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NULL SPACES DIMENSION OF THE EIGENVALUE -1 IN A GRAPH

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ABSTRACT:

In geographic, the eigenvalues and eigenvectors of transportation network provides many informations about its connectedness. It is proven that the more highly connected in a transportation network G has largest eigenvalue and hence more multiple occurrences of the eigenvalue -1. For a graph G with adjacency matrix A, the multiplicity of the eigenvalue -1 equals the dimension of the null space of the matrix A + I.

In this paper, we constructed a high closed zero sum weighting of G and by which its proved that, the dimension of the null space of the eigenvalue -1 is the same as the number of independent variables used in a non-trivial high closed zero sum weighting of the graph. Multiplicity of -1 as an eigenvalue of known graphs and of corona product of certain classes of graphs are determined and two classes of -1- nut graphs are constructed.

KEYWORDS: Graph Theory, High Zero Sum Weighting, Adjacency Matrix, Nullity, Corona Product.

1. INTRODUCTION

A simple graph G with vertex set V ,V = v_1, v_2, … , v_n , is singular provided that its adjacency matrix A(G) is a singular matrix. The eigenvalues $\lambda_1, \lambda_2, \dots, \lambda_n$ of A(G) are said to be the eigenvalues of the graph G, which form the spectrum of G, denoted by Sp(G). The occurrence of zero as an eigenvalue in the spectrum of the graph G is called its nullity and is denoted by η or *η(G).* The nullity η (G) is the dimension of the null space of A(G) (Cheng and Liu, 2007). The nullity η (G) = n if, and only if, G is the empty (null) graph. Let G = $G_1 \cup G_2 \cup G_3 \cup, \dots, \cup G_k$ then, $\eta(G)=\sum_{i=1}^{k} G_i$ where G_1, G_2, … ,G_k are connected components of G See (Ali, *et al.,* 2016a; Ali *et al.,* 2019).

A vertex weighting of a graph G is a function f : V(G)→ ℝ where ℝ is the set of real numbers, which assigns real numbers (weights) to each vertex. The weighting of G is said to be non-trivial if there is at least one vertex v ∈V (G) for which f(v) ≠ 0. A non-trivial vertex weighting of a graph G is called a zero-sum weighting provided that for each v ∈ V (G), $\sum f(u)$=0, where the summation is taken over all u ∈ N(v). Out of all zero-sum weightings of a graph G, a high zero-sum weighting (hzsw) of G is one that uses a maximum number of non-zero independent variables (Sharaf and Rashed, 2002).

A graph G is a Smith graph if 2 is an eigenvalue of G, a 2-weighting technique is applied to characterize some classes of Smith graphs as well as to study the nullity of vertex identification of such graphs. It is also proved that under certain conditions the vertex identification of some Smith graphs is a Smith graph (Mohiaddin and Khidir, 2018).

In other words a graph with nullity η contains cores determined by a basis for the null space of A. A singular graph, on at least two vertices, with a kernel eigenvector having no zero entries, is said to be a core graph.

The removal of a vertex v from a graph G results in a subgraph G – v of G consisting of all vertices of G except v, and all edges not incident with v. For two distinct vertices a and b of a graph G, we define G + ab to be a graph obtained from G by adding a new vertex ab which is adjacent to all vertices in N(a) ∪ N(b), the graph G.ab is G + ab - a - b.

Definition 1.1 (Mohan *et al.,* 2017). Let G_1 and G_2 be two graphs with vertex set V(G_1) ={v_1 , v_2, … , v_{n1}} and V(G_2) = {u_1, u_2, … , u_{n2}} respectively, the corona of G_1 and G_2 denoted by G_1O G_2 is defined as taking one copy of G_1 and n_1 copies of G_2 by adjoining i^{th} vertex of G_1 to each vertex of G_2 in i^{th} copy.

Many invariant of the corona product G_1O G_2 such as nullity, domination number, colorability are related to the invariants of G_1 and of G_2.

Theorem 1.2 *(Interlacing Theorem)*
*(*Brouwer and Haemers, *2011*). *Let G be a graph with eigenvalues $\lambda_1 \geq \lambda_2 \geq \dots \geq \lambda_n$, and let the eigenvalues of G - v be $\mu_1 \geq \mu_2 \geq \dots \geq \mu_{n-1}$. Then the eigenvalues of G-v are interlaced with the eigenvalues of G, that is, $\lambda_1 \geq \mu_1 \geq \lambda_2 \geq \mu_2 \geq \dots \geq \mu_{n-1} \geq \lambda_n$.*

A vertex of a graph can be a core vertex if, on deleting the vertex, the nullity decreases, or a Fiedler vertex, otherwise. They adopt a graph theoretical approach to determine conditions required for the identification of a pair of prescribed types of root vertices of two graphs to form a cut-vertex of unique type in the coalescence. Moreover, the nullity of subgraphs obtained by perturbations of the coalescence graph is determined relative to its nullity (Ali *et al.,* 2016a).

The change in nullity when graphs with a cut-edge, and others derived from them, undergo geometrical operations, the deletion of edges and vertices, the contraction of edges and the insertion of an edge at a coalescence vertex are studied in (Ali *et al.,* 2016b). It is proved that, two connected labeled graphs H_1 and H_2 of nullity one, with identical one-vertex deleted subgraphs H_1 -z_1 and H_2 -z_2 and having a common eigenvector in the nullspace of their 0 - 1 adjacency matrix, can be overlaid to produce the superimposition graph G. The graph G is H_1+z_2 and also H_2+z_1 whereas G+e is obtained from G by adding the edge z_1z_2 (Ali *et al.,* 2019).

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In this paper the dimension of the null space of the eigenvalue -1 of a simple graph by introducing a new idea, denoted by high closed zero sum weighting is studied.
Finally, K_n, $K_{a,b}$, C_n, P_n are complete, complete bipartite, cycle and path graphs of order n, for more definition and details see (Cvetkovic *et al.,* 1979). All graphs mentioned in the following are connected graphs except where stated.

2. MULTIPLICITY OF THE EIGENVALUE -1 IN A GRAPH

In this section, we define the occurrence of -1 as an eigenvalue of a graph G. It is clear that -1 is an eigenvalue of a matrix A if and only if determinant of the matrix A + I is equal to zero.

Definition 2.1 A non-trivial vertex weighting of a graph G is called a closed zero-sum weighting provided that for each $v \in V(G)$, then $\sum f(u)=0$, where the summation is taken over all $u \in N[v]$. Out of all closed zero-sum weightings of a graph G, a high closed zero-sum weighting (hczsw) of G is one that uses maximum number of non-zero independent variables.

Proposition 2.2 *For any graph G, -1 is an eigenvalue of G iff G possesses a closed zero-sum weighting.*
Proof. Let A be the adjacency matrix of G. If -1 is an eigenvalue of G then, the matrix A+I is a singular matrix and hence there exist a non-zero vector X such that (A + I)X = 0. This, means that G possesses a closed zero-sum weighting. On the other hand, if G possesses a closed zero-sum weighting,
then, (A+I)X = 0, where X is a non zero vector. Hence AX = -X which proves that -1 is an eigenvalue of G. □

It is easy to show that, for the next largest eigegvalue $\lambda_2(G)$ to be -1 the graph must be complete and for the third largest eigenvalue $\lambda_3(G)$ it is known that $\lambda_3(G)$ = -1 iff G^c is isomorphic to the union of a complete bipartite graph and some isolated vertices.
Note : We denote the multiplicity of -1 as an eigenvalue of a graph G by $m_{-1}(G)$.
Here we give some illustrated graphs with hczsws, which classify that $m_{-1}(C_3) = 2$ and $m_{-1}(P_6 + e) = 2$, where $e = v_4v_6$ in the usual labeling of the path P_6.

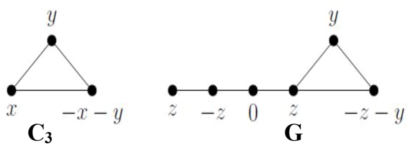

Figure 1. hczsw for C_3 and G.

Theorem 2.3 *In a graph G, $m_{-1}(G)$ equals the maximum number of independent variables in a hczsw of G and conversely.*
Proof. Let A be the adjacency matrix of G, and I be the identity matrix. -1 is an eigenvalue of G with multiplicity m, if and only if zero is an eigenvalue of the matrix A + I with the same multiplicity m. There exist a hzsw of G that uses exactly m independent variables, Say X. Thus (A + I)X = 0 gives AX = -IX, then the same number of independent variables is used in a hczsw for G. □

Corollary 2.4 *The following are equivalent :*
i) -1 is an eigenvalue of a graph G.
ii) The matrix A + I is a singular matrix.
iii) There exist a non zero eigenvector X such that AX = -X.
iv) G possesses a closed zero-sum weighting.
Proof. Is direct. □

Theorem 2.5 *If G is r-regular graph and $r = \lambda_1 \geq \lambda_2 \geq \ldots \geq \lambda_n$ are the eigenvalues of G, then the eigenvalues of G^c are n-1-r and $\{-1-\lambda_i ; 2 \leq i \leq n\}$.*
Proof. Because $A(G^c) = J - I - A(G)$, where J is an n×n matrix in which each entry is one and I is the identity matrix; G is (n-1-r)-regular, so the largest eigenvalue is n-1-r with corresponding eigenvalue $Y = [1, 1, \ldots, 1]^t$.
Any other eigenvalue λ has an eigenvector X orthogonal to Y , and hence $A(G^c)X = (JI - A(G))X = (0 - 1 - \lambda)X$. □

Corollary 2.6 *For any graph G having at least 2 vertices $|\eta(G) - m_{-1}(G^c)| \leq 1$ strictly holds if $G = P_{4n+3}$. Equality holds if $G = K_n$.*
Proof. Direct from Theorem 2.3. □

3. MULTIPLICITY OF -1 AS AN EIGENVALUE IN SOME KNOWN GRAPHS

Let H be any graph in which the multiplicity of -1 is m, then we prove the following:
Lemma 3.1 *If G is a graph obtained by identifying an end vertex of P_4 with any vertex of a graph H then, $m_{-1}(G) = m_{-1}(H)$.*
Proof. Label the vertices of P_4 in a usual way v_1, v_2, v_3 and v_4 where v_4 is the identified vertex with a vertex of the graph H. In a hczsw of G there are two possibilities for the weighting of the identified vertex $v_4 \equiv v$.
a) If w(v) = 0 then, $w(v_1) = w(v_2) = w(v_3) = 0$ and hence removing v_1; v_2 and v_3 will not change any weight in the hczsw of H so $m_{-1}(G) = m_{-1}(H)$.
b) If w(v) ≠ 0 then, $w(v_1) = x$, $w(v_2) = -x$ and $w(v_3) = 0$ and again removing v_1, v_2 and v_3 will also not change any weight in the hczsw of H so, $m_{-1}(G) = m_{-1}(H)$. See Figure 2.

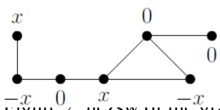

Figure 2. hczsw of the graph G.

Proposition 3.2 *Null spaces dimension of -1 as an eigenvalue of a complete graph K_n is defined as:*

$$m_{-1}(K_n) = \begin{cases} 0 & if \quad n = 1 \\ n-1 & if \quad n \neq 1 \end{cases}$$

Proof. Is direct, from the number of distinct variables in a hczsw of K_n, namely $x_1, x_2, \ldots, x_{n-1}, -x_1, - x_2, \ldots -x_{n-1}$. □

Proposition 3.3 *Null spaces dimension of -1 as an eigenvalue of a path P_n is defined as:*

$$m_{-1}(P_n) = \begin{cases} 1 & if \quad n = 2 \bmod 3 \\ 0 & otherwise \end{cases}$$

Proof. Assume that n = 2, then a hczsw of P2 is x, -x and hence -1 is an eigenvalue of P_2 with multiplicity 1. For n = 2 mod 3, any hczsw of P_n is of the form x, -x, 0, x, -x, …,0, x, -x where the triple 0, x, -x is repeated (n-2)/3 times. Hence, dimension of the null space of -1 is 1. If n ≠2 mod 3, then there exist no non trivial hczsw for P_n. That is, the dimension of the null space of -1 is zero. □

Proposition 3.4 *Null spaces dimension of -1 as an eigenvalue of a cycle C_n is defined as:*

$$m_{-1}(C_n) = \begin{cases} 2 & if \quad n = 3 \\ 1 & if \ n = 0 \bmod 3 \ and \ n \neq 3 \\ 0 & otherwise \end{cases}$$

Proof. Assume that n = 3, then a hczsw of C_3 is x, y,-x - y which uses 2 independent variables, hence -1 is an eigenvalue of C_3 with multiplicity 2.

For n = 0 mod 3, any hczsw of C_n is of the form x, -x, 0, x, -x, … , 0, x, -x, 0 where the triple x,-x, 0, is repeated n/3 times. Thus, dimension of the null space of -1 is 1. If $n \neq 0$ mod 3 then, there exist no non trivial hczsw for C_n, that is, the dimension of the null space of -1 is zero.

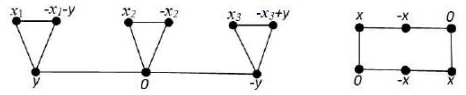

Figure 3. A hczsw for a graph with null space of dimension 4 and a hczsw of C_6.

For the complete graph K_n, there exist only one hczsw that uses exactly n-1 independent variables namely, $x_1, x_2, x_3, \ldots, x_{n-1}, -x_1-x_2-x_3-\ldots-x_{n-1}$, this proves that the dimension of the null space of -1 of the complete graph K_n is n - 1. By Interlacing Theorem (adding or removing a vertex to a graph changes the dimension of the null space of any eigenvalue by at most one) So, if t isolated vertices are adjacent to distinct vertices of a complete graph K_n, where $t \leq n-1$, to form a graph K_n^t. Then
$m_{-1}(K_n^t)$ == n-1-t. See Figure 3 where n = 4 and t = 2.

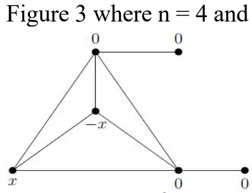

Figure 4. hczsw for K_4^2.

Proposition 3.5 The dimension of the nullspace of -1 as an eigenvalue of a complete bipartite graph $K_{a,b}$ is:

$$m_{-1}(K_{a,b}) = \begin{cases} 1 & if \quad a = b = 1 \\ 0 & otherwise \end{cases}$$

Proof. Any hczsw of $K_{a,b}$ must satisfy: $x_i + \sum y_j = 0$ and $y_j + \sum x_i = 0$, for i = 1, 2, … , a and j = 1, 2, … , b which gives $x_1 = x_2 = \ldots = x_a$ and $y_1 = y_2 = \ldots = y_b$. Then $x_1 = -ay_1$ and $y_1 = -bx_1$, thus $x_1 = bax_1$, hence ba = 1 that is a = b = 1. So, the dimension of the nullspace of -1 as an eigenvalue of a complete bipartite graph $K_{1,1}$ is one and zero otherwise. □

Existence of a triangle in a graph G is not a sufficient condition to have -1 as an eigenvalue, but if the graph G is of the form $C_3 \bullet H$, then -1 is an eigenvalue because of the czsw say x,-x, 0 of vertices of C_3 where weight of the identified vertex is 0. This gives the prove of the next corollary.

Corollary 3.6 For the graph $C_3 \bullet H$ we have:

$$m_{-1}(C_3 \bullet H) \leq m_{-1}(H) + 1$$

Now, given any number n it is easy to construct a tree T in which the dimension of the nullspace of -1 of T is n by using hczsws technique. To do this insert a new vertex to each edge of the star graph $S_{1,2n}$, then the hczsw of this tree is 0; $x_1, -x_1; -x_1$; x_1, $x_2, -x_2, -x_2, x_2, \ldots, x_n, -x_n, -x_n, x_n$, where 0 is the weight of the central vertex and other vertices, are alternatively positive
and negative for each variable.

4. VERTEX IDENTIFICATION

The following results will be proved for a vertex identi_cation of two graphs G_1 and G_2 with rooted vertices v_1 and v_2.

Definition 4.1 In a czsw of a graph G each vertex with non zero weight is called a *-1 – core vertex*, and the vector whose entries are weights of such czsw is called a *-1 – core vector* of G.

Definition 4.2 In a hczsw of a graph G each vertex with zero weight is called a *-1 – core forbidden vertex*.

So all vertices of a graph are either -1 -core or -1 - core forbidden vertices and these two types of vertices plays the main rule on the null spaces dimension of the eigenvalue -1 in the vertex identification graph $G_1 \bullet G_2$ of two graphs G_1 and G_2.

Theorem 4.3 *For any two graphs G_1 and G_2 with rooted vertices v_1 and v_2 we have:*

$$m_{-1}(G_1)+m_{-1}(G_2) - 2 \leq m_{-1}(G_1 \bullet G_2) \leq m_{-1}(G_1)+m_{-1}(G_2) + 1$$

Equality holds in the left where $G_1 = G_2$ = triangle and on the right where $G_1 = G_2$ = triangle with a pendant.

Proof. Let G_1 and G_2 be two graphs with rooted vertices v_1 and v_2 with hczsws X and Y . With out loss of generality, assume that $w(v_1) = x_1$ and $w(v_2) = y_1$ in the above hczsws. Let z be a hczsw of $G_1 \bullet G_2$ and z_1 be the weight of the identified vertex v ($v_1 \equiv v_2$). So, we have the following cases:

1) If $x_1 = y_1 = z_1 = 0$ then $m_{-1}(G_1 \bullet G_2) = m_{-1}(G_1) + m_{-1}(G2)$ or $m_{-1}(G_1 \bullet G_2) = m_{-1}(G_1) + m_{-1}(G_2) + 1$.
2) If $x_1 = y_1 = 0$ and $z_1 \neq 0$ then $m_{-1}(G_1 \bullet G_2) = m_{-1}(G1) + m_{-1}(G2) + 1$, because a new variable is introduced.
3) If $x_1 \neq 0$ and $y_1 = 0$ or conversely and $z_1 = 0$ then $m_{-1}(G_1 \bullet G_2) = m_{-1}(G_1) + m_{-1}(G_2) - 1$, because a variable is vanished.
4) If $x_1 \neq 0$ and $y_1 = 0$ or conversely and $z_1 \neq 0$ then $m_{-1}(G_1 \bullet G_2) = m_{-1}(G_1) + m_{-1}(G_2)$.
5) If $x_1 \neq 0$ and $y_1 \neq 0$ and $z_1 = 0$ then $m_{-1}(G_1 \bullet G_2) = m_{-1}(G_1) + m_{-1}(G_2) - 2$, because 2 variables are vanished.
6) If $x_1 \neq 0$ and $y_1 \neq 0$ and $z_1 \neq 0$ then $m_{-1}(G_1 \bullet G_2) = m_{-1}(G_1) + m_{-1}(G_2) - 1$, because 2 variables are replaced by one. But this case does not exist because of the hczsws of the new neighbors of the vertex identified and hczsws of olds one. To get such contradiction, assume that such z exist, then z = ax = by for some a and b where, x = $w(v_1)$ in G_1 and y = $w(v_2)$ in G_2. Then $\sum w(N(v_1 \equiv v_2))$ in $G_1 \bullet G_2$ must equal to zero that is ax + $\sum_{G_1} w(N(v_1))$ + by + $\sum_{G_2} w(N(v_2))$ = 0 which implies that x + $\sum_{G_1} w(N(v_1))$ + (1-a)x + $\sum_{G_2} w(N(v_2))$ = 0 which is 0+1-by+ $\sum_{G_2} w(N(v_2))$= 0 therefore 1 - by - y = 0 thus 1 = (1 + b)y where is any variable, this is not possible.

All cases are discussed and prove is complete. □

We conclude that vertex identi_cation of two -1 - core vertices cannot give a -1-core vertex. One can answer? does there exist graphs that satisfy the above 6 cases, the answer is illustrated by the next table.

Table 1. Illustration of cases of Theorem 4.3

Case	G_1	X_1	G_2	Y_1	G	Z_1	$m_1(G_1 \bullet G_2)$
1	P_3	0	P_4	0	P_6	0	0
	P_3	0	P_3	0	P_5	z	+1
2	$K_{1,3}$ v_1 has degree 1	0	P_3 v_2 has degree 2	0	$K_{1,3} \bullet P_3$	z	+1
3	P_2	x	P_3	0	P_4	0	-1
4	P_2	x	P_4	0	P_5	z	0
5	C_3	x,u	C_3	y,v	$C_3 \bullet C_3$	x,y	-2
6	P_2	x	P_2	y	P_3	z=ax+by=0	-2

Theorem 4.4 *For any tree T having at least 2 vertices, whose root is an end vertex v then,*

$$m_{-1}(C_3 \bullet T) \geq m_{-1}(T)$$

strictly holds where $T = P_{4n}$, and equality holds where $T = P_2$

Proof. Label the vertices of C_3 as u_1, u_2, and $u_3 = v_1$(the root vertex) then,in any hczsw of $C_3 \bullet T$ put $w(u_1) = x$ and $w(u_2) = -x$. If $w(v_1) = 0$ then it is clear that when we remove u_1 and u_2 from the composite graph $C_3 \bullet T$ the variable x is removed so the number of independent variables in the hczsw is reduced by 1. If $w(u_3) = y$, in the hczsw of T, then put $w(u_1) = x$ and $w(u2) = -x-y$. Again removing u_1 and u_2 from the composite graph $C_3 \bullet T$ only the variable x is removed so the number of independent variables in the hczsw is again reduced by 1. Finally, -1 is not an eigenvalue for P4n, but a hczsw for $C_3 \bullet P_{4n}$ that uses 2 variables, hence $m_{-1}(C_3 \bullet P_{4n}) = m_{-1}(P_{4n}) + 2$ Also it is clear that $m_{-1}(C_3 \bullet P_2) = m_{-1}(P_2) = 1$. □

Theorem 4.5 *For any connected graph G having at least 2 vertices, with rooted vertex v, then*

$m_{-1}(P_{4n} \bullet G) = m_{-1}(G)$

where the root vertex of P_{4n} is one of its end vertices.

Proof. Label the vertices of P_{4n} as $u_{1,1}, u_{1,2}, u_{1,3}, u_{1,4}, u_{2,1}, u_{2,2}, u_{2,3}, u_{2,4}, \dots, u_{n,1}, u_{n,2}, u_{n,3}, u_{n,4} = v$(the root vertex) then, in any hczsw of $P_{4n} \bullet G$, if $w(u_{n,4}) = x$, the vertices of P_{4n} will be weighted as x, -x, 0, x, -x, 0, x, -x, 0, x so, it is clear that when we remove $u_{1,1}, u_{1,2}, u_{1,3}, u_{1,4}, u_{2,1}, u_{2,2}, u_{2,3}, u_{2,4}, \dots, u_{n,1}, u_{n,2}, u_{n,3}$ from the composite graph $P_{4n} \bullet G$ the variable x(or -x) is not removed so the number of independent variables in the hczsw remains the same. If $w(u_{n,4}) = 0$, in the hczsw of G, the vertices of P4n will be weighted as 0, 0, ... , 0, Again removing the above vertices from the composite graph $P_{4n} \bullet G$ no variable will be removed or increased so the number of independent variables in the hczsw again remains the same. □

As a usual question, one can ask: given any number k does there exist a graph or a tree or under a limited diameter for which the null spaces dimension of of -1 is k. So, the following cases are easily constructed:

1) The complete graph with order $n = k + 1$ is such a graph of diameter 1.
2) The vertex identification of the central vertices of k copies of P_5 is such a graph (tree with diameter 4).
3) The vertex identification of the central vertices of k copies P_{5+6t}, t = 1, 2, 3 , ... is such a graph (tree with diameter 4 + 6t).
4) The vertex identification of the (third) vertices of k copies of P_8 is such a graph (tree with diameter 10).
5) The vertex identification of the (third) vertices of k copies P_{8+6t}, t = 1, 2,3, ... is such a graph (tree with diameter 12t + 10).

A Fan graph F with n arms is de_ned to be the graph obtained from identifying a vertex of n copies of C_3, then $|F| = 2n + 1$ with dimension of -1 to be n, not a tree, connected graph with minimum diameter 2. Also, consider the generalized Fan graph F*, with a, b and c arms (Where at least non two of them are zero), which is also a connected graph with diameter 3 and $m_{-1}(F^*) = a + b + c$.

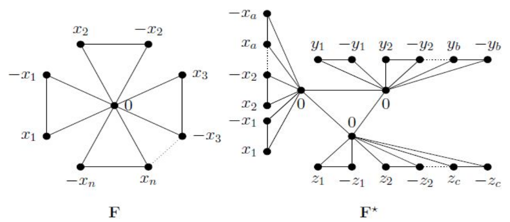

Figure 5. hczsw of the Fan graph F and generalized Fan graphs F*

5. NULL SPACES DIMENSION OF -1 IN THE CORONA PRODUCT OF TWO GRAPHS

In this section, we determine the dimension of the null space of -1 as an eigenvalue in the corona product of two graphs.

Proposition 5.1 *For paths P_n, $n > 0$ and P_t, $t > 0$ we have:*

$$m_{-1}(P_t O P_n) = \begin{cases} t+1 & if \quad n = 2 \bmod 3 \text{ and } t \text{ is odd} \\ t & if \quad n = 2 \bmod 3 \text{ and } t \text{ is even} \\ 1 & if \quad n \neq 2 \bmod 3 \text{ and } t \text{ is odd} \\ 0 & if \quad n \neq 2 \bmod 3 \text{ and } t \text{ is even} \end{cases}$$

Proof. Label the vertices of the graph P_tOP_n as $v_{1,1}, v_{1,2}, \dots, v_{1,n}, v_{2,1}, v_{2,2}, \dots, v_{2,n}, v_{t,1}, v_{t,2}, \dots, v_{t,n}$ and of P_t as $u_1, u_2, \dots u_t$. For n = 2 mod 3 and t is odd the vertices of the ith copy of P_n are weighted x_{ji}, $-x_{ji} - y$, 0, x_{ji}, $-x_{ji} - y$, 0, ... , x_{ji}, $-x_{ji} - y$ where $w(u_j) = y$, while the vertices of P_n are labeled y, 0, -y, 0, ... , +y or -y. Hence, there exist t + 1 independent variables in a hczsw of P_t O P_n.

If t is even then, y = 0 and the number of independent variables is reduced to t. For n ≠2 mod 3 and t is odd each $x_i = 0$ and the number of independent variables is reduced to 1, while if n ≠2 mod 3 and t is even each $x_i = 0$ and y = 0, then the number of independent variables is reduced to 0.

Proposition 5.2 *For any cycle C_n and path P_t then:*

$$m_{-1}(P_t O C_n) = \begin{cases} 2t+1 & if \quad n = 3 \text{ and } t \text{ is odd} \\ 2t & if \quad n = 3 \text{ and } t \text{ is even} \\ t & if \quad n > 3 \text{ and } n = 0 \bmod 3 \\ 0 & otherwise \end{cases}$$

Proof. Label the vertices of the graph P_tOC_n as $v_{1,1}, v_{1,2}, \dots, v_{1,n}, v_{2,1}, v_{2,2}, \dots, v_{2,n}, \dots, v_{t,1}, v_{t,2}, \dots, v_{t,n}$ and of P_t as $u_1, u_2, \dots, u_t$. For n = 3 and t is odd the vertex weighting of P_tOC_3 is $x_{1,1}$, $x_{2,1}$, $-x_{1,1} - x_{2,1} - y$, $x_{1,2}$, $x_{2,2}$, $-x_{1,2} - x_{2,2}$, $x_{1,3}$, $x_{2,3}$, $-x_{1,3} - x_{2,3} + y$, ... , $x_{1,i}$, $x_{2,i}$, $-x_{1,i} - x_{2,i} - y$ (or +y), ... , $x_{1,t}$, $x_{2,t}$, $-x_{1,t} - x_{2,t} - y$ (or +y), while the vertices of P_n are weighted by, y, 0, □y, 0, : : :, +y (or □y). That is the vertex uj is weighted by, y if, i = 1 mod 3 or 0 if i = 2 mod 3 or -y if i = 0 mod 3, respectively. Hence, there exist 2t + 1 independent variables in a hczsw of P_tOC_3.

If n = 3 and t is even then the variable y is vanished. If n > 3 and n = 0 mod 3 then y will vanish and $x_{2,i} = -x_{1,i}$ so another t variables will vanish. If n ≠ 0 mod 3 then there exist no non trivial hczsw for P_tOC_n and in this case $m_{-1}(P_tOC_n) = 0$.

Proposition 5.3 *For any complete graph K_n:*

$$m_{-1}(P_t\, O\, K_n) = \begin{cases} (n-1)t+1 & if \quad t \text{ is odd} \\ (n-1)t & if \quad t \text{ is even} \end{cases}$$

Proof. It is an extension of the prove of the first part of Proposition 5.1.

6. CONSTRUCTING -1-NUT GRAPHS

In this section, two classes of -1-nut graphs are constructed in the manner of nut graphs defined by (Sciriha, 2007)

Definition 6.1 A graph G is said to be a -1 - nut graph if -1 is an eigenvalue of G with multiplicity 1 such that every vertex of G has a non zero weight in its hczsw.

Thus, if G is a -1 - nut graph then there exist only one core vector inthe null space of the matrix A(G) + I whose size is n.

Proposition 6.2 *The graph $C_nOK_3^c$ is a -1-nut graph for each n, n > 2.*

Proof. For each n the graph $C_nOK_3^c$ is a semi-regular graph with degree set {1,5} and order 4n. There is only a unique hczsw for this graph which uses only one variable say x, where x is the weight of each vertex of degree 5 and -x is the weight of each vertex of degree one. Thus, the graph $C_nOK_3^c$ is a -1 - nut graph. □

Proposition 6.3 *The graph $C_nOK_n^c$ is a -1-nut graph for each n, n > 0.*

Proof. For each n the graph $C_nOK_3^c$ is a semi-regular graph with degree set {1, 2n – 1} and order $n^2 + n$. There is only a unique hczsw for this graph which uses only one variable say x, where x is the weight of each vertex of degree 2n - 1 and -x is the weight of each vertex of degree one. Thus, the graph $C_nO\ K_3^c$ is a-1-nutgraph. □

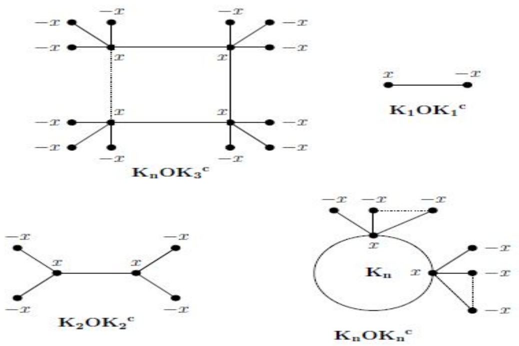

Figure 5. Two classes of -1 - nut graphs.

So, the order of the smallest nut graph is 2 and out of non trees -1-nut graphs, the minimum order is 12.

Moreover, the only -1-nut graphs which are tree are namely P_2 and $K_{1,3}O\ P_3$ mentioned in the above table.

Also it is clear that, the vertex identification of any two -1 - nut graphs is not a -1 - nut graph.

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