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NONSTANDARD VERSION OF INTERMEDIATE VALUE PROPERTY

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ABSTRACT:
In this paper some new properties and results about Intermediate Value Property (IVP) via nonstandard concepts are given, and modifying some existing results to show the advantage role of nonstandard analysis tools for obtaining differed nonstandard distinguished results.

KEYWORDS: IVP functions, CIVP functions, Functions with perfect road, Bair one functions, monad, s-continuity.

1. INTRODUCTION

During the past centuries there were several studies about intermediate value property of functions such as; investigations, new properties, new types, ...etc (see: Banaszewski,1997; Brown and Laczkovich,1988; Bruckner, 1982; Gibson and Natkaniec, 1996; Gibson and Rouch, 1982; Maliszewski, 2006; Pawlak, 1987). A little of those studies on intermediate value property includes the concepts of nonstandard analysis(Arkeryd and Henson, 1997; Goldblatt, 1998; Hurd and Loeb, 1985; Palmgren, 1998). In this paper we will restrict our attention to show how nonstandard tools will simplify some of the results obtained previously by mathematicians about intermediate value property and to present some other nonstandard results.
Classically a function is said to have the *Intermediate Value Property* (**IVP**) provided that if $a,b \in \mathrm{I}$ are real numbers such that $a \neq b$ and $f(a) < f(b)$, then for every λ ; $f(a) \leq \lambda \leq f(b)$, there exists a number Z ; $a \leq z \leq b$ such that $f(z) = \lambda$.
Darboux(1875) showed that there are functions with the intermediate value property that are not continuous. Young (1907) studied real valued functions defined on an interval I with the following property: for every $x \in \mathrm{R}$ there exist sequences x_n and y_n such that $x_n \to x$, $y_n \to x$ and both $f(x_n)$ and $f(y_n)$ converge to $f(x)$. Maximoff (1936) showed that for real valued Baire class 1 functions defined on an interval, functions with intermediate value property and functions with a perfect road are equivalent (Maximoff, 1936). Most of the following definitions and notations can be found in (Bruckner et al, 1997; Hurd, and Loeb, 1985; Kusraev and Kutateladze, 1994; Maximoff, 1936; Nelson, 1977; Robinson, 1970; Rosen, 2003):

A point $(x, y) \in \mathrm{R}^2$ is a *bilateral limit point* of the graph of $f:\mathrm{R} \to \mathrm{R}$ if for every open neighborhood U of (x, y), both $\left((-\infty, x)\times \mathrm{R}\right)\cap U \cap f$ and $\left((x,\infty)\times \mathrm{R}\right)\cap U \cap f$ are infinite. Let E be a subset of R. The set E is *perfect* if it is nonempty, closed, and has no isolated points, that is, the set E is perfect if it is closed and each point of E is a limit point of E, and E is *nowhere dense* if $\overline{E}$ contains no open intervals. A function f has a *perfect road* if for every $x \in \mathrm{R}$, there exists a perfect set P having x as a bilateral limit point such that $f\,|_P$ is continuous at x. A function f is said to be in the *first class of Baire* or *Baire 1* if it can be written as the pointwise limit of a sequence of continuous functions. A function f is *upper semicontinuous* at a point X if for every $\varepsilon > 0$ there is a $\delta > 0$ so that if $|x-y|<\delta$ then $f(y) < f(x) - \varepsilon$. If $f:[a,b] \to \mathrm{R}$ is bounded function, the *Oscillation* of f on $[a,b]$ is define to be $\omega(f,[a,b]) = sup\,|\,f(x) - f(y)\,|: x, y \in [a,b]$. If $c \in (a,b)$, then the *Oscillation* of the function at the point C is defined to be $\omega(f,c) = \lim_{\delta \to 0^+} \omega(f,[c-\delta, c+\delta])$. The graph of f is *bilaterally dense in itself* if every point $(x, f(x))$is a bilateral limit point of f. A function $f:\mathrm{R} \to \mathrm{R}$ is *symmetrically continuous* if for each $x \in \mathrm{R}$, $\lim_{h\to 0}(f(x+h) - f(x-h) = 0$.
A function f has a *Cantor Intermediate Value Property* (**CIVP**) if for all $p, q \in \mathrm{R}$ with $p \neq q$ and $f(p) \neq f(q)$ and for every Cantor set K between $f(p)$ and $f(q)$, there exists a Cantor set C between p and q such that $f(C) \subset K$.
Every set defined in **ZFC**, *Zermelo-Fraenckel Set Theory with Axiom of Choice*, is *standard* and we recall that in **ZFC** every mathematical object: a real number, function, $\cdots$, etc in **ZFC** is regarded as a set. In 1977, Nelson presented an axiomatic approach to the Robinson's nonstandard analysis (Robinson, 1970), based on a theory called Internal Set Theory (**IST**), any set or formula in **IST** is called internal in case if it does not defined with predicate *"standard"*, otherwise it is called external. A real number x is called *limited* if $|x| \leq r$ for some positive standard real numbers r.

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Otherwise is called *unlimited*. A real number x is called *infinitesimal* if $|x| < r$ for all positive standard real numbers r. Two real numbers x and y are said to be *infinitely close* if $x - y$ is infinitesimal and denoted by $x \approx y$. If x is a real number, then the set of all numbers which are infinitely close to x is called the *monad* of x and denoted by $m(x)$. By $m^{+}(x)$ or $(x >;0)$ we mean the set of all numbers which are greater than and infinitely close to x. Similarly we can define $m^{-}(x)$. The *monad* of a set $\boldsymbol{A} \subset \boldsymbol{E}$, where $\boldsymbol{E}$ is any metric space, is equal to the set of all elements of $\boldsymbol{E}$ which are infinitely close to some element members of $\boldsymbol{A}$, and denoted by $m(A)$. If x is a limited real number, then it is infinitesimal close to a unique standard real number called *standard part* or *shadow* of X and denoted by $st(x)$, or $sh(x)$. The standard part of any set $\boldsymbol{A}$ is equal to the set of all its standard elements, and denoted by ${}^{st}\boldsymbol{A}$, or sh **(*A*)**. If x is a real number, then the set of all numbers y such that $x - y$ is limited is called the *galaxy* of X and denoted by $gal(x)$, by $gal^{+}(x)$ ($gal^{-}(x)$) we mean the set of all numbers which are greater(less) than X.

Let f be a real valued function then:

1. f is called continuous at X_o if f and x_o are standards and for all $x, x \approx x_o$ implies that $f(x) \approx f(x_o)$
2. f is called s-continuous at X_o if for all $x, x \approx x_o$, then. $f(x) \approx f(x_o)$

Transfer Principle (**TP**): Let $A(x, t_1, \ldots, t_k)$ be an internal formula with free variables $x, t_1, \ldots, t_k$ and no other free variables. Then

$$\forall^{st} t_1 \cdots \forall^{st} t_k \, [\forall^{st} x \, A(x, t_1, \ldots, t_k) \Leftrightarrow \forall x (x, t_1, \ldots, t_k)]$$

Let $\boldsymbol{E}$ be a subset of R. If p is a limit point of $\boldsymbol{E}$, then there exists a sequence $\{p_n\}_{n \in \mathrm{N}}$ in $\boldsymbol{E}$ with $p_n \neq p$ for all $n \in \mathrm{N}$, such that $\lim\limits_{n \to \infty} p_n = p$, that is $p_n \approx p$ for every unlimited n.

An internal function f is s-differentiable at $x \in I$ if $f^{'}(x) = sh\left(\dfrac{f(x+\delta) - f(x)}{\delta} \right)$ for all nonzero infinitesimal δ. Some times **IVP** is defined via set properties as follows:

A function $f : I \subset \mathrm{R} \to \mathrm{R}$ satisfies the intermediate value property if and only if $f(J)$ is an interval for a subinterval $J \subset I$ (Maliszewski, 2006).

) points is to be used.

2. MAIN RESULTS

Definition 2.1 (Intermediate Value Property- Standard Version, SV) Let f be a standard function defined on a standard interval I. Then we say that f satisfies IVP if for all standard $a, b \in \mathrm{I}$ and for every standard real number λ such that $f(x) \leq \lambda \leq f(b)$, there exists a number z; $a \leq z \leq b$ such that $f(z) = \lambda$.

Lemma 2.2 Standard version of **IVP** is equivalent to the classical one. Moreover, for all $z \in \mathrm{I}$; such that $a \leq z \leq b$, where $a, b \in \mathrm{I}$, Z is standard.

Proof. Let $z \in \mathrm{I}$; $a \leq z \leq b$ where $a, b \in \mathrm{I}$, be a standard. First assume that Definition of **IVP** holds. To prove Definition 2.1, suppose that $f : [a,b] \subset \mathrm{R} \to \mathrm{R}$ is a function satisfies the conventional intermediate value property. By applying backward direction of (**TP**) on Definition **IVP** we get the following statement:

[A standard function $f : [a,b] \subset \mathrm{R} \to \mathrm{R}$ satisfies the intermediate value property if for all standard real numbers λ between standard values $f(a)$ and $f(b)$, there exists a standard number Z; $z \in \mathrm{I}$; $a \leq z \leq b$ such that $f(z) = \lambda$].

Take I to be the interval $[a,b]$ and $x, y \in \mathrm{I}$. Since both f and λ are standards, then Z; $a \leq z \leq b$, is also standard because standard functions cannot take standard value at nonstandard variables. Conversely, by applying forward direction of (**TP**) on Definition 2.1 we get Definition of **IVP**.

The Transfer Principle (**TP**) is one of the main tools for exchanging the valid statements between standard and nonstandard senses. Lemma 2.2 shows how Definition 2.1 made up on it. In a similar manner, we proved the valid of the nonstandard version of some definitions given in the introduction of the paper and it takes the following forms:

A standard function f is upper semicontinuous at a standard point X if $f(y) \in m^{+}(f(x))$ whenever $x \approx y$. Oscillation of f at a point X is defined as the quantity $\omega_f(x) = sup\{|f(y) - f(x)|\}$ for all $y \in m(x)$. A function $f : \mathrm{R} \to \mathrm{R}$ is called symmetrically s-continuous if for each $x \in \mathrm{R}$, $f(x+h) \approx f(x-h)$ for every $h \approx 0$.

Theorem 2.3 Every continuous function is satisfy **IVP-SV**.

Proof. Let $f : \mathrm{R} \to \mathrm{R}$ be a a standard continuous function and λ be a standard real numbers such that:

$f(a) \leq \lambda \leq f(b)$ for standard $a, b \in \mathrm{R}$. To find Z; $a \leq z \leq b$ such that $f(z) = \lambda$,

take $S = {}^{st}\{x \in [a,b] : f(x) \square \lambda\} \subset \mathrm{R}$, then $S \neq \varnothing$ because $\exists a \in S$ and S is bounded by a standard element b.

Since R is complete, then S has a least upper bound (lub) in R, standard or nonstandard. Let $z = lub(S)$. Since b is an upper bound, $z \leqq b$.

We now show that $f(z) = \lambda$. Since $z = lub(S)$, either $z \in S$ or $z \in m(S)$. If $z \in S$, then $f(z) \leqq \lambda$. If $z \in m(S)$, then there exists a standard sequence $\{t_n\}_{n \in \mathrm{N}} \in S$ such that $t_\omega \approx z$ for all unlimited ω. Since f and t_ω are standards, $t_\omega \in S$, then $f(t_\omega) \leq \lambda$. Since f is continuous, then $f(z) \approx f(t_\omega) \leqq \lambda$ for all unlimited ω.

Thus in either case $f(z) \square \lambda$.

To prove $f(z)=\lambda$. First suppose $f(z)<\lambda$. Since f is continuous at z, therefore for all $x\approx z$ we have $f(x)\approx f(z)$.

That is, for all standard $\varepsilon>0$, [there exist a standard $\delta>0:|f(x)-f(z)|<\varepsilon$, whenever $|x-z|<\delta$]. Using (**TP**) twice, we get for all $\varepsilon>0$, [there exist $\delta>0:|f(x)-f(z)|<\varepsilon$, whenever $|x-z|<\delta$]. Take $\varepsilon=\dfrac{\lambda-f(z)}{\omega}$, then for all $x\in m(z)\cap[a,b]$, we see that

$f(z)-\varepsilon<f(x)<f(z)+\varepsilon$

Since $f(z)<\lambda$, $z\neq b$, then $m(z)\cap(z,b]\neq\varnothing$. Thus, for any $x\in(z,b]$ with $x\in m^{+}(z)$, we have

$$f(x)<f(z)+\varepsilon=f(z)+\frac{\lambda-f(z)}{\omega}=\frac{\lambda}{\omega}+f(z)\left(1-\frac{1}{\omega}\right)<\lambda$$

That is $f(x)<\lambda$. But then $x\in S$ and $x>z$, which contradicts $z=lub(S)$. Therefore $f(z)=\lambda$.

Remark 2.4 The intermediate value Property is one of the fundamental properties of calculus. The property implies that if I is an interval and $f:\mathrm{I}\subset\mathrm{R}\to\mathrm{R}$ is continuous, then $f(\mathrm{I})$ is an interval. The following corollary is a nonstandard equivalent version of the above remark.

Corollary 2.5 If $\mathrm{I}\subset\mathrm{R}$ is a standard interval and $f:\mathrm{I}\to\mathrm{R}$ is continuous on I, then $f(\mathrm{I})$ is a standard interval.

Proof. Let α,β be standard numbers belongs to $f(\mathrm{I})$ with $\alpha<\beta$, and let $a,b\in\mathrm{I}$ with $a\neq b$ be such that $f(a)=\alpha$ and $f(b)=\beta$. Suppose λ satisfies $\alpha<\lambda<\beta$. If $a<b$, then since f is continuous on $[a,b]$ by Theorem 2.3 there exists $z\in(a,b)$ such that $f(z)=\lambda$. Thus $\lambda\in f(\mathrm{I})$. A similar argument also holds if $a>b$.

Corollary 2.6 Every s-continuous function satisfies **IVP-SV**.
Proof. From the fact that every s-continuous is continuous and by Theorem 2.3 the result follows.

Remark 2.7 Classically the converse of Theorem 2.3 does not holds generally. See the following example.
Example 2.8 Let the function $f:\mathrm{R}\to\mathrm{R}$ defined by $f(x)=\begin{cases}\sin\left(\dfrac{1}{x}\right) & x\neq 0\\ 0 & x=0\end{cases}$ satisfies IVP but is not continuous.
The following lemma shows that s-continuity is stronger than continuity in the sense of **IVP-SV**. With s-continuity the values of variables a and b given in the Definition 2.1 are need not necessary to be standards.

Lemma 2.9 Let f be a s-continuous function from a metric space (X,d) into a metric space (Y,ρ). Then for every $a,b\in X$ and standard real number λ such that $f(a)<\lambda<f(b)$ there exist a standard $z\in(a,b)$ such that $f(z)=\lambda$.

Proof. Let t be a greatest point in X such that $f(t)<\lambda$. Then for $a\leq t\leq b$ we have $f(t)\leq\lambda<f(t+\alpha)$, where α is positive infinitesimal. Since $t\approx t+\alpha\approx st(t)$ and f is continuous then $f(t)\approx f(t+\alpha)\approx f(st(t))$. Since λ is standard, then $\lambda=f(st(t))$. Put $z=st(t)$, the result follows.

In the following theorem and lemmas we present some applications of **IVP-SV** concerning the differentiation concepts.

Theorem 2.10 Let f be an internal continuous function defined on an interval I. Then $\forall x,y\in\mathrm{I};x\neq y$ such that $x\in m(y)$, the shadow of the set $S=\left\{\dfrac{f(x)-f(y)}{x-y}:x,y\in\mathrm{I};x\neq y\right\}$ is a galaxy interval of points on the graph of the function f.

Proof. Let $S^{*}=\{\lambda\in S:\lambda\ is\ standard\}$.

Fix a point $\lambda\in S^{*}$, then $\exists^{st}x,y\in\mathrm{I},x<y$ such that $\lambda=\dfrac{f(y)-f(x)}{y-x}$, then λ is standard. Let $\lambda^{*}=\dfrac{f(v)-f(u)}{v-u},u<v.$ Since $x<y$ and $u<v$ then for all $t\in[0,1]$ we have $(1-t)(y-x)+t(v-u)\neq 0$. Define $g:[0,1]\to S$ as follows:

$$g(t)=\frac{f(\{1-t\}y+tv)-f(\{1-t\}x-tu)}{\{1-t\}y+tv-\{1-t\}x-tu},t\in[0,1].$$

It's clear that

$$g(0)=\lambda, and\ \ g(1)=\lambda^{*} \qquad (1)$$

Since f is continuous, then so is g.
Thus, by Corollary 2.5 we obtain that $g([0,1])$ is an interval, From Equation (1) we obtain that λ and any other points λ^{*} of S belong to a bounded interval subset S. That is, S^{*} is a standard interval for all points belong to S. Thus, S is also bounded interval.
Therefore, $gal(\lambda)=S$ and $S^{*}=st(S)$.

Lemma 2.11 Let $f:\mathrm{I}\to\mathrm{R}$ be s-differentiable function with standard differentiation, defined on an internal open interval I. Then for every nonstandard number λ such that $f'(a)\leq\lambda\leq f'(b)$ there exist a standard z; $a\leq z\leq b$ such that $f'(z)=\lambda$.

Proof. Let $E=\left\{\dfrac{f(x+\delta)-f(x)}{\delta}:x\in\mathrm{I},\delta\ is\ infinitesimal\right\}$.

Since f is continuous then by Theorem 2.10 we obtain that E is an interval. Let $S=\{f'(x):x\in\mathrm{I}\}$. To prove $f'(z)=\lambda$, by Corollary 2.5 it is only required to prove that S is an interval. Let $\mathrm{H}=m(E)$. Then $S\subset\mathrm{H}$, but $E\subset S\subset\mathrm{H}$, by Theorem 2.10 we have E is an interval and definition of monad of sets implies that S is an interval.

Lemma 2.12 Let f and g be two internal functions defined on an open interval I into R satisfying **IVP**. If f is limited and g is infinitesimal, then the shadow of $f+g$ satisfy **IVP**.

Proof. Since both f and g are satisfying **IVP**, then there exist Z and u between x and y where $x, y \in \mathrm{I}$, such that $f(z)=\lambda$ and $g(u)=\gamma$ for $f(x) \le \lambda \le f(y)$ and $g(x) \le \gamma \le g(y)$.

Let $(f+g)(x) \le \mu \le (f+g)(y)$. Since f is limited and g is infinitesimal, then $(f+g)(x) \approx f(x)$ for all x .Sin $\lambda = f(z)$,then $(f+g)(z) \approx f(z) = \lambda$, that is $(f+g)(z) \approx \lambda$.

Thus ${}^{o}(f+g)(z) = \lambda$. Hence, choose $w = z$,the result follows.

Lemma 2.13 If f is a standard function satisfying Definition 2.1 and not continuous at a standard point x, then there is a an interval (a,b) such that for all standard $y \in (a,b)$ there is a sequence x_n such that $x_n \to x$ and $f(x_n) = y$ for all unlimited n.

Proof. Since f is not continuous at x, then there are sequences y_n and t_n such that $y_n \approx x \approx t_n$ for all unlimited n and $f(x_n) \approx a < b \approx f(t_n)$ for some unlimited n. That is, $f(y_n) < f(t_n)$. For all standard $y; a < y < b$ we obtain that $f(y_n) < y < f(t_n)$. Use Definition 2.1 and from being $y_n \approx x \approx t_n$ we deduce that there is a sequence $x_n \in m(x)$ such that $y_n < x_n < t_n$ and $f(x_n) = y$.

Gibson and Natkaniecn (1996) gave a proof of the following theorem in a standard method, here we modify and reprove it by using nonstandard tools and showing that there are more types of Cantor sets satisfying the requirement of the given theorem.

Theorem 2.14 If $f : \mathrm{R} \to \mathrm{R}$ is a function satisfying **CIVP**, then f is perfect road function.

Proof. Let $x \in \mathrm{R}$ such that the function f is constant on $\mathrm{R} \setminus m^{-}(x)$. Let X_n be a standard increasing sequence in $m^{-}(x)$ such that $x_n \to x$, $f(x_n) \ne f(x)$ and $f(x_n) \to f(x)$, that is $x_n \in m^{-}(x)$ and $f(x_n) \in m(f(x))$ for all unlimited n such that $|f(x_n) - f(x)| >; 0$. from definition of s-continuity, we obtain that the function f is s-continuity and for every unlimited n not both $f(x_n)$ and $f(x)$ are standards at the same time, because on contrary we have $f(x_n) = f(x)$. Therefore, for some standard n, select any Cantor set K_n between $f(x_n)$ and $f(x)$. According to the value of n, the Cantor set K_n will be as follows

$$K_n = \begin{cases} \textit{infinitesimal set} & n \textit{ unlimited} \\ \textit{appreciable set} & n \textit{ appreciable} \\ \phi & n, x, f \textit{ are standards} \end{cases}$$

Since f satisfying **CIVP**, then there exists a Cantor set C_n between x_n and X as follows

$$C_n = \begin{cases} \textit{infinitesimal set} & n \textit{ unlimited} \\ \textit{appreciable set} & n \textit{ appreciable} \\ \phi & n, x, f \textit{ are standards} \end{cases}$$

such that $f(C_n) \subseteq K_n$. Let $A = \bigcup_n C_n \cup \{x\}$ for standard n. Then A is perfect and $f|_A$ is s-continuous on $m^{-}(x)$. Similarly we can find perfect set B such that $f|_B$ is s-continuous on $m^{+}(x)$. Let $P = A \cup B$. Then the function f is s-continuous on $m(x) = m^{-}(x) \cup m^{+}(x)$ and it is also perfect.

Theorem 2.15 If $f : \mathrm{R} \to [0,1]$ is a bilaterally dense in itself, symmetrically s-continuous and the set of s-continuous points of f is subset of $f^{-1}(0)$, then the oscillation function, ω_f, is satisfying Definition 2.1.

Proof. Let $C(f)$ be the set of all s- continuity points of f. By Theorem 2.5 in [20] the oscillation function ω_f is upper semicontinuous, symmetrically continuous and $C(f) = \omega_f^{-1} = C(f) \subset f^{-1}(0)$. Since f is symmetrically continuous, then C(f) is dense in R. To see that the graph of ω_f is bilaterally dense in itself, let $x \in \mathrm{R}$ and $\varepsilon \approx 0^{+}$ .Since $C(f) \subset f^{-1}(0)$ and $C(f)$ is dense in R, then $\omega_f^{-1} \approx sup\{| f(y) - f(z) |: y, z \in m(x)\}$
$\approx sup\{f(y) : y \in m(x)\} \boxed{\ge}\approx f(x)$.

Since f is symmetrically continuous at X and ω_f^{-1} is upper semicontinuous, there exits $\delta \approx 0^{+}$ such that: if $h \in m^{+}(\delta)$, then $f(x+h) \approx f(x-h)$ and if $x \approx y$, then $\omega_f^{-1}(y) <; \omega_f^{-1}(y)$. Since f is bilaterally dense in itself and $C(f) \subset f^{-1}(0)$, then for every $h \in m^{+}(\delta)$ there exist $\alpha, \beta \in m(x) \setminus \{x\}$ symmetric with respect to X. That is, $\alpha + \beta = 2x$ such that $f(\alpha) \approx \omega_f^{-1}(x)$ and $f(\delta) \approx f(\beta)$. Therefore $\omega_f^{-1}(x) \approx f(\beta)$.

Now, since $\alpha \in m(x) \setminus \{x\}$ and from being $\omega_f^{-1}(y) <\approx \omega_f^{-1}(y)$ we deduce that $\omega_f^{-1}(\alpha) <\approx \omega_f(x)$ $\omega_f^{-1}(\beta) <\approx \omega_f(x)$.

Due to the fact that ω_f satisfies Definition 2.1 and its graph having each $(x, \omega_f(x))$ as a bilateral limit point(Kuratowski and Sierpinski, 1922).

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کورتیا لێکولینێ:

لەم توێژینەوەیە, هەندێك لە سـیفەت و ئەنجامی نوێ دەربارەی تایبەتمەندی نرخی ناوەندینە IVP بەدەسـت گەیشـت بەهۆی بەکارهێنانی چەمکی نا پێوانەیی و گۆرانکاریکردنی هەندێك لە ئەنجامەکانی پێشوو و دیاریکردنی رۆڵی چەمکی شیکاری ناپێوانەیی بۆ بەدەستهێنانی ئەنجامی نا پێوانەیی.

خلاصة البحث:

في هذه البحث ، يتم إعطاء بعض الخصـائص والنتائج الجديدة عن خاصـية القيمة المتوسـطة IVP من خلال مفاهيم غير قياسـية وتعديل بعض النتائج الحالية لإظهار دور و مميزات أدوات التحليل غير القياسية للحصول على نتائج غير قياسية متميزة.