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EXISTENCE, UNIQUENESS AND STABILITY OF PERIODIC SOLUTION FOR NONLINEAR SYSTEM OF INTEGRO-DIFFERENTIAL EQUATIONS

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ABSTRACT:

In this paper, we investigate the existence, uniqueness and stability of the periodic solution for the system of nonlinear integro-differential equations by using the numerical-analytic methods for investigate the solutions and the periodic solutions of ordinary differential equations, which are given by A. Samoilenko.

KEYWORDS: Existence, Uniqueness & Stability solution; Periodic Solution; Nonlinear System of Integro-differential Equations, Numerical-Analytic Methods.

1. INTRODUCTION

Analysis of various processes in mechanics, physics, biology and celestial mechanics, and other branches of science and engineering requires investigation into the existence and uniqueness solutions of differential equations of various types and systems (Apostol, 1973; Burrill, 1969). In particular, there developed many problems of the existence of periodic solutions and construction business. Among them, one should mention especially numerical-analytical method of successive approximations (Samoilenko, 1965; Tidke & More,2015).

The conditions of existence solutions that are important in the theories of analysis and results are often obtained by using fixed-point theorems (Banach, Schauder) and successive approximation (Hu & Li,2005; Rama, 1981). The study theories about the existence, uniqueness and stability of differential equations (David & Williams, 2015; Korol, 2010). There are many details to the investigation of different aspects of this theory (see, for example, (Butriset al., 2015; Byszewski et al., 1997; Korol, 2009). Previously, Samoilenko in many original works referred to as the numerical-analytic method based upon successive approximations (Samoilenko, 1965; Samoilenko, 1966). Tidke and More (Tidke & More, 2015), investigated the existence and uniqueness of solution of Volterra integro-differential equation with nonlocal condition in cone metric space. The result is obtained by using some extensions of Banach's contraction principle in complete cone metric space of the form.

$$x'(t) = A(t)x(t) + f\big(t, x(t)\big) + \int_0^t k(s, x(s))ds\ , \quad t \in J = [0, b] \text{ with } x(0) + g(x) = x_0 \quad \ldots (P1)$$

In (Aziz, 2006) Aziz, studied the periodic solutions for some systems of nonlinear ordinary differential equations which has the form:

$$\left.\begin{aligned} \frac{dx(t)}{dt} &= (A + B(t))x + f(t, x, y) \\ \frac{dy(t)}{dt} &= (C + D(t))y + g(t, x, y) \end{aligned}\right\} \qquad (P2)$$

In this work, we study the existence, uniqueness and stability of solution of nonlinear system of integro-differential equations which has the form:-

$$\left.\begin{aligned} \frac{dx}{dt} &= Ax + f\Big(t, \varepsilon, \int_{-\infty}^{t} R(t-\tau)\big(x(\tau) - y(\tau)\big)d\tau\Big) \\ \frac{dy}{dt} &= By + g\Big(t, \mu, \int_{-\infty}^{t} G(t-\tau)\big(x(\tau) - y(\tau)\big)d\tau\Big) \end{aligned}\right\}, \qquad (1.1)$$

and also investigate the existence and uniqueness of periodic solution of (1.1), where $\varepsilon = \int_a^b \lambda\big(x(\tau) - y(\tau)\big)d\tau$, $\mu = \int_c^d \gamma\big(x(\tau) - y(\tau)\big)d\tau$, λ and γ are constants, and $x \in D \subset R^n$; $y \in D_1 \subset R^n$, D and D_1 are closed and bounded domain. Assume that the vector functions $f(t, \varepsilon, u)$ and $g(t, \mu, v)$ where $u = \int_{-\infty}^{t} R(t-\tau)\big(x(\tau) - y(\tau)\big)d\tau$ and $v = \int_{-\infty}^{t} G(t-\tau)\big(x(\tau) - y(\tau)\big)d\tau$, are defined on the domain.

$$[t, x, y] \in R^1 \times D \times D_1, \qquad (1.2)$$

and continuous functions in t, x, y and satisfy the following inequalities:-

$$\left.\begin{aligned} &\|f(t, \varepsilon, u)\| \le M \\ &\|g(t, \mu, v)\| \le N \\ &\|f(t, \varepsilon_1, u_1) - f(t, \varepsilon_2, u_2)\| \le j_1\|\varepsilon_1 - \varepsilon_2\| + j_2\|u_1 - u_2\| \\ &\|g(t, \mu_1, v_1) - g(t, \mu_2, v_2)\| \le k_1\|\mu_1 - \mu_2\| + k_2\|v_1 - v_2\| \end{aligned}\right\} \qquad (1.3)$$

for all $t \in R^1$, $x \in D$, $y \in D_1$, where $M, N, j_1, j_2, k_1,$ and k_2 are positive constants. Suppose that the positive matrices A and B satisfy the condition:

$$\left.\begin{aligned} \big\|e^{A(t-\tau)}\big\| &\le Q_1 \\ \big\|e^{B(t-\tau)}\big\| &\le Q_2 \end{aligned}\right\} \qquad (1.4)$$

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where $Q_1$and Q_2 are positive constant. Also, the functions $R(t-\tau)$ and $G(t-\tau)$ satisfying conditions

$$\left.\begin{aligned} \|R(t-\tau)\| &\le \frac{he^{-\alpha(t-\tau)}}{(t-\tau)^{1-\alpha}} \\ \|\mathrm{G}(\mathrm{t}-\mathrm{s})\| &\le \frac{\sigma e^{-\beta(t-\tau)}}{(t-\tau)^{1-\beta}} \end{aligned}\right\} \quad (1.5)$$

where $-\infty \le 0 \le \tau \le t \le T \le \infty, 0 \le \alpha \le 1, 0 \le \beta \le 1$, h and σ are positive constants, and the integrals of $\|R(t-\tau)\|$ and $\|G(t-\tau)\|$, and by using gamma function,

we can find that

$\int_{-\infty}^{t}\|R(t-\tau)\|d\tau = p_\alpha$ and $\int_{-\infty}^{t}\|\mathrm{G}(\mathrm{t}-\mathrm{s})\|d\tau = p_\beta$

where $p_\alpha = \alpha^{1-\alpha}h\Gamma(a)$ and $p_\beta = \beta^{1-\beta}\sigma\Gamma(\beta)$.

The nonempty sets are as follow:-

$$\left.\begin{aligned} D_f &= D_0 - (I_1 + TQ_1M) \\ D_g &= D_1 - (I_2 + TQ_2N) \end{aligned}\right\}, \quad (1.6)$$

where $I_1 = \|x_0\|(\|e^{At}\| + \|E\|)$ and $I_2 = \|y_0\|(\|e^{Bt}\| + \|E\|)$

Moreover, assume that the maximum eigen-value of the matrix $\Lambda = \begin{pmatrix} TQ_1p_\alpha^\lambda & TQ_1p_\alpha^\lambda \\ TQ_2p_\beta^\gamma & TQ_2p_\beta^\gamma \end{pmatrix}$ is less than unity, i.e.

$$TQ_1p_\alpha^\lambda + TQ_2p_\beta^\gamma < 1. \quad (1.7)$$

2. EXISTENCE, UNIQUENESS AND STABILITY SOLUTION OF THE SYSTEM (1.1)

Theorem 2.1 (Existence theorem). Let the system (1.1) satisfy the inequalities (1.3)-(1.5) and the condition (1.6) and (1.7). Then the sequences of vectors functions $\{x_m(t)\}_{m=0}^\infty$ and $\{y_m(t)\}_{m=0}^\infty$ are defined by the following:

$$x_{m+1}(t,x_0,y_0) = x_0e^{At} + \int_0^t e^{A(t-\tau)}f(t,\varepsilon_{m+1},u_{m+1})d\tau, x(0) = x_0, \mathrm{m} = 0,1,2,..., \quad (2.1)$$

and,

$$y_{m+1}(t,x_0,y_0) = y_0e^{Bt} + \int_0^t e^{B(t-\tau)}g(t,\mu_{m+1},v_{m+1})d\tau, y(0) = y_0, \mathrm{m} = 0,1,2,..., \quad (2.2)$$

where $\varepsilon_{m+1} = \int_a^b \lambda\big(x_m(\tau,x_0,y_0) - y_m(\tau,x_0,y_0)\big)d\tau$, $\mu_{m+1} = \int_c^d \gamma\big(x_m(\tau,x_0,y_0) - y_m(\tau,x_0,y_0)\big)d\tau$, $u_{m+1} = \int_{-\infty}^t R(t-\tau)\big(x_m(\tau,x_0,y_0) - y_m(\tau,x_0,y_0)\big)d\tau$, $v_{m+1} = \int_{-\infty}^t G(t-\tau)\big(x_m(\tau,x_0,y_0) - y_m(\tau,x_0,y_0)\big)d\tau$,

convergent uniformly when $m \to \infty$ in the domain (1.2) to the limit vectors functions $x^0(t,x_0,y_0)$ and $y^0(t,x_0,y_0)$ which are satisfying the following integral equations:-

$$x(t,x_0,y_0) = x_0e^{At} + \int_0^t e^{A(t-\tau)}f(t,\varepsilon,u)d\tau, x(0) = x_0 \quad (2.3)$$

$$y(t,x_0,y_0) = y_0e^{Bt} + \int_0^t e^{B(t-\tau)}g(t,\mu,v)d\tau, y(0) = y_0 \quad (2.4)$$

where $\varepsilon = \int_a^b \lambda\big(x(\tau,x_0,y_0) - y(\tau,x_0,y_0)\big)d\tau$,

$$\mu = \int_c^d \gamma\big(x(\tau,x_0,y_0) - y(\tau,x_0,y_0)\big)d\tau,$$

$$u = \int_{-\infty}^t R(t-\tau)\big(x(\tau,x_0,y_0) - y(\tau,x_0,y_0)\big)d\tau,$$

$$v = \int_{-\infty}^t G(t-\tau)\big(x(\tau,x_0,y_0) - y(\tau,x_0,y_0)\big)d\tau,$$

and it's a unique solution to the system (1.1), provided that

$$\begin{pmatrix} \|x^0(t,x_0,y_0) - x_m(t,x_0,y_0)\| \\ \|y^0(t,x_0,y_0) - y_m(t,x_0,y_0)\| \end{pmatrix} \le \Lambda^m(E-\Lambda)^{-1}W_0 \quad (2.5)$$

where $W_0 = \begin{pmatrix} I_1 + TQ_1M \\ I_2 + TQ_2N \end{pmatrix}$,

$\Lambda^m = \begin{pmatrix} TQ_1p_\alpha^\lambda & TQ_1p_\alpha^\lambda \\ TQ_2p_\beta^\gamma & TQ_2p_\beta^\gamma \end{pmatrix}^m$, E is an identity matrix, for all $m \ge 1$ and $t \in R^1$.

Proof: From (2.1) and (2.2) when m=0, we find that:

$\|x_1(t,x_0,y_0) - x_0\| \le I_1 + TQ_1M$, for all $t \in R^1$, $x_0 \in D_f$.

$\|y_1(t,x_0,y_0) - y_0\| \le I_2 + TQ_2N$, for all $t \in R^1$, $y_0 \in D_g$.

By mathematical induction and from (2.1) and (2.2) , we get:-

$$\|x_\mathrm{m}(t,x_0,y_0) - x_0\| \le I_1 + TQ_1M \quad (2.6)$$

$$\|y_\mathrm{m}(t,x_0,y_0) - y_0\| \le I_2 + TQ_2N \quad (2.7)$$

i. e. $x_m(t) \in D_0$, $y_m(t) \in D_1$, $x_0 \in D_f$, $y_0 \in D_g$ for all $t \in R^1$.

After that, we shall prove the sequences of vectors functions from (2.1) and (2.2) uniformly converges on domain(1.2), when $\mathrm{m} > 1$, we have

$$\|x_2(t,x_0,y_0) - x_1(t,x_0,y_0)\| \le TQ_1p_\alpha^\lambda(\|x_1(t,x_0,y_0) - x_0\| + \|y_1(t,x_0,y_0) - y_0\|) \quad (2.8)$$

where $p_\alpha^\lambda = j_1\lambda(b-a) + j_2p_\alpha$

Similarly

$$\|y_2(t,x_0,y_0) - y_1(t,x_0,y_0)\| \le TQ_2p_\beta^\gamma(\|x_1(t,x_0,y_0) - x_0\| + \|y_1(t,x_0,y_0) - y_0\|) \quad (2.9)$$

Where $p_\beta^\gamma = k_1\gamma(d-c) + k_2p_\beta$

By mathematical induction, we can get:-

$$\|x_{m+1}(t,x_0,y_0) - x_m(t,x_0,y_0)\| \le TQ_1p_\alpha^\lambda(\|x_m(t,x_0,y_0) - x_{m-1}(t,x_0,y_0)\| + \|y_m(t,x_0,y_0) - y_{m-1}(t,x_0,y_0)\|) \quad (2.10)$$

and

$$\|y_{m+1}(t,x_0,y_0) - y_m(t,x_0,y_0)\| \le TQ_2p_\beta^\gamma(\|x_m(t,x_0,y_0) - x_{m-1}(t,x_0,y_0)\| + \|y_m(t,x_0,y_0) - y_{m-1}(t,x_0,y_0)\|) \quad (2.11)$$

We rewrite the inequalities (2.10) and (2.11) in vector form, we find that

$$W_{m+1}(t) \le \Lambda\, W_m(t) \quad (2.12)$$

where $\mathrm{W_{m+1}(t)} = \begin{pmatrix} \|x_{m+1}(t,x_0,y_0) - x_m(t,x_0,y_0)\| \\ \|y_{m+1}(t,x_0,y_0) - y_m(t,x_0,y_0)\| \end{pmatrix}$,

$\mathrm{W_m(t)} = \begin{pmatrix} \|x_m(t,x_0,y_0) - x_{m-1}(t,x_0,y_0)\| \\ \|y_m(t,x_0,y_0) - y_{m-1}(t,x_0,y_0)\| \end{pmatrix}$, and

$$\Lambda = \begin{pmatrix} TQ_1 p_\alpha^\lambda & TQ_1 p_\alpha^\lambda \\ TQ_2 p_\beta^\gamma & TQ_2 p_\beta^\gamma \end{pmatrix}$$

Also from (2.12). We get

$$W_{m+1} \le \Lambda^m W_0 \qquad (2.13)$$

and hence

$$\sum_{j=1}^{m} W_j \le \sum_{j=1}^{m} \Lambda^{j-1} W_0 \qquad (2.14)$$

By condition (1.7), the inequality (2.14) is uniformly convergent when, $m \to \infty$, and

$$\lim_{m\to\infty} \sum_{j=1}^{m} \Lambda^{j-1} W_0 = \sum_{j=1}^{\infty} \Lambda^{j-1} W_0 = (E-\Lambda)^{-1} W_0 \qquad (2.15)$$

Thus the sequence of vectors functions (2.1) and (2.1) are uniformly convergent. Suppose that

$$\left.\begin{aligned} \lim_{m\to\infty} x_m(t,x_0,y_0) &= x^0(t,x_0,y_0) \\ \lim_{m\to\infty} y_m(t,x_0,y_0) &= y^0(t,x_0,y_0) \end{aligned}\right\} \qquad (2.16)$$

Since the sequences of function (2.1) and (2.2) are defined and continuous in the domain, $(\mathrm{t,x,y}) \in [0,\mathrm{T}] \times \mathrm{D} \times \mathrm{D}$, then the limiting functions $x^0(t,x_0,y_0)$ and $y^0(t,x_0,y_0)$ are also defined and continuous in the same domain.

Moreover, by (2.15) and proceeding (2.1) and (2.2) to the limit when $m \to \infty$ this show that the limiting functions $x^0(t,x_0,y_0)$ and $y^0(t,x_0,y_0)$ are the solutions of the integral equations (2.3) and (2.4).

Finally, we have to show that the solutions of system (1.1) are unique, suppose that $\tilde{\mathrm{x}}(t,x_0,y_0)$ and $\tilde{\mathrm{y}}(t,x_0,y_0)$ be another solution to the system (1.1), then

$$\tilde{x}(t,x_0,y_0) = x_0 e^{At} + \int_0^t e^{A(t-\tau)} f(t,\tilde{\varepsilon},\tilde{\mathrm{u}})\,d\tau, \qquad (2.17)$$

$$\tilde{y}(t,x_0,y_0) = y_0 e^{Bt} + \int_0^t e^{B(t-\tau)} g(t,\tilde{\mu},\tilde{v})\,d\tau, \qquad (2.18)$$

where

$$\tilde{\varepsilon} = \int_a^b \lambda\big(\tilde{\mathrm{x}}(\tau,x_0,y_0) - \tilde{\mathrm{y}}(\tau,x_0,y_0)\big)d\tau\,;\ \tilde{\mathrm{u}} = \int_{-\infty}^{t} R(t-\tau)\big(\tilde{\mathrm{x}}(\tau,x_0,y_0) - \tilde{\mathrm{y}}(\tau,x_0,y_0)\big)d\tau,$$

$$\tilde{\mu} = \int_c^d \gamma\big(\tilde{\mathrm{x}}(\tau,x_0,y_0) - \tilde{\mathrm{y}}(\tau,x_0,y_0)\big)d\tau\,;\ \tilde{v} = \int_{-\infty}^{t} G(t-\tau)\big(\tilde{\mathrm{x}}(\tau,x_0,y_0) - \tilde{\mathrm{y}}(\tau,x_0,y_0)\big)d\tau,$$

Then,

$$\begin{aligned} \|\mathrm{x}(\mathrm{t},x_0,y_0) - \tilde{\mathrm{x}}(\mathrm{t},x_0,y_0)\| &\le TQ_1 p_\alpha^\lambda (\|x(\mathrm{t},x_0,y_0) - \tilde{\mathrm{x}}(\mathrm{t},x_0,y_0)\| + \|y(\mathrm{t},x_0,y_0) - \tilde{\mathrm{y}}(\mathrm{t},x_0,y_0)\|) \qquad (2.19) \\ \|\mathrm{y}(\mathrm{t},x_0,y_0) - \tilde{\mathrm{y}}(\mathrm{t},x_0,y_0)\| &\le TQ_2 p_\beta^\gamma (\|x(\mathrm{t},x_0,y_0) - \tilde{\mathrm{x}}(\mathrm{t},x_0,y_0)\| + \|y(\mathrm{t},x_0,y_0) - \tilde{\mathrm{y}}(\mathrm{t},x_0,y_0)\|) \qquad (2.20) \end{aligned}$$

Rewrite the inequalities (2.19) and (2.20) in a vector form, we get

$$\begin{pmatrix} \|\mathrm{x}(\mathrm{t},x_0,y_0) - \tilde{\mathrm{x}}(\mathrm{t},x_0,y_0)\| \\ \|\mathrm{y}(\mathrm{t},x_0,y_0) - \tilde{\mathrm{y}}(\mathrm{t},x_0,y_0)\| \end{pmatrix} \le \Lambda \begin{pmatrix} \|\mathrm{x}(,x_0,y_0\mathrm{t}) - \tilde{\mathrm{x}}(\mathrm{t},x_0,y_0)\| \\ \|\mathrm{y}(\mathrm{t},x_0,y_0) - \tilde{\mathrm{y}}(\mathrm{t},x_0,y_0)\| \end{pmatrix}. \qquad (2.21)$$

where $\Lambda = \begin{pmatrix} TQ_1 p_\alpha^\lambda & TQ_1 p_\alpha^\lambda \\ TQ_2 p_\beta^\gamma & TQ_2 p_\beta^\gamma \end{pmatrix}$

Also from (2.21), we have:

$$\begin{pmatrix} \|\mathrm{x}(\mathrm{t},x_0,y_0) - \tilde{\mathrm{x}}(\mathrm{t},x_0,y_0)\| \\ \|\mathrm{y}(\mathrm{t},x_0,y_0) - \tilde{\mathrm{y}}(\mathrm{t},x_0,y_0)\| \end{pmatrix} \le \Lambda^{\mathrm{m}} \begin{pmatrix} \|\mathrm{x}(\mathrm{t},x_0,y_0) - \tilde{\mathrm{x}}(\mathrm{t},x_0,y_0)\| \\ \|\mathrm{y}(\mathrm{t},x_0,y_0) - \tilde{\mathrm{y}}(\mathrm{t},x_0,y_0)\| \end{pmatrix}. \qquad (2.22)$$

From the condition(1.7), we conclude that $x(t,x_0,y_0) = \tilde{x}(t,x_0,y_0)$ and $y(t,x_0,y_0) = \tilde{y}(t,x_0,y_0)$, then the solutions of (1.1) are unique. Thus the proof is complete.

Theorem 2.2. Let $x(t,x_0,y_0)$ and $y(t,x_0,y_0)$ are solutions of the system (1.1), and all condition of the theorem 2.1 are satisfied, then the following inequalities holds

$$\begin{aligned} \|\mathrm{x}(\mathrm{t},x_0^1,y_0^1) - \mathrm{x}(\mathrm{t},x_0^2,y_0^2)\| &\le Q_1F_1F_3\|x_0^1 - x_0^2\| + TQ_1Q_1p_\alpha^\lambda F_1F_2F_3\|y_0^1 - y_0^2\| \qquad (2.23) \\ \|\mathrm{y}(\mathrm{t},x_0^1,y_0^1) - \mathrm{y}(\mathrm{t},x_0^2,y_0^2)\| &\le TQ_1Q_1p_\beta^\gamma F_1F_2F_3\|x_0^1 - x_0^2\| + Q_2F_2F_3\|y_0^1 - y_0^2\| \qquad (2.24) \end{aligned}$$

for all $t \in R^1$, $x_0^1, x_0^2 \in D_f$ and $y_0^1, y_0^2 \in D_g$ where $F_1 = (1 - TQ_1p_\alpha^\lambda)^{-1}$, $F_2 = (1 - TQ_2p_\beta^\gamma)^{-1}$ and $F_3 = (1 - T^2Q_1Q_1p_\alpha^\lambda p_\beta^\gamma F_1F_2)^{-1}$.

Proof: Let $x(t,x_0^1,y_0^1)$, $y(t,x_0^1,y_0^1)$ and $x(t,x_0^2,y_0^2)$, $y(t,x_0^2,y_0^2)$ are two solutions of the system (1.1), then

$$\|\mathrm{x}(\mathrm{t},x_0^1,y_0^1) - \mathrm{x}(\mathrm{t},x_0^2,y_0^2)\| \le \|x_0^1 - x_0^2\|Q_1 + TQ_1p_\alpha^\lambda(\|\mathrm{x}(\mathrm{t},x_0^1,y_0^1) - \mathrm{x}(\mathrm{t},x_0^2,y_0^2)\| + \|\mathrm{y}(\mathrm{t},x_0^1,y_0^1) - \mathrm{y}(\mathrm{t},x_0^2,y_0^2)\|) \qquad (2.25)$$

Or

$$\|\mathrm{x}(\mathrm{t},x_0^1,y_0^1) - \mathrm{x}(\mathrm{t},x_0^2,y_0^2)\| \le \|x_0^1 - x_0^2\|Q_1F_1 + TQ_1p_\alpha^\lambda F_1\|\mathrm{y}(\mathrm{t},x_0^1,y_0^1) - \mathrm{y}(\mathrm{t},x_0^2,y_0^2)\| \qquad (2.26)$$

and

$$\|\mathrm{y}(\mathrm{t},x_0^1,y_0^1) - \mathrm{y}(\mathrm{t},x_0^2,y_0^2)\| \le \|y_0^1 - y_0^2\|Q_2 + TQ_2p_\beta^\gamma(\|\mathrm{x}(\mathrm{t},x_0^1,y_0^1) - \mathrm{x}(\mathrm{t},x_0^2,y_0^2)\| + \|\mathrm{y}(\mathrm{t},x_0^1,y_0^1) - \mathrm{y}(\mathrm{t},x_0^2,y_0^2)\|) \qquad (2.27$$

or

$$\|\mathrm{y}(\mathrm{t},x_0^1,y_0^1) - \mathrm{y}(\mathrm{t},x_0^2,y_0^2)\| \le \|y_0^1 - y_0^2\|Q_2F_2 + TQ_2p_\beta^\gamma F_2\|\mathrm{x}(\mathrm{t},x_0^1,y_0^1) - \mathrm{x}(\mathrm{t},x_0^2,y_0^2)\| \qquad (2.28)$$

Substitute (2.28) in (2.26), we get (2.23) and substitute (2.26) in (2.28), we get (2.24). Hence, the proof is complete.

3. EXISTENCE, UNIQUENESS AND STABILITY OF PERIODIC SOLUTION OF THE SYSTEM (1.1)

In this section, we investigate the existence, uniqueness and stability of periodic solution of the problem of nonlinear system of differential equation (1.1).
Let the vector functions $f(t,\varepsilon,u)$ *and* $g(t,\mu,v)$ in (1.1) are defined on the domain

$$(t,x_0,y_0)\in[0,T]\times D_2\times D_3, \qquad (3.1)$$

where D_2 and D_3 are closed and bounded subsets of Euclidean spaces and $u=\int_{-\infty}^{t}R(t-\tau)\big(x(\tau,x_0,y_0)-y(\tau,x_0,y_0)\big)d\tau$, *and* $v=\int_{-\infty}^{t}G(t-\tau)\big(x(\tau,x_0,y_0)-y(\tau,x_0,y_0)\big)d\tau$, continuous functions in t,x,y and periodic in t of period T, i.e. $f(t,\varepsilon,u)=f(t+T,\varepsilon,u)$ and $g(t,\mu,v)=g(t+T,\mu,v)$. Also the vector functions $f(t,\varepsilon,u)$ *and* $g(t,\mu,v)$ satisfy (1.3) for all $t\in[0,T]$, and the matrices A and B satisfy the inequality (1.4), where $-\infty\le 0\le\tau\le t\le T\le\infty$.
We define the non-empty sets:

$$\left.\begin{aligned}D_{1f}&=D_2-Q_1TM\\ D_{1g}&=D_3-Q_2TN\end{aligned}\right\} \qquad (3.2)$$

Furthermore, we suppose that the greatest eigenvalue $\lambda_{\max}$ of the matrix $\Lambda(t)=\begin{pmatrix}\sigma_1(t)Q_1p_\alpha^\lambda & \sigma_1(t)Q_1p_\alpha^\lambda\\ \sigma_2(t)Q_2p_\beta^\gamma & \sigma_2(t)Q_2p_\beta^\gamma\end{pmatrix}$, does not exceed unity, i.e. satisfies the inequality (1.7)

Lemma 3.1: Let the vector functions $f(t,\varepsilon,u)$ *and* $g(t,\mu,v)$ are defined and continuous on the domain (3.1) and satisfy (1.3), then the inequality

$$\begin{pmatrix}\|F_1(t,x_0,y_0)\|\\ \|G_1(t,x_0,y_0)\|\end{pmatrix}\le\begin{pmatrix}\sigma_1(t)Q_1M\\ \sigma_2(t)Q_2N\end{pmatrix}, \qquad (3.3)$$

holds for $0\le t\le T,\ \sigma_1(t)\le T$, *and* $\sigma_2(t)\le T$ where

$$F_1(t,x_0,y_0)=\int_0^t e^{A(t-\tau)}\Big(f(\tau,\varepsilon,u)-\frac{A}{e^{AT}-E}\int_0^T e^{A(T-\tau)}f(\tau,\varepsilon,u)d\tau\Big)d\tau \qquad (3.4)$$

$$G_1(t,x_0,y_0)=\int_0^t e^{B(t-\tau)}\Big(g(\tau,\varepsilon,u)-\frac{B}{e^{BT}-E}\int_0^T e^{A(T-\tau)}g(\tau,\varepsilon,u)d\tau\Big)d\tau \qquad (3.5)$$

$$\sigma_1(t)=\frac{t\big(e^{\|A\|T}-2e^{\|A\|t}+\|E\|\big)+T\big(e^{\|A\|t}-\|E\|\big)}{e^{\|A\|T}-\|E\|} \qquad (3.6)$$

$$\sigma_2(t)=\frac{t\big(e^{\|B\|T}-2e^{\|B\|t}+\|E\|\big)+T\big(e^{\|Bb\|t}-\|E\|\big)}{e^{\|B\|T}-\|E\|} \qquad (3.7)$$

Proof: From (3.4), we have

$$\|F_1(t,x_0,y_0)\|\le\left(\|E\|-\frac{e^{\|A\|t}-\|E\|}{e^{\|A\|T}-\|E\|}\right)\int_0^t\big\|e^{A(t-\tau)}\big\|\|f(\tau,\varepsilon,u)\|\,d\tau+\frac{e^{\|A\|t}-\|E\|}{e^{\|A\|T}-\|E\|}\int_t^T\big\|e^{A(T-\tau)}\big\|\|f(\tau,\varepsilon,u)\|\,d\tau\le\sigma_1(t)Q_1M \qquad (3.8)$$

Similarly, we get

$$\|F_1(t,x_0,y_0)\|\le\sigma_2(t)Q_2N \qquad (3.9)$$

from (3.8) and (3.9), we conclude that the inequality (3.3) holds for $0\le t\le T,\ \sigma_1(t)\le T$, *and* $\sigma_2(t)\le T$.

3.1 Approximation of periodic solution

Theorem 3.2: Let $f(t,\varepsilon,u)$ *and* $g(t,\mu,v)$ are defined on the domain (3.1), which are continuous in t,x,y and periodic in t of period T, also satisfying the inequalities (1.3) and the conditions (3.2) and (1.7), then the sequences of function defined by

$$x_{m+1}(t,x_0,y_0)=x_0+\int_0^t e^{A(t-\tau)}\Big(f(\tau,\varepsilon_{m+1},u_{m+1})-\frac{A}{e^{AT}-E}\int_0^T e^{A(T-\tau)}f(\tau,\varepsilon_{m+1},u_{m+1})d\tau\Big)\,d\tau,$$

with $x(0)=x_0$, $\quad m=0,1,2,\dots,\qquad (3.10)$
and

$$y_{m+1}(t,x_0,y_0)=y_0+\int_0^t e^{B(t-\tau)}\Big(g(\tau,\mu_{m+1},v_{m+1})-\frac{B}{e^{BT}-E}\int_0^T e^{B(T-\tau)}g(\tau,\mu_{m+1},v_{m+1})d\tau\Big)\,d\tau,$$

$y(0)=y_0$, $\quad m=0,1,2,\dots\qquad (3.11)$

Where $\varepsilon_{m+1}=\int_a^b\lambda\big(x_m(\tau,x_0,y_0)-y_m(\tau,x_0,y_0)\big)d\tau$,

$$\mu_{m+1}=\int_c^d\gamma\big(x_m(\tau,x_0,y_0)-y_m(\tau,x_0,y_0)\big)d\tau,$$

$$u_{m+1}=\int_{-\infty}^t R(t-\tau)\big(x_m(\tau,x_0,y_0)-y_m(\tau,x_0,y_0)\big)d\tau,$$

$$v_{m+1}=\int_{-\infty}^t G(t-\tau)\big(x_m(\tau,x_0,y_0)-y_m(\tau,x_0,y_0)\big)d\tau,$$

are periodic in t of period T, uniformly convergent as $m\to\infty$ in the domain (3.1) to the limit functions $x^0(t,x_0,y_0)$ *and* $y^0(t,x_0,y_0)$ respectively, which are periodic in t of period T and satisfy the system of integral equations:

$$x(t,x_0,y_0)=x_0+\int_0^t e^{A(t-\tau)}\Big(f(\tau,\varepsilon,u)-\frac{A}{e^{AT}-E}\int_0^T e^{A(T-\tau)}f(\tau,\varepsilon,u)d\tau\Big)\,d\tau, \qquad (3.12)$$

and

$$y(t,x_0,y_0)=y_0+\int_0^t e^{B(t-\tau)}\Big(g(\tau,\mu,v)-\frac{B}{e^{BT}-E}\int_0^T e^{B(T-\tau)}g(\tau,\mu,v)d\tau\Big)\,d\tau, \qquad (3.13)$$

where $\varepsilon=\int_a^b\lambda\big(x(\tau,x_0,y_0)-y(\tau,x_0,y_0)\big)d\tau$,

$$\mu=\int_c^d\gamma\big(x(\tau,x_0,y_0)-y(\tau,x_0,y_0)\big)d\tau,$$

$$u=\int_{-\infty}^t R(t-\tau)\big(x(\tau,x_0,y_0)-y(\tau,x_0,y_0)\big)d\tau,$$

$$v=\int_{-\infty}^t G(t-\tau)\big(x(\tau,x_0,y_0)-y(\tau,x_0,y_0)\big)d\tau,$$

its unique solution of (1.1) and satisfy the following inequality

$$\begin{pmatrix}\|x^0(t,x_0,y_0)-x_m(t,x_0,y_0)\| \\ \\ \|y^0(t,x_0,y_0)-y_m(t,x_0,y_0)\|\end{pmatrix} \le \Lambda^m (E-\Lambda)^{-1} V_0 \qquad (3.14)$$

Where $V_0=\begin{pmatrix}TQ_1M \\ \\ TQ_2N\end{pmatrix}$, E is an identity matrix.

Proof: The sequences of functions $x_m(t,x_0,y_0)$ and $y_m(t,x_0,y_0)$ are defined in (3.10) and (3.11) are defined and continuous in the domain (3.1) and periodic in t of period T. Now by lemma 3.1 and from (3.10) when m=0, we get

$\|x_1(t,x_0,y_0)-x_0\| \le \sigma_1(t)Q_1M \le \mathrm{T}Q_1M$

also from (3.10) and by lemma 3.1,we obtain

$\|y_1(t,x_0,y_0)-y_0\| \le \sigma_2(t)Q_2N \le \mathrm{T}Q_2N$

and by using mathematical induction we can prove the following inequalities for $m \ge 1$.

$$\|x_m(t,x_0,y_0)-x_0\| \le \mathrm{T}Q_1M \qquad (3.15)$$

and

$$\|y_m(t,x_0,y_0)-y_0\| \le \mathrm{T}Q_2N \qquad (3.16)$$

From (3.15) and (3.16), $x_m(t,x_0,y_0) \in D_2$ *and* $y_m(t,x_0,y_0) \in D_3$ for all $t \in [0,T]$, $x_0 \in D_{1f}$ *and* $y_0 \in D_{1g}$.

Now, we prove that the sequences $\{x_m(t,x_0,y_0)\}_{m=0}^{\infty}$ and $\{y_m(t,x_0,y_0)\}_{m=0}^{\infty}$ are uniformly convergent in the domain (3,1).

By using lemma 3.1 and from (3.10) and (3.11) when m=1, we obtain:

$$\begin{aligned}\|x_2(t,x_0,y_0)-x_1(t,x_0,y_0)\| &\le \sigma_1(t)Q_1p_\alpha^\lambda(\|x_1(t,x_0,y_0)-x_0\| \\ &\quad + \|y_1(t,x_0,y_0)-y_0\|), \qquad (3.17)\end{aligned}$$

and

$$\begin{aligned}\|y_2(t,x_0,y_0)-y_1(t,x_0,y_0)\| &\le \sigma_2(t)Q_2p_\beta^\gamma(\|x_1(t,x_0,y_0)-x_0\| \\ &\quad + \|y_1(t,x_0,y_0)-y_0\|) \qquad (3.18)\end{aligned}$$

Also by using mathematical induction, we can prove the following inequalities:

$$\begin{aligned}\|x_{m+1}(t,x_0,y_0)-x_m(t,x_0,y_0)\| &\le \sigma_1(t)Q_1p_\alpha^\lambda(\|x_m(t,x_0,y_0) \\ &\quad - x_{m-1}(t,x_0,y_0)\| \\ &\quad + \|y_m(t,x_0,y_0) \\ &\quad - y_{m-1}(t,x_0,y_0)\|) \qquad (3.19)\end{aligned}$$

$$\begin{aligned}\|y_{m+1}(t,x_0,y_0)-y_m(t,x_0,y_0)\| &\le \sigma_2(t)Q_2p_\beta^\gamma(\|x_m(t,x_0,y_0) \\ &\quad - x_{m-1}(t,x_0,y_0)\| \\ &\quad + \|y_m(t,x_0,y_0) \\ &\quad - y_{m-1}(t,x_0,y_0)\|) \qquad (3.20)\end{aligned}$$

we rewrite (3.19) and (3.20) with vectors form and on the following mode:

$$V_{m+1}(t) \le \Lambda(t)V_m(t) \qquad (3.21)$$

where

$$\mathrm{V_{m+1}}=\begin{pmatrix}\|x_{m+1}(t,x_0,y_0)-x_m(t,x_0,y_0)\| \\ \\ \|y_{m+1}(t,x_0,y_0)-y_m(t,x_0,y_0)\|\end{pmatrix},$$

$$\mathrm{V_m}=\begin{pmatrix}\|x_m(t,x_0,y_0)-x_{m-1}(t,x_0,y_0)\| \\ \\ \|y_m(t,x_0,y_0)-y_{m-1}(t,x_0,y_0)\|\end{pmatrix}$$

and $\Lambda(\mathrm{t})=\begin{pmatrix}\sigma_1(t)Q_1p_\alpha^\lambda & \sigma_1(t)Q_1p_\alpha^\lambda \\ \\ \sigma_2(t)Q_2p_\beta^\gamma & \sigma_2(t)Q_2p_\beta^\gamma\end{pmatrix}.$

Now, we take the maximum value for the two sides of the inequality (3.21), we get:

$$V_{m+1} \le \Lambda\ V_m \qquad (3.22)$$

Where $\Lambda = \max_{t\in[0,T]} \Lambda(t)$, we obtain

$\Lambda=\begin{pmatrix}TQ_1p_\alpha^\lambda & TQ_1p_\alpha^\lambda \\ \\ TQ_2p_\beta^\gamma & TQ_2p_\beta^\gamma\end{pmatrix}$, Also (3.22) , we find

$$V_{m+1} \le \Lambda^m\ V_0 \qquad (3.23)$$

where $V_0=\begin{pmatrix}TQ_1M \\ \\ TQ_2N\end{pmatrix}$ and also we get

$$\sum_{i=1}^{m} V_i \le \sum_{i=1}^{m} \Lambda^{i-1}\, V_0 \qquad (3.24)$$

by condition (1.7) then the sequence (3.24) is uniformly convergent that is

$$\lim_{m\to\infty}\sum_{i=1}^{m} \Lambda^{i-1}\, V_0 = \sum_{i=1}^{\infty} \Lambda^{i-1}\, V_0 = (E-\Lambda)^{-1}V_0 \qquad (3.25)$$

Let

$$\left.\begin{aligned}\lim_{m\to\infty} x_m(t,x_0,y_0) &= x^0(t,x_0,y_0) \\ \lim_{m\to\infty} y_m(t,x_0,y_0) &= y^0(t,x_0,y_0)\end{aligned}\right\} \qquad (3.26)$$

Since the sequences of function (3.10) and (3.11) are defined and continuous in the domain $(\mathrm{t,x,y}) \in [0,\mathrm{T}] \times \mathrm{D_2} \times \mathrm{D_3}$, then the limiting functions $x^0(t,x_0,y_0)$ and $y^0(t,x_0,y_0)$ are also defined and continuous in the same domain.

Moreover, from (3.25), (3.10) and (3.11), when $m \to \infty$, the limiting functions $x^0(t,x_0,y_0)$ and $y^0(t,x_0,y_0)$ are the solution of the integral equations (3.12) and (3.13) respectively.

Finally, we have to show that $x(t,x_0,y_0)$ and $y(t,x_0,y_0)$ are unique of periodic solution of the system (1.1).

Let $x_\infty(t,x_0,y_0)$ and $y_\infty(t,x_0,y_0)$ be another solution of (1.1), i.e.

$$\begin{aligned}x_\infty(t,x_0,y_0) &= x_0 + \int_0^t e^{A(t-\tau)}(f(\tau,\varepsilon,u) \\ &\quad - \frac{A}{e^{AT}-E}\int_0^T e^{A(T-\tau)}f(\tau,\varepsilon,u)d\tau)\,d\tau, \qquad (3.27)\end{aligned}$$

and

$$\begin{aligned}y_\infty(t,x_0,y_0) &= y_0 + \int_0^t e^{B(t-\tau)}(g(\tau,\mu,v) \\ &\quad - \frac{B}{e^{BT}-E}\int_0^T e^{B(T-\tau)}g(\tau,\mu,v)d\tau)\,d\tau, \qquad (3.28)\end{aligned}$$

where

$$\varepsilon = \int_a^b \lambda\big(x_\infty(\tau,x_0,y_0)-y_\infty(\tau,x_0,y_0)\big)d\tau,$$

$$\mu = \int_c^d \gamma\big(x_\infty(\tau,x_0,y_0)-y_\infty(\tau,x_0,y_0)\big)d\tau,$$

$$u = \int_{-\infty}^t R(t-\tau)\big(x_\infty(\tau,x_0,y_0)-y_\infty(\tau,x_0,y_0)\big)d\tau,$$

$$v = \int_{-\infty}^t G(t-\tau)\big(x_\infty(\tau,x_0,y_0)-y_\infty(\tau,x_0,y_0)\big)d\tau,$$

From (3.12) and (3.13) the following inequalities holds

$$\|x(t,x_0,y_0)-x_\infty(t,x_0,y_0)\| \le \sigma_1(t)Q_1p_\alpha^\lambda(\|x(t,x_0,y_0)-x_\infty(t,x_0,y_0)\| + \|y(t,x_0,y_0)-y_\infty(t,x_0,y_0)\|) \quad (3.29)$$

$$\|y(t,x_0,y_0)-y_\infty(t,x_0,y_0)\| \le \sigma_2(t)Q_2p_\beta^\gamma(\|x(t,x_0,y_0)-x_\infty(t,x_0,y_0)\| + \|y(t,x_0,y_0)-y_\infty(t,x_0,y_0)\|) \quad (3.30)$$

Thus

$$\begin{pmatrix}\|x(t,x_0,y_0)-x_\infty(t,x_0,y_0)\| \\ \|y(t,x_0,y_0)-y_\infty(t,x_0,y_0)\|\end{pmatrix} \le \Lambda \begin{pmatrix}\|x(t,x_0,y_0)-x_\infty(t,x_0,y_0)\| \\ \|y(t,x_0,y_0)-y_\infty(t,x_0,y_0)\|\end{pmatrix} \quad (3.31)$$

by iteration we find that

$$\begin{pmatrix}\|x(t,x_0,y_0)-x_\infty(t,x_0,y_0)\| \\ \|y(t,x_0,y_0)-y_\infty(t,x_0,y_0)\|\end{pmatrix} \le \Lambda^m \begin{pmatrix}\|x(t,x_0,y_0)-x_\infty(t,x_0,y_0)\| \\ \|y(t,x_0,y_0)-y_\infty(t,x_0,y_0)\|\end{pmatrix} \quad (3.32)$$

But from the condition (1.7), we get $\Lambda^m \to 0$, when $m \to \infty$, hence we obtain that $x(t,x_0,y_0) = x_\infty(t,x_0,y_0)$ and $y(t,x_0,y_0) - y_\infty(t,x_0,y_0)$ which proves that the solution is unique..

3.2 Existence of periodic solution

The problem of existence solution of the system (1.1) is uniquely connected with the existence of zeros of the functions $\Delta_1(x_0,y_0) \in D_{1f} \times D_{1g} \to R$ and $\Delta_2(x_0,y_0) \in D_{1f} \times D_{1g} \to R$ which has the form

$$\Delta_1(x_0,y_0) = Ax_0 + \frac{A}{e^{AT}-E}\int_0^T e^{A(T-\tau)} f(\tau,\varepsilon,u)d\tau, \quad (3.33)$$

$$\Delta_2(x_0,y_0) = By_0 + \frac{B}{e^{BT}-E}\int_0^T e^{B(T-\tau)} g(\tau,\mu,v)d\tau, \quad (3.34)$$

where $\varepsilon = \int_a^b \lambda\big(x(\tau,x_0,y_0) - y(\tau,x_0,y_0)\big)d\tau$,

$$\mu = \int_c^d \gamma\big(x(\tau,x_0,y_0) - y(\tau,x_0,y_0)\big)d\tau,$$

$$u = \int_{-\infty}^t R(t-\tau)\big(x(\tau,x_0,y_0) - y(\tau,x_0,y_0)\big)d\tau,$$

$$v = \int_{-\infty}^t G(t-\tau)\big(x(\tau,x_0,y_0) - y(\tau,x_0,y_0)\big)d\tau,$$

Since these functions (3.33) and (3.34) are approximately determined from the sequences of functions.

$$\Delta_{1m}(x_0,y_0) = Ax_0 + \frac{A}{e^{AT}-E}\int_0^T e^{A(T-\tau)} f(\tau,\varepsilon_{m+1},u_{m+1})d\tau, \quad (3.35)$$

$$\Delta_{2m}(x_0,y_0) = By_0 + \frac{B}{e^{BT}-E}\int_0^T e^{B(T-\tau)} g(\tau,\mu_{m+1},v_{m+1})d\tau, \quad (3.36)$$

where

$$\varepsilon_{m+1} = \int_a^b \lambda\big(x_m(\tau,x_0,y_0) - y_m(\tau,x_0,y_0)\big)d\tau,$$

$$\mu_{m+1} = \int_c^d \gamma\big(x_m(\tau,x_0,y_0) - y_m(\tau,x_0,y_0)\big)d\tau,$$

$$u_{m+1} = \int_{-\infty}^t R(t-\tau)\big(x_m(\tau,x_0,y_0) - y_m(\tau,x_0,y_0)\big)d\tau$$

$$v_{m+1} = \int_{-\infty}^t G(t-\tau)\big(x_m(\tau,x_0,y_0) - y_m(\tau,x_0,y_0)\big)d\tau,$$

for $m = 0,1,2,\ldots$

Theorem 3.3: Let all assumptions and conditions of theorem 3.2 be satisfied, then the following inequalities hold

$$\|\Delta_1(x_0,y_0) - \Delta_{1m}(x_0,y_0)\| \le \langle\big(L_1Q_1p_\alpha^\lambda \quad L_1Q_1p_\alpha^\lambda\big), \Lambda^m(E-\Lambda)^{-1}V_0\rangle = \omega_{1m} \quad (3.37)$$

$$\|\Delta_2(x_0,y_0) - \Delta_{2m}(x_0,y_0)\| \le \langle\big(L_2Q_2p_\beta^\gamma \quad L_2Q_2p_\beta^\gamma\big), \Lambda^m(E-\Lambda)^{-1}V_0\rangle = \omega_{2m} \quad (3.38)$$

where

$L_1 = \frac{\|A\|T}{e^{\|A\|T}-\|E\|}$, $L_2 = \frac{\|B\|T}{e^{\|B\|T}-\|E\|}$ and ω_{1m} , ω_{2m} are positive constant satisfy for $m \ge 0$, $x_0 \in D_{1f}$, $y_0 \in D_{1g}$ and $\langle . \rangle$ denots to the non-cross product in the Euclidean space R^n.

Proof:

By equations (3.33) and (3.35) and using (3.14) , we have

$$\|\Delta_1(x_0,y_0) - \Delta_{1m}(x_0,y_0)\| \le \frac{\|A\|}{e^{\|A\|T}-\|E\|}[\sigma_1(t)Q_1p_\alpha^\lambda(\|x(t,x_0,y_0) - x_m(t,x_0,y_0)\| + \|y(t,x_0,y_0) - y_m(t,x_0,y_0)\|)] \le \langle\big(L_1Q_1p_\alpha^\lambda \quad L_1Q_1p_\alpha^\lambda\big), \Lambda^m(E-\Lambda)^{-1}V_0\rangle = \omega_{1m}$$

also from the equations (3.34) and (3.36) and using (3.14) , we get

$$\|\Delta_2(x_0,y_0) - \Delta_{2m}(x_0,y_0)\| \le \frac{\|B\|}{e^{\|B\|T}-\|E\|}[\sigma_2(t)Q_2p_\beta^\gamma(\|x(t,x_0,y_0) - x_m(t,x_0,y_0)\| + \|y(t,x_0,y_0) - y_m(t,x_0,y_0)\|)] \le \langle\big(L_2Q_2p_\beta^\gamma \quad L_2Q_2p_\beta^\gamma\big), \Lambda^m(E-\Lambda)^{-1}V_0\rangle = \omega_{2m}$$

Now, we prove the following theorem taking in to account that the inequalities (3.37) and (3.38) will be satisfied for all $m \ge 0$.

Theorem 3.4: Suppose that for $m \ge 0$ the sequences of functions $\Delta_{1m}(x_0,y_0)$ and $\Delta_{2m}(x_0,y_0)$, which are defined in (3.35) and (3.36) satisfy the inequalities:

$$\left.\begin{aligned} \min_{\substack{a1+TQ_1M\le x_0\le b1-TQ_1M \\ c1+TQ_2N\le y_0\le d1-TQ_2N}} \Delta_{1m}(x_0,y_0) &\le -\omega_{1m} \\ \max_{\substack{a1+TQ_1M\le x_0\le b1-TQ_1M \\ c1+TQ_2N\le y_0\le d1-TQ_2N}} \Delta_{1m}(x_0,y_0) &\ge \omega_{1m} \end{aligned}\right\} \quad (3.39)$$

$$\left.\begin{aligned} \min_{\substack{a1+TQ_1M\le x_0\le b1-TQ_1M \\ c1+TQ_2N\le y_0\le d1-TQ_2N}} \Delta_{2m}(x_0,y_0) &\le -\omega_{2m} \\ \max_{\substack{a1+TQ_1M\le x_0\le b1-TQ_1M \\ c1+TQ_2N\le y_0\le d1-TQ_2N}} \Delta_{2m}(x_0,y_0) &\ge \omega_{2m} \end{aligned}\right\} \quad (3.40)$$

Then the system (1.1) has a periodic solutions $x = x(t,x_0,y_0)$ and $y = y(t,x_0,y_0)$ such that $x_0 \in [a1 + TQ_1M, b1 - TQ_1M]$, $y_0 \in [c1 + TQ_2N, d1 - TQ_2N]$.

Proof: Let x_1, x_2 be any points in the interval $[a1+TQ_1M, b1-TQ_1M]$, and y_1, y_2 be any points in the interval $[c1+TQ_2N, d1-TQ_2N]$ such that

$$\left.\begin{aligned}\Delta_{1m}(x_1,y_1) &= \min_{\substack{a1+TQ_1M\le x_0\le b1-TQ_1M\\ c1+TQ_2N\le y_0\le d1-TQ_2N}} \Delta_{1m}(x_0,y_0)\\ \Delta_{1m}(x_2,y_2) &= \max_{\substack{a1+TQ_1M\le x_0\le b1-TQ_1M\\ c1+TQ_2N\le y_0\le d1-TQ_2N}} \Delta_{1m}(x_0,y_0)\end{aligned}\right\} \quad (3.41)$$

$$\left.\begin{aligned}\Delta_{2m}(x_1,y_1) &= \min_{\substack{a1+TQ_1M\le x_0\le b1-TQ_1M\\ c1+TQ_2N\le y_0\le d1-TQ_2N}} \Delta_{2m}(x_0,y_0)\\ \Delta_{2m}(x_2,y_2) &= \max_{\substack{a1+TQ_1M\le x_0\le b1-TQ_1M\\ c1+TQ_2N\le y_0\le d1-TQ_2N}} \Delta_{2m}(x_0,y_0)\end{aligned}\right\} \quad (3.42)$$

By using the inequalities (3.37) and (3.38), we have

$$\left.\begin{aligned}\Delta_1(x_1,y_1) &= \Delta_{1m}(x_1,y_1) + \big(\Delta_1(x_1,y_1) - \Delta_{1m}(x_1,y_1)\big) < 0\\ \Delta_1(x_2,y_2) &= \Delta_{1m}(x_2,y_2) + \big(\Delta_1(x_2,y_2) - \Delta_{1m}(x_2,y_2)\big) > 0\end{aligned}\right\} \quad (3.43)$$

$$\left.\begin{aligned}\Delta_2(x_1,y_1) &= \Delta_{2m}(x_1,y_1) + \big(\Delta_2(x_1,y_1) - \Delta_{2m}(x_1,y_1)\big) < 0\\ \Delta_2(x_2,y_2) &= \Delta_{2m}(x_2,y_2) + \big(\Delta_2(x_2,y_2) - \Delta_{2m}(x_2,y_2)\big) > 0\end{aligned}\right\} \quad (3.44)$$

and from the continuity of the functions $\Delta_1(x_0,y_0)$ and $\Delta_2(x_0,y_0)$ and the inequalities (3.43) and (3.44), then there exists an isolated singular point $(x^0,y^0) = (x_0,y_0)$ and $x^0 \in [x_1,x_2]$, $y^0 \in [y_1,y_2]$ where $\Delta_1(x_0,y_0)$ and $\Delta_2(x_0,y_0)$ are equals to zero, this mean that the system (1.1) has periodic solutions $x = x(t,x_0,y_0)$ and $y = y(t,x_0,y_0)$ for $x_0 \in [a1+TQ_1M, b1-TQ_1M]$, $y_0 \in [c1+TQ_2N, d1-TQ_2N]$.

Theorem 3.5: Suppose that $\Delta_1(x_0,y_0) \in D_{1f}\times D_{1g} \to R$ and $\Delta_2(x_0,y_0) \in D_{1f}\times D_{1g} \to R$ which has the form

$$\Delta_1(x_0,y_0) = Ax_0 + \frac{A}{e^{AT}-E}\int_0^T e^{A(T-\tau)} f(\tau,\varepsilon^0,u^0)\,d\tau, \qquad (3.45)$$

$$\Delta_2(x_0,y_0) = By_0 + \frac{B}{e^{BT}-E}\int_0^T e^{B(T-\tau)} g(\tau,\mu^0,v^0)\,d\tau, \qquad (3.46)$$

where

$$\varepsilon^0 = \int_a^b \lambda\big(x^0(\tau,x_0,y_0) - y^0(\tau,x_0,y_0)\big)d\tau,$$

$$\mu^0 = \int_c^d \gamma\big(x^0(\tau,x_0,y_0) - y^0(\tau,x_0,y_0)\big)d\tau,$$

$$u^0 = \int_{-\infty}^t R(t-\tau)\big(x^0(\tau,x_0,y_0) - y^0(\tau,x_0,y_0)\big)d\tau,$$

$$v^0 = \int_{-\infty}^t G(t-\tau)\big(x^0(\tau,x_0,y_0) - y^0(\tau,x_0,y_0)\big)d\tau,$$

where $x^0(t,x_0,y_0)$ and $y^0(t,x_0,y_0)$ are the limit of the sequences of functions (3.10) and (3.11) respectively, then the following inequalities hold:

$$\|\Delta_1(x_0,y_0)\| \le I_3 + L_1Q_1M\,, \qquad (3.47)$$

$$\|\Delta_2(x_0,y_0)\| \le I_4 + L_2Q_2N\,, \qquad (3.48)$$

$$\|\Delta_1(x_0^1,y_0^1) - \Delta_1(x_0^2,y_0^2)\| \le W_4\|x_0^1 - x_0^2\| + W_5\|y_0^1 - y_0^2\|\,, \qquad (3.49)$$

$$\|\Delta_2(x_0^1,x_0^1) - \Delta_2(x_0^2,x_0^2)\| \le W_6\|x_0^1 - x_0^2\| + W_7\|y_0^1 - y_0^2\|\,, \qquad (3.50)$$

where $I_3 = \|A\|\|x_0\|$, $I_4 = \|B\|\|y_0\|$, $W_1 = (1 - TQ_1p_\alpha^\lambda)^{-1}$, $W_2 = \left(1 - TQ_2p_\beta^\gamma\right)^{-1}$, $W_3 = (1 - T^2Q_1Q_2p_\alpha^\lambda p_\beta^\gamma W_1W_2)^{-1}$, $W_4 = \|A\| + L_1Q_1p_\alpha^\lambda W_1W_3(1 + TQ_2p_\beta^\gamma W_2)$

$W_5 = L_1Q_1p_\alpha^\lambda W_2W_3(1 + TQ_1p_\alpha^\lambda W_1)$, $W_6 = L_2Q_2p_\beta^\gamma W_1W_3\left(1 + TQ_2p_\beta^\gamma W_2\right)$, $W_7 = \|B\| + L_2Q_2p_\beta^\gamma W_2W_3(1 + TQ_1p_\alpha^\lambda W_1)$

Proof: From the properties of the functions $x^0(t,x_0,y_0)$ and $y^0(t,x_0,y_0)$ were fixed in the theorem 3.2, the function $\Delta_1(x_0,y_0)$ *and* $\Delta_2(x_0,y_0)$ are continuous and bounded by $I_3 + L_1Q_1M$, and $I_4 + L_2Q_2N$ respectively in the domain (3.1). By using (3.45) and (3.46), we get

$$\|\Delta_1(x_0,y_0)\| \le \|A\|\|x_0\| + \frac{\|A\|}{e^{\|A\|T}-E}\int_0^T \left\|e^{A(T-\tau)}\right\|\|f(\tau,\varepsilon^0,u^0)\|d\tau \le I_3 + L_1Q_1M$$

and

$$\|\Delta_2(x_0,y_0)\| \le \|B\|\|y_0\| + \frac{\|B\|}{e^{\|B\|T}-E}\int_0^T \left\|e^{B(T-\tau)}\right\|\|g(\tau,\mu^0,v^0)\|d\tau \le I_4 + L_2Q_2N$$

By using (3.45) and (3.46), we obtain

$$\begin{aligned}\|\Delta_1(x_0^1,y_0^1) - \Delta_1(x_0^2,y_0^2)\| &\le \|A\|\|x_0^1 - x_0^2\| + L_1Q_1p_\alpha^\lambda(\|x(t,x_0^1,y_0^1) - x(t,x_0^2,y_0^2)\|\\ &\quad + \|y(t,x_0^1,y_0^1) - y(t,x_0^2,y_0^2)\|)\,, \end{aligned} \qquad (3.51)$$

$$\begin{aligned}\|\Delta_2(x_0^1,y_0^1) - \Delta_2(x_0^2,y_0^2)\| &\le \|B\|\|x_0^1 - x_0^2\| + L_2Q_2p_\beta^\gamma(\|x(t,x_0^1,y_0^1) - x(t,x_0^2,y_0^2)\|\\ &\quad + \|y(t,x_0^1,y_0^1) - y(t,x_0^2,y_0^2)\|)\,, \end{aligned} \qquad (3.52)$$

The norm $\|x(t,x_0^1,y_0^1) - x(t,x_0^2,y_0^2)\|$ and $\|y(t,x_0^1,y_0^1) - y(t,x_0^2,y_0^2)\|$ in (3.51) and (3.52), were gain from the following equations:

$$x(t,x_0^1,y_0^1) = x_0^1 + \int_0^t e^{A(t-\tau)}(f(\tau,\varepsilon,u) - \frac{A}{e^{AT}-E}\int_0^T e^{A(T-\tau)}f(\tau,\varepsilon,u)d\tau)\,d\tau, \qquad (3.53)$$

and

$$y(t,x_0^1,y_0^1) = x_0^1 + \int_0^t e^{B(t-\tau)}(g(\tau,\mu,v) - \frac{B}{e^{BT}-E}\int_0^T e^{B(T-\tau)}g(\tau,\mu,v)d\tau)\,d\tau, \qquad (3.54)$$

where

$$\varepsilon = \int_a^b \lambda\Big(x(\tau,x_0^1,y_0^1) - y(\tau,x_0^1,y_0^1)\Big)d\tau,$$

$$\mu = \int_c^d \gamma\Big(x(\tau,x_0^1,y_0^1) - y(\tau,x_0^1,y_0^1)\Big)d\tau,$$

$$u = \int_{-\infty}^t R(t-\tau)\Big(x(\tau,x_0^1,y_0^1) - y(\tau,x_0^1,y_0^1)\Big)d\tau,$$

$$v = \int_{-\infty}^t G(t-\tau)\Big(x(\tau,x_0^1,y_0^1) - y(\tau,x_0^1,y_0^1)\Big)d\tau,$$

Now, use the equation (3.53) as follows

$$\begin{aligned}\|x(t,x_0^1,y_0^1)-x(t,x_0^2,y_0^2)\| &\le \|x_0^1-x_0^2\| \\ &+ TQ_1 p_\alpha^\lambda(\|x(t,x_0^1,y_0^1)-x(t,x_0^2,y_0^2)\| \\ &+ \|y(t,x_0^1,y_0^1)-y(t,x_0^2,y_0^2)\|),\end{aligned}$$

then we can write

$$\begin{aligned}\|x(t,x_0^1,y_0^1)-x(t,x_0^2,y_0^2)\| &\le W_1\|x_0^1-x_0^2\| \\ &+ TQ_1 p_\alpha^\lambda W_1\|y(t,x_0^1,y_0^1) \\ &- y(t,x_0^2,y_0^2)\|\,, \qquad (3.55)\end{aligned}$$

Also, use the equation (3.54), we get

$$\begin{aligned}\|y(t,x_0^1,y_0^1)-y(t,x_0^2,y_0^2)\| &\le \|y_0^1-y_0^2\| \\ &+ TQ_2 p_\beta^\gamma(\|x(t,x_0^1,y_0^1)-x(t,x_0^2,y_0^2)\| \\ &+ \|y(t,x_0^1,y_0^1)-y(t,x_0^2,y_0^2)\|),\end{aligned}$$

then we write this equation as follows

$$\begin{aligned}\|y(t,x_0^1,y_0^1)-y(t,x_0^2,y_0^2)\| &\le W_2\|y_0^1-y_0^2\| \\ &+ TQ_2 p_\beta^\gamma W_2\|x(t,x_0^1,y_0^1) \\ &- x(t,x_0^2,y_0^2)\|\,, \qquad (3.56)\end{aligned}$$

Now, substitute (3.56) in (3.55), we find that

$$\begin{aligned}\|x(t,x_0^1,y_0^1)-x(t,x_0^2,y_0^2)\| &\le W_1W_3\|x_0^1-x_0^2\| \\ &+ TQ_1 p_\alpha^\lambda W_1W_2W_3\|y_0^1 \\ &- y_0^2\|\,, \qquad (3.57)\end{aligned}$$

and substitute (3.55) in (3.56), we get

$$\begin{aligned}\|y(t,x_0^1,y_0^1)-y(t,x_0^2,y_0^2)\| &\le TQ_2 p_\beta^\gamma W_1W_2W_3\|x_0^1-x_0^2\| \\ &+ W_2W_3\|y_0^1-y_0^2\|\,, \qquad (3.58)\end{aligned}$$

Finally, we substitute the inequalities (3.57) and (3.58) in (3.51) we get (3.49) and substitute the inequalities (3.57) and (3.58) in (3.52), we get (3.50).

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كورتيا لێكولينێ:

دڤێ ڤەكولينێدا, خاندنا هبوون و تاك وجيگێرا شيكاررييێن خولى بو سيستەمێت هاوكێشێن جياكارى وتەواوكارى ييًت نه هێلى. پشت بەستن ل سەر رێكێن شلوڤەكرييێن ژمارەيى بو خاندنا شيكارێت خولى بو هاوكێشێن جياكارييێن سادە ژ لايێ A. Samoilenko هاتييه پێشكێشكرن.

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