SUCCESSIVE AND FINITE DIFFERENCE METHOD FOR GRAY SCOTT MODEL

Saad **A**bdullah **M**anaa and **J**oli **R**asheed
Department of Mathematics, Faculty of Science, University of Zakho, Kurdistan Region, Iraq.

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Abstract:

In this paper, Gray-Scott model has been solved numerically for finding an approximate solution by Successive approximation method and Finite difference method. Example showed that Successive approximation method is much faster and effective for this kind of problems than Finite difference method.

KEYWORDS: Partial Differential Equations, Gray-Scott equation, Successive approximation method. .

1. Introduction:

Nonlinear first-order partial differential equations arise in a variety of physical theories, primarily in dynamics (to generate canonical transformations), continuum mechanics (to record conservation of mass, momentum, energy, etc.) and optics (to describe wave fronts). Although the strong nonlinearity generally precludes our deriving any simple formulas for solutions, we can, remarkably, often employ calculus to glean fairly detailed information about solutions (Lawrence, 2010).

The basic idea of the method of finite difference is to cast the continuous problem described by the PDE and auxiliary conditions into a discrete problem that can be solved by computer in finitely many steps. The discretization is accomplished by restricting the problem to a set of discrete points. By systematic procedure, we then calculate the unknown function at those discrete points. Consequently, a finite difference technique yields a solution only at discrete points in the domain of interest rather than, as we expect for an analytical calculation, a formula or closed-form solution valid at all points of the domain (Logan, 1987).

Solution by the finite difference method, although more general, will involve stability and convergence problems, may require special handling of boundary conditions, and may require large computer storage and execution time. The problem of numerical dispersion for finite difference solutions is also difficult to overcome (Guymon, 1970).

There is another approximation method for solving integral equations and differential equations. This method starts by using the constant function as an approximation to a solution. We substitute this approximation into the right side of the given equation and use the result as a next approximation to the solution. Then we substitute this approximation into the right side of the given equation to obtain what we hope is a still better approximation and we continuing the process. Our goal is to find a function with the property that when it is substituted in the right side of the given equation the result is the same function. This procedure is known as successive approximation method (Brauer & Nohel, 1973).

Mathematical Model:

A general class of nonlinear-diffusion system is in the form

$$\frac{\partial u}{\partial t} = d_1 \Delta u + a_1 u + b_1 v + f(u,v) + g_1(x)$$

$$\frac{\partial v}{\partial t} = d_2 \Delta v + a_2 u + b_2 v - f(u,v) + g_2(x)$$

with homogenous Dirchlet or Neumann boundary condition on a bounded domain Ω , $n \leq 3$, with locally Lipschitz continuous boundary. It is well known that reaction and diffusion of chemical or biochemical species can produce a variety of spatial patterns. This class of reaction diffusion systems includes some significant pattern formation equations arising from the modeling of kinetics of chemical or biochemical reactions and from the biological pattern formation theory.

In this group, the following four systems are typically important and serve as mathematical models in physical chemistry and in biology:

Brusselator model:

$$a_1 = -(b+1),\ b_1 = 0,\ a_2 = b,\ b_2 = 0,\ f = u^2 v,\ g_1 = a,\ g_2 = 0$$

where *a* and *b* are positive constants.

Gray-Scott model:

$$a_1 = -(F+k),\ b_1 = 0,\ a_2 = 0,\ b_2 = -F,\ f = u^2 v,\ g_1 = 0,\ g_2 = F$$

where *F* and *k* are positive constants.

Glycolysis model:

$$a_1 = -1,\ b_1 = k,\ a_2 = 0,\ b_2 = -k,\ f = u^2 v,\ g_1 = p,\ g_2 = \delta$$

where *k, p* and δ are positive constants.

Schnackenberg model:

$$a_1 = -k,\ b_1 = a_2 = b_2 = 0,\ f = u^2 v,\ g_1 = a,\ g_2 = b$$

where k, a and b are positive constants. (Saeed, 2006; Temam, 1997)

Then one obtains the following system of two nonlinearly coupled reaction-diffusion equations,

$$\frac{\partial u}{\partial t} = d_1 \Delta u - (F+k)u + u^2 v \quad , t>0 \quad x \in \Omega$$

$$\frac{\partial v}{\partial t} = d_1 \Delta v + F(1-v) - u^2 v \quad , t>0 \quad x \in \Omega \tag{1}$$

With initial and boundary conditions:

$$u(t,x) = v(t,x) = 0, \quad t > 0, \quad x \in \partial\Omega$$

$$u(0,x) = u_0(x),\ v(0,x) = v_0(x), \quad x \in \Omega \tag{2}$$

And with Neumann boundary conditions:

$$\frac{\partial u}{\partial x} = \frac{\partial v}{\partial x} = 0 \quad at \quad x = 0 \quad and \quad x = L \tag{3}$$

Where d_1, d_2, F, k *and* L are positive constants (Temam, 1997).

Most chemical reactions can present rich phenomena in vessels, such as chemical oscillations, periodic doubling, chemical waves, and chaos.

analysis of forced nonlinear oscillations plays an important role in understanding their dynamical phenomena of electronic generators, mechanical, chemical and biological systems. Even small external disturbances are likely to change behaviors of dynamical systems (Mingjing et al., 2008).

Reaction-diffusion (RD) systems arise frequently in the study of chemical and biological phenomena and are naturally modeled by parabolic partial differential equations (PDEs). The dynamics of RD systems has been the subject of intense research activity over the past decades. The reason is that RD system exhibit very rich dynamic behavior including periodic and quasi-periodic solutions (Brauer & Nohel, 1973; Shanthakumar, 1987).

In an experimental application of linear modal feedback control for suppressing chaotic temporal fluctuation of spatiotemporal thermal pattern on a catalytic wafer was reported in (Chakravarti et al., 1995).

Various orders are self-organized far from the chemical equilibrium. The theoretical procedures and notions to describe the dynamics of patterns formation have been developed for the last three decades (Nicolis & Prigogine, 1977). Attempts have also been made to understand morphological orders in biology (Cross & Hohenberg, 1993). Clarification of the mechanisms of the formation of orders and the relationship among them has been one of the fundamental problems in non-equilibrium statistical physics (Murray, 2007).

Finite Difference Approximations

Assume that the rectangle $R=\{(x,t):0\le x\le a,\quad 0\le t\le b\}$ is subdivided into n-1 by m-1 rectangle with sides $\Delta x=h$ and $\Delta t=k$, as shown in Fig.(1). Start at the bottom row, where $t = t_1 = 0$, and the solution is $u(x_p,t_1)=f(x_p)$.

A method for computing the approximations to $u(x,t)$ at grid points in successive rows $\{u(x_p,t_q):p=1,2,...,n\}$, for $q=2,3,...,m$. The difference formulas used for approximation $u_t(x,t), u_x(x,t)\, and \quad u_{xx}(x,t)$ are

$$u_t(x,t)=\frac{u(x,t+k)-u(x,t)}{k}+O(k) \tag{4}$$

$$u_x(x,t)=\frac{u(x+h,t)-u(x,t)}{h}+O(h) \tag{5}$$

$$u_{xx}(x,t)=\frac{u(x-h,t)-2u(x,t)+u(x+h,t)+}{h^2}+O(h^2) \tag{6}$$

The grid spacing is uniform in every row: $x_{p+1}=x_p+h$ and ($x_{p-1}=x_p-h$), and it is uniform in every column: $t_{q+1}=t_q+k$ and ($t_{q-1}=t_q-k$). Next, we drop the terms $O(k),O(h)$ and $O(h^2)$ (Leon & George, 1982; Mathews, 2004), and use the approximation $u_{p,q}$ for $u(x_p,t_q)$ in equation (4-6) and substituted into equation (1) to obtain

By finite difference:

$$\frac{\partial u}{\partial t}=\frac{u_{i,j+1}-u_{i,j}}{\Delta t} \tag{7}$$

$$\frac{\partial v}{\partial t}=\frac{v_{i,j+1}-v_{i,j}}{\Delta t} \tag{8}$$

$$\frac{\partial^2 u}{\partial x^2}=\frac{u_{i-1,j}-2u_{i,j}+u_{i+1,j}}{(\Delta x)^2} \tag{9}$$

$$\frac{\partial^2 v}{\partial x^2}=\frac{v_{i-1,j}-2v_{i,j}+v_{i+1,j}}{(\Delta x)^2} \tag{10}$$

The grid spacing is uniform in every row: $x_{p+1}=x_p+h$ and ($x_{p-1}=x_p-h$), and it is uniform in every column: $t_{q+1}=t_q+k$ and ($t_{q-1}=t_q-k$) (Mathews, 2004).

and substitute (7-10) in the Gray-Scott model (1.1) to get

$$\frac{u_{i,j+1}-u_{i,j}}{\Delta t}=d_1\frac{u_{i-1,j}-2u_{i,j}+u_{i+1,j}}{(\Delta x)^2}-(F+k)u_{i,j}+(u_{i,j})^2v_{i,j}$$

$$\frac{v_{i,j+1}-v_{i,j}}{\Delta t}=d_2\frac{v_{i-1,j}-2v_{i,j}+v_{i+1,j}}{(\Delta x)^2}+F(1-v_{i,j})-(u_{i,j})^2v_{i,j}$$

$$u_{i,j+1}-u_{i,j}=\frac{d_1\Delta t}{(\Delta x)^2}[u_{i-1,j}-2u_{i,j}+u_{i+1,j}]-\Delta t(F+k)u_{i,j}+\Delta t(u_{i,j})^2v_{i,j}$$

$$v_{i,j+1}-v_{i,j}=\frac{d_2\Delta t}{(\Delta x)^2}[v_{i-1,j}-2v_{i,j}+v_{i+1,j}]+\Delta tF(1-v_{i,j})-\Delta t(u_{i,j})^2v_{i,j}$$

Let $r_1 = \frac{d_1 \Delta t}{(\Delta x)^2}$ and $r_2 = \frac{d_2 \Delta t}{(\Delta x)^2}$ then

$$u_{i,j+1} - u_{i,j} = r_1[u_{i-1,j} - 2u_{i,j} + u_{i+1,j}] - \Delta t(F+k)u_{i,j} + \Delta t(u_{i,j})^2 v_{i,j}$$

$$v_{i,j+1} - v_{i,j} = r_2[v_{i-1,j} - 2v_{i,j} + v_{i+1,j}] + \Delta tF(1 - v_{i,j}) - \Delta t(u_{i,j})^2 v_{i,j}$$

$$u_{i,j+1} = [1 - 2r_1 - \Delta t(F+k)]u_{i,j} + r_1[u_{i-1,j} + u_{i+1,j}] + \Delta t(u_{i,j})^2 v_{i,j}$$

$$v_{i,j+1} = \Delta tF + [1 - 2r_2 - \Delta tF]v_{i,j} + r_2[v_{i-1,j} + v_{i+1,j}] - \Delta t(u_{i,j})^2 v_{i,j} \qquad (11)$$

From the boundary conditions we have

$$\frac{\partial u}{\partial x} = \frac{u_{i,j} - u_{i-1,j}}{2\Delta x} = 0 \qquad u_{i-1,j} = u_{i,j} \Rightarrow \quad u_{1,j} = u_{2,j} \qquad (12)$$

And

$$\frac{\partial u}{\partial x} = \frac{u_{i+1,j} - u_{i,j}}{2\Delta x} = 0 \qquad u_{i+1,j} = u_{i,j} \Rightarrow \therefore u_{11,j} = u_{10,j} \qquad (13)$$

And also for v

$$\frac{\partial v}{\partial x} = \frac{v_{i,j} - v_{i-1,j}}{2\Delta x} = 0 \qquad v_{i-1,j} = v_{i,j} \Rightarrow v_{1,j} = v_{2,j} \qquad (14)$$

And

$$\frac{\partial v}{\partial x} = \frac{v_{i+1,j} - v_{i,j}}{2\Delta x} = 0 \qquad v_{i+1,j} = v_{i,j} \quad \Rightarrow v_{11,j} = v_{10,j} \qquad (15)$$

And from the initial condition:

$$u_{1,1} = u_{2,1} = u_{3,1} = u_{4,1} = u_{5,1} = u_{6,1} = u_{7,1} = u_{8,1} = u_{9,1} = u_{10,1} = u_{11,1} = u_0(x)$$

$$v_{1,1} = v_{2,1} = v_{3,1} = v_{4,1} = v_{5,1} = v_{6,1} = v_{7,1} = v_{8,1} = v_{9,1} = v_{10,1} = v_{11,1} = v_0(x)$$

The result equation (11) is the finite difference method for the Gray-Scott model.

Successive Method (SAM)

The method of SAM provides a method that can, in principle, be used to solve any initial value problem (Chakravarti, et al., 1995; Jerri, 1985; Otto & Denier, 2005)

$$u' = f(t,u); \qquad u(t_0) = u_0 \qquad (16)$$

It starts by observing that any solution to (12) must also be a solution to

$$u(t) = u_0 + \int_{t_0}^{t} f(s, u(s))ds \qquad (17)$$

And then iteratively constructs a sequence of solutions that get closer and closer to the actual (exact) solutions of (17). The SAM is based on the integral equation (17) as follows:

$$u_0(t) = u_0$$

$$u_1(t) = u_0 + \int_{t_0}^{t} f(s, u_0) ds$$

$$u_2(t) = u_0 + \int_{t_0}^{t} f(s, u_1(s)) ds$$

$$u_3(t) = u_0 + \int_{t_0}^{t} f(s, u_2(s)) ds$$

This process can be continued to obtain the n^{th} approximation,

$$u_n(t) = u_0 + \int_{t_0}^{t} f(s, u_{n-1}(s)) ds, \qquad n = 1, 2, ...$$

Then determine whether $u_n(x)$ approaches the solution $u(x)$ as n increases. This will be done by proving the following:

- The sequence $\{u_n(x)\}$ converges to a limit $u(x)$, that
  $\lim_{n\to\infty} u_n(x) = u(x) \qquad a \le x \le b$.
- The limiting function $u(x)$ is a solution of (13) on the interval $a \le x \le b$.
- The solution $u(x)$ of (13) is unique.

A proof of these results can be constructed along the lines of the corresponding proof for ordinary differential equations (Coddington, 1961).

By integrating both sides from of (1) with respect to s, from 0 to t, we get:

$$\begin{aligned} u(t,x) &= u(0,x) + d_1 \int_0^t \frac{\partial^2 u}{\partial x^2} ds - (F+k) \int_0^t u ds + \int_0^t (u^2 v) ds \\ v(t,x) &= v(0,x) + d_2 \int_0^t \frac{\partial^2 v}{\partial x^2} ds + F \int_0^t (1-v) ds - \int_0^t (u^2 v) ds \end{aligned} \qquad (18)$$

Using the initial conditions in (3) we get:

$$\begin{aligned} u(t,x) &= u_0(x) + d_1 \int_0^t \frac{\partial^2 u(s,x)}{\partial x^2} ds - (F+k) \int_0^t u(s,x) ds + \int_0^t [u^2(s,x) v(s,x)] ds \\ v(t,x) &= v_0(x) + d_2 \int_0^t \frac{\partial^2 v(s,x)}{\partial x^2} ds + Ft - F \int_0^t v(s,x) ds - \int_0^t [u^2(s,x) v(s,x)] ds \end{aligned} \qquad (19)$$

The general form is:

$$\begin{aligned} u_n(t,x) &= u_0(x) + d_1 \int_0^t \frac{\partial^2 u_{n-1}(s,x)}{\partial x^2} ds - (F+k) \int_0^t u_{n-1}(s,x) ds + \int_0^t [u_{n-1}^2(s,x) v_{n-1}(s,x)] ds \\ v_n(t,x) &= v_0(x) + Ft + d_2 \int_0^t \frac{\partial^2 v_{n-1}(s,x)}{\partial x^2} ds - F \int_0^t v_{n-1}(s,x) ds - \int_0^t [u_{n-1}^2(s,x) v_{n-1}(s,x)] ds \end{aligned} \qquad (20)$$

$n = 1, 2, 3, 4, ...$

For n=1

$$u_1(t,x) = u_0(x) + d_1\int_0^t \frac{\partial^2 u_0(x)}{\partial x^2} ds - (F+k)\int_0^t u_0(x) ds + \int_0^t [u_0^2(x) v_0(x)] ds$$

$$= u_0 + d_1 \frac{\partial^2 u_0}{\partial x^2} t - (F+k) u_0 t + u_0^2 v_0 t$$

$$\therefore u_1(t,x) = u_0 + [d_1 \frac{\partial^2 u_0}{\partial x^2} - (F+k) u_0 + u_0^2 v_0] t \qquad (21)$$

$$v_1(t,x) = v_0(x) + Ft + d_2\int_0^t \frac{\partial^2 v_0(x)}{\partial x^2} ds - F\int_0^t v_0(x) ds - \int_0^t [u_0^2(x) v_0(x)] ds$$

$$= v_0 + Ft + d_2 \frac{\partial^2 v_0}{\partial x^2} t - F v_0 t - u_0^2 v_0 t$$

$$\therefore v_1(t,x) = v_0 + [F + d_2 \frac{\partial^2 v_0}{\partial x^2} - F v_0 - u_0^2 v_0] t \qquad (22)$$

For n=2

$$u_2(t,x) = u_0 + d_1\int_0^t \frac{\partial^2 u_1(s,x)}{\partial x^2} ds - (F+k)\int_0^t u_1(s,x) ds + \int_0^t [u_1^2(s,x) v_1(s,x)] ds$$

By eqs (4) and (5)

$$u_2(t,x) = u_0 + d_1\int_0^t \frac{\partial^2}{\partial x^2}(u_0 + [d_1 \frac{\partial^2 u_0}{\partial x^2} - (F+k) u_0 + u_0^2 v_0] s) ds - (F+k)\int_0^t (u_0 + [d_1 \frac{\partial^2 u_0}{\partial x^2}$$
$$- (F+k) u_0 + u_0^2 v_0] s) ds + \int_0^t u_1^2 v_1 ds$$

$$= u_0 + d_1 \frac{\partial^2 u_0}{\partial x^2} t + d_1 [d_1 \frac{\partial^4 u_0}{\partial x^4} - (F+k) \frac{\partial^2 u_0}{\partial x^2} + \frac{\partial^2}{\partial x^2}(u_0^2 v_0)] \frac{t^2}{2} - (F+k) u_0 t - (F+k)[d_1 \frac{\partial^2 u_0}{\partial x^2}$$
$$- (F+k) u_0 + u_0^2 v_0] \frac{t^2}{2} + \int_0^t (u_1^2 v_1) ds \qquad . \qquad (23)$$

$$\int_0^t (u_1^2 v_1)ds = \int_0^t (u_0 + [d_1 \frac{\partial^2 u_0}{\partial x^2} - (F+k)u_0 + u_0^2 v_0]s)^2 (v_0 + [F + d_2 \frac{\partial^2 v_0}{\partial x^2} - Fv_0 - u_0^2 v_0]s)ds$$

$$= \int_0^t (u_0^2 + 2u_0[d_1 \frac{\partial^2 u_0}{\partial x^2} - (F+k)u_0 + u_0^2 v_0]s + [d_1 \frac{\partial^2 u_0}{\partial x^2} - (F+k)u_0 + u_0^2 v_0]^2 s^2)(v_0$$

$$+ [F + d_2 \frac{\partial^2 v_0}{\partial x^2} - Fv_0 - u_0^2 v_0]s)ds$$

$$= \int_0^t (u_0^2 v_0 + 2u_0 v_0 [d_1 \frac{\partial^2 u_0}{\partial x^2} - (F+k)u_0 + u_0^2 v_0]s + v_0[d_1 \frac{\partial^2 u_0}{\partial x^2} - (F+k)u_0 + u_0^2 v_0]^2 s^2$$

$$+ u_0^2[F + d_2 \frac{\partial^2 v_0}{\partial x^2} - Fv_0 - u_0^2 v_0]s + 2u_0[d_1 \frac{\partial^2 u_0}{\partial x^2} - (F+k)u_0 + u_0^2 v_0][F + d_2 \frac{\partial^2 v_0}{\partial x^2} - Fv_0 - u_0^2 v_0]s^2$$

$$+ [d_1 \frac{\partial^2 u_0}{\partial x^2} - (F+k)u_0 + u_0^2 v_0]^2 [F + d_2 \frac{\partial^2 v_0}{\partial x^2} - Fv_0 - u_0^2 v_0]s^3)ds$$

$$\therefore \int_0^t (u_1^2 v_1)ds = u_0^2 v_0 t + 2u_0 v_0 [d_1 \frac{\partial^2 u_0}{\partial x^2} - (F+k)u_0 + u_0^2 v_0]\frac{t^2}{2} + v_0[d_1 \frac{\partial^2 u_0}{\partial x^2} - (F+k)u_0 + u_0^2 v_0]^2 \frac{t^3}{3}$$

$$+ u_0^2[F + d_2 \frac{\partial^2 v_0}{\partial x^2} - Fv_0 - u_0^2 v_0]\frac{t^2}{2} + 2u_0[d_1 \frac{\partial^2 u_0}{\partial x^2} - (F+k)u_0 + u_0^2 v_0][F + d_2 \frac{\partial^2 v_0}{\partial x^2} - Fv_0 - u_0^2 v_0]\frac{t^3}{3}$$

$$+ [d_1 \frac{\partial^2 u_0}{\partial x^2} - (F+k)u_0 + u_0^2 v_0]^2 [F + d_2 \frac{\partial^2 v_0}{\partial x^2} - Fv_0 - u_0^2 v_0]\frac{t^4}{4} \qquad (24)$$

Substitute eq. (24) in (23) we get :

$$u_2(t,x) = u_0 + d_1 \frac{\partial^2 u_0}{\partial x^2} t + d_1[d_1 \frac{\partial^4 u_0}{\partial x^4} - (F+k)\frac{\partial^2 u_0}{\partial x^2} + \frac{\partial^2}{\partial x^2}(u_0^2 v_0)]\frac{t^2}{2} - (F+k)u_0 t$$

$$- (F+k)[d_1 \frac{\partial^2 u_0}{\partial x^2} - (F+k)u_0 + u_0^2 v_0]\frac{t^2}{2} + u_0^2 v_0 t + 2u_0 v_0[d_1 \frac{\partial^2 u_0}{\partial x^2} - (F+k)u_0 + u_0^2 v_0]\frac{t^2}{2}$$

$$+ v_0[d_1 \frac{\partial^2 u_0}{\partial x^2} - (F+k)u_0 + u_0^2 v_0]^2 \frac{t^3}{3} + u_0^2[F + d_2 \frac{\partial^2 v_0}{\partial x^2} - Fv_0 - u_0^2 v_0]\frac{t^2}{2} + 2u_0[d_1 \frac{\partial^2 u_0}{\partial x^2}$$

$$- (F+k)u_0 + u_0^2 v_0][F + d_2 \frac{\partial^2 v_0}{\partial x^2} - Fv_0 - u_0^2 v_0]\frac{t^3}{3} + [d_1 \frac{\partial^2 u_0}{\partial x^2} - (F+k)u_0 + u_0^2 v_0]^2[F + d_2 \frac{\partial^2 v_0}{\partial x^2}$$

$$- Fv_0 - u_0^2 v_0]\frac{t^4}{4}$$

$$\therefore u_2(t,x) = u_0 + [d_1 \frac{\partial^2 u_0}{\partial x^2} - (F+k)u_0 + u_0^2 v_0]t + [d_1 \frac{\partial^2}{\partial x^2}(d_1 \frac{\partial^2 u_0}{\partial x^2} - (F+k)u_0 + u_0^2 v_0)$$

$$- (F+k)(d_1 \frac{\partial^2 u_0}{\partial x^2} - (F+k)u_0 + u_0^2 v_0) + 2u_0 v_0(d_1 \frac{\partial^2 u_0}{\partial x^2} - (F+k)u_0 + u_0^2 v_0) + u_0^2(F + d_2 \frac{\partial^2 v_0}{\partial x^2}$$

$$- Fv_0 - u_0^2 v_0)]\frac{t^2}{2} + [v_0(d_1 \frac{\partial^2 u_0}{\partial x^2} - (F+k)u_0 + u_0^2 v_0)^2 + 2u_0(d_1 \frac{\partial^2 u_0}{\partial x^2} - (F+k)u_0 + u_0^2 v_0)(F$$

$$+ d_2 \frac{\partial^2 v_0}{\partial x^2} - Fv_0 - u_0^2 v_0)]\frac{t^3}{3} + [d_1 \frac{\partial^2 u_0}{\partial x^2} - (F+k)u_0 + u_0^2 v_0]^2[F + d_2 \frac{\partial^2 v_0}{\partial x^2} - Fv_0 - u_0^2 v_0]\frac{t^4}{4}$$

For simplicity we assume that:

$$U1 = d_1 \frac{\partial^2 u_0}{\partial x^2} - (F+k)u_0 + u_0^2 v_0 \qquad (25)$$

$$V1 = F + d_2 \frac{\partial^2 v_0}{\partial x^2} - Fv_0 - u_0^2 v_0 \qquad (26)$$

Then

$$\therefore u_2(t,x) = u_0 + U1t + [d_1 \frac{\partial^2}{\partial x^2} U1 - (F+k)U1 + 2u_0 v_0 U1 + u_0^2 V1]\frac{t^2}{2} + [v_0 (U1)^2 + 2u_0 U1V1]\frac{t^3}{3}$$
$$+ (U1)^2 V1 \frac{t^4}{4} \qquad (27)$$

$$v_2(t,x) = v_0 + Ft + d_2 \int_0^t \frac{\partial^2 v_1(s,x)}{\partial x^2} ds - F\int_0^t v_1(s,x) ds - \int_0^t [u_1^2(s,x) v_1(s,x)] ds$$
$$= v_0 + Ft + d_2 \int_0^t \frac{\partial^2}{\partial x^2}(v_0 + [F + d_2 \frac{\partial^2 v_0}{\partial x^2} - Fv_0 - u_0^2 v_0]s) ds - F\int_0^t (v_0 + [F + d_2 \frac{\partial^2 v_0}{\partial x^2} - Fv_0$$
$$- u_0^2 v_0]s) ds - \int_0^t u_1^2 v_1 ds$$
$$= v_0 + Ft + d_2 \frac{\partial^2 v_0}{\partial x^2} t + d_2 \frac{\partial^2}{\partial x^2}(F + d_2 \frac{\partial^2 v_0}{\partial x^2} - Fv_0 - u_0^2 v_0)\frac{t^2}{2} - Fv_0 t - F[F + d_2 \frac{\partial^2 v_0}{\partial x^2}$$
$$- Fv_0 - u_0^2 v_0]\frac{t^2}{2} - \int_0^t u_1^2 v_1 ds$$

Then by eq. (24)

$$v_2(t,x) = v_0 + Ft + d_2 \frac{\partial^2 v_0}{\partial x^2} t + d_2 \frac{\partial^2}{\partial x^2}(F + d_2 \frac{\partial^2 v_0}{\partial x^2} - Fv_0 - u_0^2 v_0)\frac{t^2}{2} - Fv_0 t - F[F + d_2 \frac{\partial^2 v_0}{\partial x^2}$$
$$- Fv_0 - u_0^2 v_0]\frac{t^2}{2} - u_0^2 v_0 t - 2u_0 v_0 [d_1 \frac{\partial^2 u_0}{\partial x^2} - (F+k)u_0 + u_0^2 v_0]\frac{t^2}{2} - v_0 [d_1 \frac{\partial^2 u_0}{\partial x^2} - (F+k)u_0$$
$$+ u_0^2 v_0]^2 \frac{t^3}{3} - u_0^2 [F + d_2 \frac{\partial^2 v_0}{\partial x^2} - Fv_0 - u_0^2 v_0]\frac{t^2}{2} - 2u_0 [d_1 \frac{\partial^2 u_0}{\partial x^2} - (F+k)u_0 + u_0^2 v_0][F + d_2 \frac{\partial^2 v_0}{\partial x^2}$$
$$- Fv_0 - u_0^2 v_0]\frac{t^3}{3} - [d_1 \frac{\partial^2 u_0}{\partial x^2} - (F+k)u_0 + u_0^2 v_0]^2 [F + d_2 \frac{\partial^2 v_0}{\partial x^2} - Fv_0 - u_0^2 v_0]\frac{t^4}{4}$$
$$= v_0 + [F + d_2 \frac{\partial^2 v_0}{\partial x^2} - Fv_0 - u_0^2 v_0]t + [d_2 \frac{\partial^2}{\partial x^2}(F + d_2 \frac{\partial^2 v_0}{\partial x^2} - Fv_0 - u_0^2 v_0) - F(F + d_2 \frac{\partial^2 v_0}{\partial x^2}$$
$$- Fv_0 - u_0^2 v_0) - 2u_0 v_0 (d_1 \frac{\partial^2 u_0}{\partial x^2} - (F+k)u_0 + u_0^2 v_0) - u_0^2 (F + d_2 \frac{\partial^2 v_0}{\partial x^2} - Fv_0 - u_0^2 v_0)]\frac{t^2}{2}$$
$$-[v_0 (d_1 \frac{\partial^2 u_0}{\partial x^2} - (F+k)u_0 + u_0^2 v_0)^2 + 2u_0 (d_1 \frac{\partial^2 u_0}{\partial x^2} - (F+k)u_0 + u_0^2 v_0)(F + d_2 \frac{\partial^2 v_0}{\partial x^2} - Fv_0 - u_0^2 v_0)]\frac{t^3}{3}$$
$$-[d_1 \frac{\partial^2 u_0}{\partial x^2} - (F+k)u_0 + u_0^2 v_0]^2 [F + d_2 \frac{\partial^2 v_0}{\partial x^2} - Fv_0 - u_0^2 v_0]\frac{t^4}{4}$$

Then by eqs. (25) and (26)

$$v_2(t,x) = v_0 + V1t + [d_2 \frac{\partial^2}{\partial x^2}(V1) - FV1 - 2u_0v_0U1 - u_0^2V1]\frac{t^2}{2} - [v_0(U1)^2 + 2u_0U1V1]\frac{t^3}{3} - (U1)^2V1\frac{t^4}{4} \quad (28)$$

And by the same way for n=3,4,………

Numerical Example

We solved the following example numerically to illustrate efficiency of the presented methods

***Example*:**

$$\frac{\partial u}{\partial t} = d_1\Delta u - (F+k)u + u^2v \quad , \ t>0 \quad x \in \Omega$$

$$\frac{\partial v}{\partial t} = d_1\Delta v + F(1-v) - u^2v \quad , \ t>0 \quad x \in \Omega$$

We the initial conditions

$U(x, 0) = Us + 0.01 \sin(\pi x/L)$ for $0 \le x \le L$

$V(x, 0) = Vs - 0.12 \sin(\pi x/L)$ for $0 \le x \le L$

$U(0, t) = Us$, $U(L, t) = Us$ and $V(0, t) = Vs$, $V(L, t) = Vs$

We will take

$d_1=d_2=0.01$, F= 0.09 , k=-0.004 , $U_s=0$, $V_s=1$

Table 1 Comparison between the FDM and SAM for the values of concentration V.

	t=1		t=2		t=3	
X	SAM	FDM	SAM	FDM	SAM	FDM
0	1.0000	1.0000	1.0001	1.0000	1.0003	1.0000
0.1	1.0015	1.0014	1.0015	1.0014	1.0017	1.0013
0.2	1.0028	1.0027	1.0027	1.0026	1.0028	1.0024
0.3	1.0039	1.0039	1.0037	1.0035	1.0036	1.0032
0.4	1.0048	1.0048	1.0044	1.0043	1.0040	1.0038
0.5	1.0055	1.0055	1.0049	1.0048	1.0042	1.0041
0.6	1.0061	1.0061	1.0052	1.0052	1.0042	1.0043
0.7	1.0065	1.0065	1.0053	1.0054	1.0041	1.0044
0.8	1.0068	1.0068	1.0054	1.0056	1.0040	1.0044
0.9	1.0069	1.0070	1.0054	1.0056	1.0038	1.0044
1.0	1.0070	1.0070	1.0054	1.0057	1.0038	1.0044
1.1	1.0069	1.0070	1.0054	1.0056	1.0038	1.0044
1.2	1.0068	1.0068	1.0054	1.0056	1.0040	1.0044
1.3	1.0065	1.0065	1.0053	1.0054	1.0041	1.0044
1.4	1.0061	1.0061	1.0052	1.0052	1.0042	1.0043
1.5	1.0055	1.0055	1.0049	1.0048	1.0042	1.0041
1.6	1.0048	1.0048	1.0044	1.0043	1.0040	1.0038
1.7	1.0039	1.0039	1.0037	1.0035	1.0036	1.0032
1.8	1.0028	1.0027	1.0027	1.0026	1.0028	1.0024
1.9	1.0015	1.0014	1.0015	1.0014	1.0017	1.0013
2.0	1.0000	1.0000	1.0001	1.0000	1.0003	1.0000

Table 2 Comparison between the FDM and SAM for the values of concentration U.

	t=1		t=2		t=3	
x	SAM	FDM	SAM	FDM	SAM	FDM
0	0.0003	0	0.0012	0	0.0025	0
0.1	-0.0162	-0.0163	-0.0136	-0.0139	-0.0113	-0.0119
0.2	-0.0318	-0.0317	-0.0271	-0.0271	-0.0236	-0.0230
0.3	-0.0461	-0.0461	-0.0392	-0.0390	-0.0340	-0.0331
0.4	-0.0589	-0.0589	-0.0497	-0.0495	-0.0424	-0.0419
0.5	-0.0701	-0.0700	-0.0585	-0.0585	-0.0486	-0.0493
0.6	-0.0793	-0.0793	-0.0656	-0.0659	-0.0527	-0.0553
0.7	-0.0867	-0.0867	-0.0709	-0.0716	-0.0551	-0.0600
0.8	-0.0920	-0.0920	-0.0746	-0.0757	-0.0561	-0.0633
0.9	-0.0952	-0.0952	-0.0768	-0.0782	-0.0565	-0.0653
1.0	-0.0962	-0.0962	-0.0775	-0.0790	-0.0565	-0.0659
1.1	-0.0952	-0.0952	-0.0768	-0.0782	-0.0565	-0.0653
1.2	-0.0920	-0.0920	-0.0746	-0.0757	-0.0561	-0.0633
1.3	-0.0867	-0.0867	-0.0709	-0.0716	-0.0551	-0.0600
1.4	-0.0793	-0.0793	-0.0656	-0.0659	-0.0527	-0.0553
1.5	-0.0701	-0.0700	-0.0585	-0.0585	-0.0486	-0.0493
1.6	-0.0589	-0.0589	-0.0497	-0.0495	-0.0424	-0.0419
1.7	-0.0461	-0.0461	-0.0392	-0.0390	-0.0340	-0.0331
1.8	-0.0318	-0.0317	-0.0271	-0.0271	-0.0236	-0.0230
1.9	-0.0162	-0.0163	-0.0136	-0.0139	-0.0113	-0.0119
2.0	0.0003	-0.0000	0.0012	-0.0000	0.0025	-0.0000

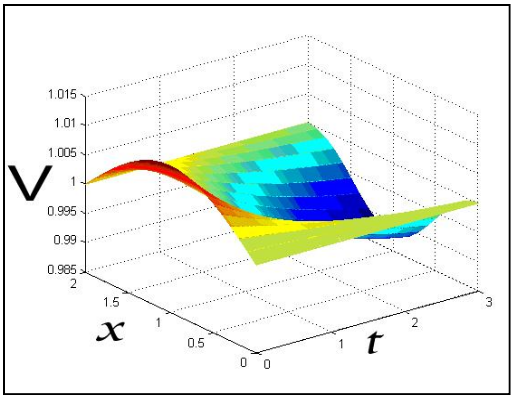

Fig. 1 FDM for the values of concentration V with $0<x<2$ and $0<t<3$

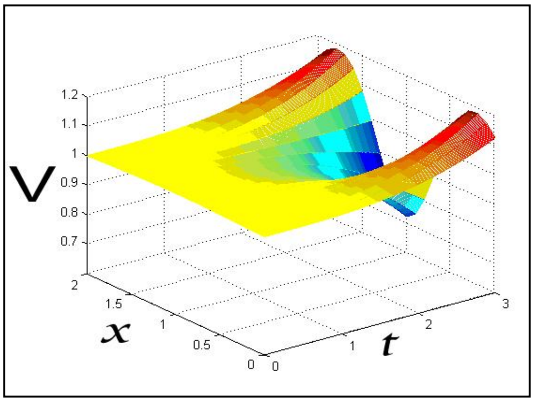

Fig. 2 SAM for the values of concentration V with $0<x<2$ and $0<t<3$

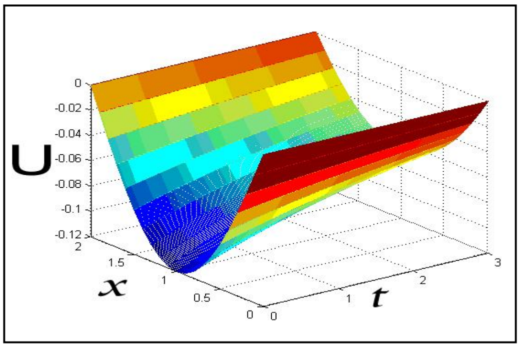

Fig. 3 FDM for the values of concentration U with $0<x<2$ and $0<t<3$

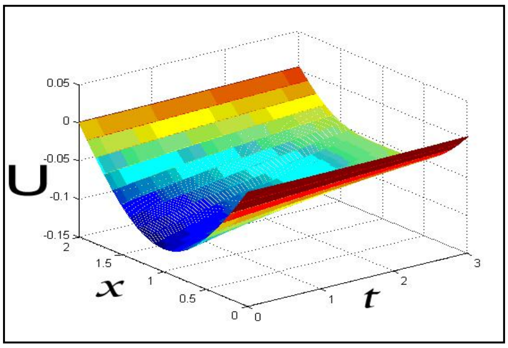

Fig. 4 SAM for the values of concentration U with $0<x<2$ and $0<t<3$

Conclusion:

We saw that Successive approximation method is more accurate than finite difference method for solving Gray-Scott model especially when we increase t as shown in figure 1-4 and tables 1-2.

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الملخص

في هذا البحث تم حل نموذج كري سكوت عدديا لايجاد الحل التقريبي بطريقتين التقريبات الناجحة والفروقات المنتهية. تم حل مثال وتبين ان طريقة التقريبات الناجحة اسرع واكثر دقة من طريقة الفروقات المنتهية.