_{i=1}^{k-1} p_ip_{i+1}
   \end{align*}
   \]

**Hypothesis 3:** Put $p_i = p$ and $q_i = q$.

**Conclusion:**

1. **Order of $G$** is $kp$,
2. **Size of $G$** is $kq + (k-1)p^2$, and
3. **Weight of $G$** is
   
   \[
   \begin{align*}
   \sum_{i=1}^{k} w(G_i) + 8(k-1)pq \\
   + 2(2k-3)p^3 - 2(k-1)p^2
   \end{align*}
   \]

**Hypothesis 4:** For $1 \leq i \leq k$, $G_i$ is $r_i$-regular with order $p_i$, where $p_ir_i$ is even.

**Conclusion:**

1. **Order of $G$** is $\sum_{i=1}^{k} p_i$,
2. **Size of $G$** is $\frac{1}{2}\sum_{i=1}^{k} p_ir_i + \sum_{i=1}^{k-1} p_ip_{i+1}$, and
3. **Weight of $G$** is
   
   \[
   \begin{align*}
   \sum_{i=1}^{k} w(G_i) + 2[p_ip_{i+1} + p_kr_kp_{k-1} \\
   + \sum_{i=2}^{k-1} p_ir_i(p_{i-1} + p_{i+1})] \\
   + p_1(p_2)^2 + (p_{k-1})^2 p_k \\
   + \sum_{i=2}^{k-1} p_i((p_{i-1})^2 + (p_{i+1})^2) \\
   + 2\sum_{i=2}^{k-1} p_{i-1}p_ip_{i+1} - 2\sum_{i=1}^{k-1} p_ip_{i+1}
   \end{align*}
   \]
Hypothesis 5: For $1 \leq i \leq k$, $G_i$ is $r_i$-regular with fixed order $p$, where $pr_i$ is even.

Conclusion: (1) order of $G$ is $kp$.
(2) Size of $G$ is
$$\frac{p}{2} \sum_{i=1}^{k} r_i + (k-1)p^2$$, and

(3) Weight of $G$ is
$$\sum_{i=1}^{k} w(G_i) + 2p^2[r_1 + r_k + 2\sum_{i=2}^{k-1} r_i] + 2(2k-3)p^3 - 2(k-1)p^2$$

Hypothesis 6: For $1 \leq i \leq k$, $G_i$ is $r$-regular with order $p_i$, where $pr_i$ is even.

Conclusion: (1) order of $G$ is $\sum_{i=1}^{k} p_i$.
(2) Size of $G$ is
$$\frac{r}{2} \sum_{i=1}^{k} p_i + \sum_{i=1}^{k-1} p_ip_{i+1}$$, and

(3) Weight of $G$ is
$$\sum_{i=1}^{k} w(G_i) + 2r[p_1p_2 + p_kp_{k-1} + \sum_{i=2}^{k-1} p_i(p_{i-1} + p_{i+1})] + p_1p_2^2 + (p_k - 2)p_k^2$$
$$+ \sum_{i=2}^{k-1} p_i((p_{i-1})^2 + (p_{i+1})^2)$$
$$+ 2\sum_{i=2}^{k-1} p_{i-1}p_ip_{i+1} - 2\sum_{i=1}^{k-1} p_ip_{i+1}$$

Hypothesis 7: For $1 \leq i \leq k$, $G_i$ is $r$-regular with fixed order $p$, where $pr$ is even.

Conclusion: (1) order of $G$ is $kp$.
(2) Size of $G$ is
$$\frac{1}{2} kpr + (k-1)p^2$$, and

(3) Weight of $G$ is
$$kpr + 4(k-1)p^2 + 2(2k-3)p^3 - 2(k-1)p^2$$

4- Products
Let $G_1 = (V_1, E_1)$ and $G_2 = (V_2, E_2)$ be vertex disjoint non-trivial graphs. The Cartesian product $G_1 \times G_2$ of the two graphs $G_1$ and $G_2$ is the graph with vertex set $V(G_1 \times G_2) = V_1 \times V_2$ and two vertices $(u_1, v_1)$ and $(u_2, v_2)$ are adjacent in $G_1 \times G_2$ if, and only if,
$$[u_1 \equiv u_2 \text{ and } v_1 \text{ adjacent with } v_2 \text{ in } G_2] \text{ or } [v_1 \equiv v_2 \text{ and } u_1 \text{ adjacent with } u_2 \text{ in } G_1].$$

The following Theorem determine the weight of $G_1 \times G_2$.

Theorem 4.1: Let $G_1$ and $G_2$ be disjoint graphs of order $p_1$ and $p_2$, and size $q_1$ and $q_2$, respectively. Then
$$w(G_1 \times G_2) = p_1w(G_1) + p_2w(G_2) + 8q_1q_2$$

Proof: Let $x \in V(G_1)$, $y \in V(G_2)$, and $(x, y) \in V(G_1 \times G_2)$. Then,
$$w(G_1 \times G_2) = \sum_{(x,y) \in V(G_1 \times G_2)} (\deg(x, y))^2$$
$$- 2q(G_1 \times G_2)$$
$$= \sum_{(x,y) \in V(G_1 \times G_2)} (\deg(x) + \deg(y))^2$$
$$- 2(p_1q_2 + p_2q_1)$$
$$= \sum_{(x,y) \in V(G_1 \times G_2)} [(\deg(x))^2 + 2\deg(x)\deg(y) + (\deg(y))^2] - 2p_1q_2 - 2p_2q_1$$
\[
\sum_{(x,y)\in E(G)} (\deg(x))^2
+ 2 \sum_{(x)\in V(G)} \deg(x) \sum_{(y)\in V(G)} \deg(y)
+ \sum_{(x,y)\in E(G)} (\deg(y))^2
\]
\[
-2p_1q_2 - 2p_2q_1
\]

Since,
\[
w(G) + 2q(G) = \sum_{v \in V(G)} (\deg(v))^2, \text{ thus }
\]
\[
w(G_1 \times G_2) = \left[p_2w(G_1) + 2q_1\right] + (2q_1) (2q_2) + p_1(w(G_2))
+ 2q_2 - 2p_1q_2 - 2p_2q_1
\]
\[
= 2p_2w(G_1) + 2p_2q_1 + 8q_1q_2 + p_1w(G_2)
+ 2p_1q_2 - 2p_2q_1
\]
\[
= p_2w(G_1) + p_1w(G_2) + 8q_1q_2
\]

**Corollary 4.2:** Let \( G_1 \) and \( G_2 \) be disjoint regular graphs of order \( p_1 \) and \( p_2 \), and degree \( r_1 \) and \( r_2 \), respectively. Then
\[
w(G_1 \times G_2) = p_1p_2(r_1 + r_2)^2.
\]

**Proof:** From Theorem 4.1,
\[
w(G_1 \times G_2) = p_2w(G_1) + p_1w(G_2) + 8q_1q_2,
\]
where \( q_1 \) and \( q_2 \) are the respectively sizes of \( G_1 \) and \( G_2 \). In this case, \( w(G_1) = p_1r_1^2 \),
\[
w(G_2) = p_2r_2^2, \quad q_1 = \frac{p_1r_1^2}{2}, \quad \text{and} \quad q_2 = \frac{p_2r_2^2}{2}.
\]
Thus,
\[
w(G_1 \times G_2) = p_2p_1r_1^2 + p_1p_2r_2^2 + 8\left(\frac{p_1r_1^2}{2}\right)\left(\frac{p_2r_2^2}{2}\right)
\]

Therefore,
\[
w(G_1 \times G_2) = p_1p_2(r_1 + r_2)^2.
\]

**Corollary 4.3:** Let \( G_1 \) and \( G_2 \) be disjoint regular graphs of the same order \( p \) and degree \( r \), then \( w(G_1 \times G_2) = p^2(3r^2 + 2r) \).

The tensor product \( G_1 \otimes G_2 \) of the two graphs \( G_1 \) and \( G_2 \) is the graph with vertex set \( V(G_1 \otimes G_2) = V_1 \times V_2 \) and two vertices \((u_1,v_1)\) and \((u_2,v_2)\) are adjacent in \( G_1 \otimes G_2 \) if, and only if, \([u_1 \text{ adjacent with } u_2 \text{ in } G_1]\) and \([v_1 \text{ adjacent with } v_2 \text{ in } G_2]\).

The following Theorem determine the weight of \( G_1 \otimes G_2 \).

**Theorem 4.4:** Let \( G_1 \) and \( G_2 \) be disjoint graphs of order \( p_1 \) and \( p_2 \), and size \( q_1 \) and \( q_2 \), respectively. Then
\[
w(G_1 \otimes G_2) = \left[ w(G_1)w(G_2) + 2q_2w(G_1) \right]
+ 2q_1(G_2)\]

**Proof:** Let \( x \in V(G_1), \ y \in V(G_2), \) and \((x,y) \in V(G_1 \times G_2)\). Then,
\[
w(G_1 \otimes G_2) = \sum_{(x,y)\in V(G_1 \otimes G_2)} (\deg(x,y))^2
- 2q(G_1 \otimes G_2)
= \sum_{(x,y)\in V(G_1 \otimes G_2)} (\deg(x)) \deg(y) - 4q_1q_2
\]

Since, \( w(G) + 2q(G) = \sum_{v \in V(G)} (\deg(v))^2 \), thus
\[
w(G_1 \otimes G_2) = \left(w(G_1) + 2q_1\right)w(G_2) + 2q_2w(G_1) - 4q_1q_2
\]

**Corollary 4.5:** Let \( G_1 \) and \( G_2 \) be disjoint regular graphs of order \( p_1 \) and \( p_2 \), and degree \( r_1 \) and \( r_2 \), respectively. Then
\[
w(G_1 \otimes G_2) = p_1p_2(r_1^2r_2^2 + r_1r_2^2 + r_1^2r_2).
\]

**Proof:** From Theorem 4.4,
\[
w(G_1 \otimes G_2) = \left[ w(G_1)w(G_2) + 2q_2w(G_1) \right]
+ 2q_1(G_2)\]

, where \( q_1 \) and \( q_2 \) are the respectively sizes of \( G_1 \) and \( G_2 \). In this case, \( w(G_1) = p_1r_1^2 \),
\[
w(G_2) = p_2r_2^2, \quad q_1 = \frac{p_1r_1^2}{2}, \quad \text{and} \quad q_2 = \frac{p_2r_2^2}{2}.
\]
Thus,
\[
w(G_1 \otimes G_2) = p_1p_2r_1^2p_2r_2^2 + 2\left(\frac{p_1r_1^2}{2}\right)\left(\frac{p_2r_2^2}{2}\right)
+ 2q_1(G_2)\]

Therefore,
\[
w(G_1 \otimes G_2) = p_1p_2(r_1^2r_2^2 + 2r_1r_2^2 + r_1^2r_2).\]
Corollary 4.6: Let \( G_1 \) and \( G_2 \) be disjoint regular graphs of the same order \( p \) and degree \( r \), then \( w(G_1 \oplus G_2) = p^2(r^4 + 2r^3) \).

References
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