ON THE PARAMETRIZATION OF NONLINEAR IMPULSIVE FRACTIONAL INTEGRO–DIFFERENTIAL SYSTEM WITH NON-SEPARATED INTEGRAL COUPLED BOUNDARY CONDITIONS

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ABSTRACT:
We give a new investigation of periodic solutions of nonlinear impulsive fractional integro-differential system with different orders of fractional derivatives with non-separated integral coupled boundary conditions. Uniformly Converging of the sequence of functions according to the main idea of the Numerical-analytic technique, from creating a sequence of functions. An example of impulsive fractional system is also presented to illustrate the theory.

KEYWORDS: Caputo fractional derivative, fractional integro-differential, integral coupled boundary conditions, Periodic solutions, successive approximation method.

1. INTRODUCTION

Fractional differential equations have been developed in the last decade as good tools to describe the mathematical modeling of systems and processes in the fields of physics, chemistry, aerodynamics, economics, control theory and image processing, etc. (Kilbas et al., 2006, Sabatier et al., 2007, Lakshmikantham et al., 2009). On the other hand, we observe periodic motions in every field of science and everywhere in real life (Farkas, 1994). The applications and theory of the fractional differential equations have recently been collected by several researchers for different problems, we refer the reader to (Ahmad and Nieto, 2014, Henderson et al., 2015). The theory of impulsive differential equations is a new and important field of differential equation theory, many authors see (Agarwal et al., 2010, Wang et al., 2012, Feckan et al., 2014), which has been an object of intensive investigation in recent years, and applications to different areas have been considered. However, the concept of solutions for impulsive fractional differential equations (Zhou and Chu, 2012, Bai et al., 2016, Bai and Zhang, 2016, Dong et al., 2017) has been argued extensively.

Also an extended this method by (Feckan and Marynets, 2017) to nonlinear system of integro-differential equation (Butris et al., 2017, Butris, and Taher, 2019) and Caputo type differential equations with periodic boundary value problems (Mahmudov et al., 2019). Furthermore, the investigate of a coupled system of fractional order is also very significant because this type of system can often occur in applications. The reader is referred to read (Su, 2009, Wang et al., 2010), and the references cited therein.

Caputo-type fractional integro-differential system with nonlinear coupled integral boundary conditions has been considered in this paper. We apply picard approximations technique proposed by (Rontou and Samoilenko, 2000, Ronto et al., 2015, Marynets, 2016, Feckan and Marynets, 2018) for investigation the existence, uniqueness and approximation of periodic solutions of nonlinear fractional integro-differential system with nonlinear coupled integral boundary conditions. Motivated by the works mentioned and many known results, we use the numerical analytic method to investigate the existence and uniqueness of periodic solutions and define our problem

\[ g D^\alpha_+ u(t) = f(t, \beta(t, a), u(t), \int_0^t K(t, s)(u(s) - z(s))ds, t \neq t_i, \] \n
\[ g D^\alpha_- u(t) = g(t, \beta(t, \gamma), y(t), \int_0^t \beta(t, s)(u(s) - z(s))ds, t \neq t_i, \] \n
\[ \Delta u|_{t=t_i} = \sum_{k=1}^n h_k(\beta(t, a), u(t)) \] \n
\[ \Delta y|_{t=t_i} = \sum_{k=1}^n \gamma_k(\beta(t, a), y(t)) \] \n
\[ u(0) + Bu(T) = \int_0^T h_1(z(s))ds, \quad \text{with} \quad \det(B) \neq 0 \] \n
\[ Cz(0) + Dz(T) = \int_0^T h_2(z(s))ds, \quad \text{with} \quad \det(D) \neq 0 \] \n
for all \( t \in [0, T] \), \( A, B, C \) and \( D \) are \( n \times n \) matrices where \( g D^\alpha_+ \) and \( g D^\alpha_- \) denote the Caputo fractional derivatives, \( 0 < \alpha \leq 1, \) also \( \beta(t, a) \) and \( \beta(t, \gamma) \) are said to be special functions provided that (Beta function), the function \( f, g \in C([0, T] \times \Omega \times D_1 \times D_2), \Omega = [0, T] \times (0, 1], D_1 \) and \( D_2 \) are compact subset of, also \( a, b, h_1 \) and \( h_2 \) are continuous functions on \([0, T]\).

2. BACKGROUND MATERIAL

In this section, some definitions of fraction calculus and lemmas are presented which are used for the statement of the problem (1.1) and (1.2).

Definition 2.1 (Kilbas et al., 2006) For a function \( g \) given on the interval \([a, b]\), the Caputo fractional order derivative of \( g \) is defined by

\[ g D^\alpha g(t) = \frac{1}{\Gamma(n-\alpha)} \int_0^t \frac{g(t-s)^{\alpha-n+1}}{(t-s)^{\alpha-n}} g(s)ds \] \n
where \( n = [\alpha] + 1 \) and \( \lfloor \alpha \rfloor \) denotes the integer part of \( \alpha, \) and \( \Gamma(\cdot) \) denotes the Gamma function.
Lemma 2.3 Let g(t) be a continuous function for \( t \in [0, T] \), then the following estimate holds
\[
\left| \frac{1}{\Gamma(a)} \int_0^1 (t-s)^{a-1} g(s) \, ds - \frac{1}{\Gamma(a)} \int_0^T (T-s)^{a-1} g(s) \, ds \right| 
\leq \omega(t) \max_{t \in [0,T]} |g(t)|, 
\]
where \( \omega(t) = \frac{2t^a}{\Gamma(a+1)} \left( 1 - \frac{t}{T} \right)^a \). 

Proof. It is obvious that
\[
\left| \frac{1}{\Gamma(a)} \int_0^1 (t-s)^{a-1} g(s) \, ds - \frac{1}{\Gamma(a)} \int_0^T (T-s)^{a-1} g(s) \, ds \right| 
\leq \frac{1}{\Gamma(a)} \left| \int_0^1 \left( (t-s)^{a-1} - (T-s)^{a-1} \right) g(s) \, ds \right| 
\leq \frac{1}{\Gamma(a)} \int_0^1 (T-s)^{a-1} |g(s)| \, ds 
\]
and
\[
\left| \frac{1}{\Gamma(a)} \int_0^T (T-s)^{a-1} g(s) \, ds - \frac{1}{\Gamma(a)} \int_0^1 (t-s)^{a-1} g(s) \, ds \right| 
\leq \frac{1}{\Gamma(a)} \int_0^T (T-s)^{a-1} |g(s)| \, ds 
\]
Form (Fcken and Marynets, 2017), the terms becomes
\[
(t-s)^{a-1} - \left( \frac{1}{T} \right)^a (T-s)^{a-1} = (t-s)^{a-1} \left( 1 + \frac{t}{T} \left( 1 - \frac{s}{T} \right)^a \right) 
\geq (t-s)^{a-1} \left( 1 - \frac{t}{T} \left( 1 - \frac{s}{T} \right)^a \right) = (t-s)^{a-1} \left( 1 - \frac{t}{T} \right)^a \geq 0 
\]
For any \( s \in [0, t] \), we obtain that
\[
\left| \frac{1}{\Gamma(a)} \int_0^1 (t-s)^{a-1} g(s) \, ds - \frac{1}{\Gamma(a)} \int_0^T (T-s)^{a-1} g(s) \, ds \right| 
\leq \frac{2t^a}{\Gamma(a+1)} \left( 1 - \frac{t}{T} \right)^a \max_{t \in [0,T]} |g(t)| 
\leq \omega(t) \max_{t \in [0,T]} |g(t)| 
\]
The proof of lemma is complete.

Definition 2.4 The solutions of the system of fractional integro-differential equations (1.1) and integral boundary conditions (1.2) are defined by the following integral equations:
\[
u(t, x_0, y_0) = u_0 + \frac{1}{\Gamma(a)} \int_0^t (t-s)^{a-1} f(s, \beta(t, \alpha), u(s, u_0, z_0), \int_0^a K(s, \xi) [u(\xi, u_0, z_0) - z(\xi, u_0, z_0)] \, d\xi) \, ds 
\]
3. Conditions for Convergence of Successive Approximation
For investigate of the successive approximation for periodic solution of the problem (1.1) and (2.2), we need some conditions, suppose that the vector functions \( f, g \in C([0, T] \times \Omega \times D_1 \times D_2, R) \), \( f, f_i, j_i \in C([0, T] \times D_1, R) \), \( \Omega = (0, T) \times (0, 1) \), \( D_1 \) and \( D_2 \) are compact subsets of \( R^k \), and \( \varphi, h_1 \) and \( h_2 \) are continuous functions on \( [0, T] \), and satisfies the following hypothesis.

H1: There exist positive constants \( M, L, K_1, K_2, L_1, L_2, M_{\beta, \alpha} \) and \( M_{\beta, \gamma} \) such that
\[
\| f(t, \beta(t, \alpha), u, x) \| \leq M_{\beta, \alpha} \quad \| f_i(t, \beta(t, \alpha), x) \| \leq M_{\beta, \alpha} \quad \| g(t, \beta(t, \gamma), z) \| \leq M_{\beta, \gamma} 
\]
(3.1) and
\[
\| f(t, \beta(t, \alpha), u, x, z) \| \leq M_{\beta, \alpha} \quad \| f_i(t, \beta(t, \alpha), x, z) \| \leq M_{\beta, \alpha} \quad \| g(t, \beta(t, \gamma), z, y) \| \leq M_{\beta, \gamma} 
\]
(3.2) respectively,
\[
\begin{align*}
& \left\| I_i(t, \beta(t, \alpha), u_1, x_1) - I_i(t, \beta(t, \alpha), u_2, x_2) \right\| \leq M_{\beta, \alpha} \\
& \left\| I_i(t, \beta(t, \alpha), u_1) - I_i(t, \beta(t, \alpha), u_2) \right\| \leq M_{\beta, \alpha} \\
& \left\| g(t, \beta(t, \gamma), z_1, y_1) - g(t, \beta(t, \gamma), z_2, y_2) \right\| \leq M_{\beta, \gamma} \\
& \left\| I(t, \beta(t, \gamma), z_1, y_1) - I(t, \beta(t, \gamma), z_2, y_2) \right\| \leq M_{\beta, \gamma} \\
& \left\| I_i(t, \beta(t, \gamma), z_1) - I_i(t, \beta(t, \gamma), z_2) \right\| \leq M_{\beta, \gamma} \\
& \left\| I(t, \beta(t, \gamma), z_1, y_1) - I(t, \beta(t, \gamma), z_2, y_2) \right\| \leq M_{\beta, \gamma} \\
\end{align*}
\]
(3.3) respectively,
Furthermore, we suppose that the largest eigenvalue of the matrix
\[ \Lambda = \begin{pmatrix} H_1 & H_2 \\ H_3 & H_4 \end{pmatrix} \]
where
\[ H_1 = \frac{T^a}{2^{a-1}T(a + 1)} M_{\beta_0} K_1 + K_2 K_3 + 2 M_{\beta_2} S_1 K_3 \]
\[ H_2 = \frac{T^a}{2^{a-1}T(a + 1)} M_{\beta_0} K_2 + \|B^{-1}\|T p_1 \]
\[ H_3 = \frac{T^a}{2^{a-1}T(y + 1)} M_{\beta_2} L_2 R_2 + \|D^{-1}\|T q_1 \]
\[ H_4 = \frac{T^a}{2^{a-1}T(y + 1)} M_{\beta_2} L_1 + L_2 R_2 + 2 M_{\beta_2} S_2 L_3 \]
do not exceed unity, i.e.
\[ \lambda_{\text{max}}(\Lambda) = \frac{m_2 + \sqrt{m_2^2 + 4(m_1 - m_2)}}{2} < 1 \quad (3.10) \]

Our main results separate to the following four parts:

**4. MAIN RESULTS**

In this section, we study the periodic approximation solutions of the system of nonlinear fractional integro-differential equations (1.1) with coupled integral boundary conditions (1.2) will be introduced by the following theorem.

In the beginning, we define the following sequence of functions \( u_m \) and \( z_m \) given by the iterative formulas
\[ u_{m+1}(t, u_0, z_0) = u_0 + \frac{1}{t^a} \int_0^t \frac{1}{(T - s)^{a-1}} \left( f(s, \beta(t, \alpha), u_m(s, u_0, z_0), \int_0^s K(s, \tau)u_m(\zeta, u_0, z_0) \right) (t - s)^{-a} ds \]
\[ z_{m+1}(t, u_0, z_0) = z_0 + \frac{1}{t^a} \int_0^t \frac{1}{(T - s)^{a-1}} \left( g(s, \beta(t, \gamma), z_m(s, u_0, z_0), \int_0^s K(s, \tau)u_m(\zeta, u_0, z_0) \right) (t - s)^{-a} ds \]

**Theorem 4.1.** If the system (1.1) with boundary conditions (1.2) satisfy the conditions \( H_1, H_2, H_3, H_4 \), \( T, T_1, T_2, T_3 \), then sequences of functions (4.1) and (4.2), which are periodic in \( t \) of period \( T \), converges uniformly as \( m \to \infty \) on the domain:
\[ (t, u_0, z_0) \in [0, T] \times D_1 \times D_2 \quad (4.3) \]
and satisfies the following integral equations:
\[ u(t, u_0, z_0) = u_0 + \frac{1}{t^a} \int_0^t \frac{1}{(T - s)^{a-1}} \left( f(s, \beta(t, \alpha), u_m(s, u_0, z_0), \int_0^s K(s, \tau)u_m(\zeta, u_0, z_0) \right) (t - s)^{-a} ds \]
\[ z(t, u_0, z_0) = z_0 + \frac{1}{t^a} \int_0^t \frac{1}{(T - s)^{a-1}} \left( g(s, \beta(t, \gamma), z_m(s, u_0, z_0), \int_0^s K(s, \tau)u_m(\zeta, u_0, z_0) \right) (t - s)^{-a} ds \]
for all \( t \in [0, T], z_0 \in D_0 \) we obtain that \( z_1(t, u_0, z_0) \in D_2 \).

Thus by the mathematical induction, we find that

\[
\| u_m(t, u_0, z_0) - u_0 \| \leq \left( \frac{T}{2^{\gamma-1}(\alpha + 1)} M_{\beta_4} M + M_4 \right) \left( \frac{T}{2^{\gamma-1}(\gamma + 1)} M_{\beta_4} L + M_5 \right) \quad \ldots (4.7)
\]

means that \( u_m(t, u_0, z_0) \in D_1, z_m(t, u_0, z_0) \in D_2, u_0 \in D_1, z_0 \in D_2 \), for all \( t \in [0, T], m = 0,1,2, \ldots \).

Now, we claim that the sequences of functions (4.1) and (4.2) are uniformly convergent on the domain (4.3).

\[
\| u_{m+1}(t, u_0, z_0) - u_{m}(t, u_0, z_0) \| \leq \omega(t) \left( f(t, \beta(t, \alpha), u_m(t, u_0, z_0), \int_0^{u_m(t, u_0, z_0)} K(t, s)(u_m(s, u_0, z_0) - z_m(s, u_0, z_0))ds - F(t, \beta(t, \alpha), u_m(t, u_0, z_0)) \right)
\]

\[
+ \sum_{t_1 < t_2 < \ldots < t_m \leq t} \int_0^{u_{m-1}(t, u_0, z_0)} K(t, s)(u_{m-1}(s, u_0, z_0) - z_{m-1}(s, u_0, z_0))ds \right) \leq \omega(t) \rho(t) M_{\beta_4}(K_1 + K_2 z_0) + 2M_{\beta_4} S_1 z_0 \times \| u_m(t, u_0, z_0) - u_{m-1}(t, u_0, z_0) \| + \| B^{-1} T \| \| z_m(t, u_0, z_0) - z_{m-1}(t, u_0, z_0) \| \quad \ldots (4.8)
\]

In the same way, we obtain that

\[
\| z_m(t, u_0, z_0) - z_{m-1}(t, u_0, z_0) \| \leq \omega(t) \rho(t) M_{\beta_4}(K_1 + K_2 z_0) + 2M_{\beta_4} S_1 z_0 \times \| u_m(t, u_0, z_0) - u_{m-1}(t, u_0, z_0) \| + \| B^{-1} T \| \| z_m(t, u_0, z_0) - z_{m-1}(t, u_0, z_0) \| \quad \ldots (4.9)
\]

We rewrite inequalities (4.8) and (4.9) in vector form to gain

\[
\| u_m(t, u_0, z_0) - u_{m-1}(t, u_0, z_0) \| + \| z_m(t, u_0, z_0) - z_{m-1}(t, u_0, z_0) \| \leq \omega(t) \rho(t) M_{\beta_4}(K_1 + K_2 z_0) + 2M_{\beta_4} S_1 z_0 \times \| u_m(t, u_0, z_0) - u_{m-1}(t, u_0, z_0) \| + \| B^{-1} T \| \| z_m(t, u_0, z_0) - z_{m-1}(t, u_0, z_0) \| \quad \ldots (4.10)
\]
\[ \|u_{m+1}(t, u_0, z_0) - u_m(t, u_0, z_0)\| \leq A^m \|u_1(t, u_0, z_0) - u_0\| \]

Now from \( m = 1, 2, \ldots \) and \( p \geq 1 \), we find that
\[
\|u_{m+p}(t, u_0, z_0) - u_m(t, u_0, z_0)\| \leq A^m(E - A)^{-1}
\]
\[
\begin{pmatrix}
\frac{2a^p - 1}{(a + 1)} M_{\beta\alpha} M_4 \\
\frac{2a^p - 1}{(a + 1)} M_{\beta\gamma} M_5
\end{pmatrix}
\]
for all \( t \in [0, T], u_0 \in D_1 \) and \( z_0 \in D_6 \).

Since \( \frac{2m_2 + \sqrt{m_2^2 + 4(m_1 - m_0)}}{2} < 1 \) and \( \lim_{m \to \infty} A^m = 0 \), so that the right side of (4.11) tends to zero. Therefore the sequence of functions \( u_m(t, u_0, z_0) \) converges uniformly on the domain (4.3) to the limit function
\[
\lim_{m \to \infty} \left( u_m(t, u_0, z_0) \right) = \left( u(t, u_0, z_0) \right)
\]
which is defined on the same domain. Let
\[
\lim_{m \to \infty} \frac{u_m(t, u_0, z_0)}{z_m(t, u_0, z_0)} = \frac{u(t, u_0, z_0)}{z(t, u_0, z_0)}
\]
Since the sequence of functions (4.1) and (4.2) are periodic in \( t \) of period \( T \), the limiting function \( z(t, u_0, z_0) \) is also periodic in \( t \) of period \( T \). Also by the Lemma 2.3 and the inequality (4.11) the inequality (4.6) hold for all \( m \geq 0 \).

By using the relation (4.12) and proceeding in (4.1) and (4.2) to limit, when \( m \to \infty \), it is convincing that the limiting function \( \frac{u(t, u_0, z_0)}{z(t, u_0, z_0)} \) is the periodic solution of the integral equations (4.4) and (4.4).

### 4.2 Uniqueness of Periodic Solution of (1.1) and (1.2)

**Theorem 4.2** All assumptions of the Theorem 4.1 are satisfy, then
\[
\frac{u(t, u_0, z_0)}{z(t, u_0, z_0)}
\]
is a unique solution of the system (1.1) with boundary conditions (1.2).

**Proof.** Assume that \( \frac{\tilde{u}(t, u_0, z_0)}{\tilde{z}(t, u_0, z_0)} \) is another solution for the system (1.1) with boundary conditions (1.2), means that
\[
\begin{align*}
\tilde{u}(t, u_0, z_0) &= u_0 + \frac{1}{\lambda(t)} \int_0^T (t - s)^{p-1} f(s, \beta(t, \alpha), \hat{u}(s, u_0, z_0), h(s, \xi) \hat{K}(s, \xi) \hat{u}(\xi, u_0, z_0)) \, ds \\
&\quad - \int_0^T \tilde{z} \left( \hat{u}(t, u_0, z_0), \hat{z}(t, u_0, z_0) \right) ds \\
&\quad - \int_0^T \tilde{z}(t, u_0, z_0) d\xi \end{align*}
\]
\[
\begin{align*}
\hat{u}(t, u_0, z_0) &= \hat{u}(t, u_0, z_0) \\
&\quad - \int_0^T \tilde{z}(t, u_0, z_0) d\xi \\
&\quad - \sum_{i=1}^m \int_0^T \tilde{z}(t, u_0, z_0) d\xi \\
&\quad + \left( \frac{1}{T} \right)^{p-1} \int_0^T h_1(s, u_0, z_0) ds - \left( B^{-1} A + I \right) u_0
\end{align*}
\]
\[
\frac{\tilde{u}(t, u_0, z_0)}{\tilde{z}(t, u_0, z_0)} = \frac{u(t, u_0, z_0)}{z(t, u_0, z_0)}
\]
\[
\frac{\tilde{u}(t, u_0, z_0)}{\tilde{z}(t, u_0, z_0)} = \frac{u(t, u_0, z_0)}{z(t, u_0, z_0)}
\]
and
\[
\tilde{z}(t, u_0, z_0) = \frac{u(t, u_0, z_0)}{z(t, u_0, z_0)}
\]
\[
\frac{\tilde{u}(t, u_0, z_0)}{\tilde{z}(t, u_0, z_0)} = \frac{u(t, u_0, z_0)}{z(t, u_0, z_0)}
\]
\[
\frac{\tilde{u}(t, u_0, z_0)}{\tilde{z}(t, u_0, z_0)} = \frac{u(t, u_0, z_0)}{z(t, u_0, z_0)}
\]
\[
\frac{\tilde{u}(t, u_0, z_0)}{\tilde{z}(t, u_0, z_0)} = \frac{u(t, u_0, z_0)}{z(t, u_0, z_0)}
\]
\[
\frac{\tilde{u}(t, u_0, z_0)}{\tilde{z}(t, u_0, z_0)} = \frac{u(t, u_0, z_0)}{z(t, u_0, z_0)}
\]
\[
\frac{\tilde{u}(t, u_0, z_0)}{\tilde{z}(t, u_0, z_0)} = \frac{u(t, u_0, z_0)}{z(t, u_0, z_0)}
\]
\[
\frac{\tilde{u}(t, u_0, z_0)}{\tilde{z}(t, u_0, z_0)} = \frac{u(t, u_0, z_0)}{z(t, u_0, z_0)}
\]
\[
\frac{\tilde{u}(t, u_0, z_0)}{\tilde{z}(t, u_0, z_0)} = \frac{u(t, u_0, z_0)}{z(t, u_0, z_0)}
\]
\[
\frac{\tilde{u}(t, u_0, z_0)}{\tilde{z}(t, u_0, z_0)} = \frac{u(t, u_0, z_0)}{z(t, u_0, z_0)}
\]
\[
\frac{\tilde{u}(t, u_0, z_0)}{\tilde{z}(t, u_0, z_0)} = \frac{u(t, u_0, z_0)}{z(t, u_0, z_0)}
\]
\[
\frac{\tilde{u}(t, u_0, z_0)}{\tilde{z}(t, u_0, z_0)} = \frac{u(t, u_0, z_0)}{z(t, u_0, z_0)}
\]
\[
\frac{\tilde{u}(t, u_0, z_0)}{\tilde{z}(t, u_0, z_0)} = \frac{u(t, u_0, z_0)}{z(t, u_0, z_0)}
\]
\[
\frac{\tilde{u}(t, u_0, z_0)}{\tilde{z}(t, u_0, z_0)} = \frac{u(t, u_0, z_0)}{z(t, u_0, z_0)}
\]
\[
\frac{\tilde{u}(t, u_0, z_0)}{\tilde{z}(t, u_0, z_0)} = \frac{u(t, u_0, z_0)}{z(t, u_0, z_0)}
\]
\[
\frac{\tilde{u}(t, u_0, z_0)}{\tilde{z}(t, u_0, z_0)} = \frac{u(t, u_0, z_0)}{z(t, u_0, z_0)}
\]
\[
\frac{\tilde{u}(t, u_0, z_0)}{\tilde{z}(t, u_0, z_0)} = \frac{u(t, u_0, z_0)}{z(t, u_0, z_0)}
\]
\[
\frac{\tilde{u}(t, u_0, z_0)}{\tilde{z}(t, u_0, z_0)} = \frac{u(t, u_0, z_0)}{z(t, u_0, z_0)}
\]
\[
\frac{\tilde{u}(t, u_0, z_0)}{\tilde{z}(t, u_0, z_0)} = \frac{u(t, u_0, z_0)}{z(t, u_0, z_0)}
\]
\[
\frac{\tilde{u}(t, u_0, z_0)}{\tilde{z}(t, u_0, z_0)} = \frac{u(t, u_0, z_0)}{z(t, u_0, z_0)}
\]
\[ \Delta_1(0, u_0, z_0) = -\frac{\alpha}{T^\alpha} \int_0^T (T - s)^{\alpha - 1} \]
\[ f(s, \beta(\tau, \alpha), u(s, u_0, z_0), p(\xi)) - z(\xi, u_0, z_0)) d\xi ds \]
\[ - \frac{\Gamma(\alpha + 1)}{T^\alpha} \sum_{t=1}^{s_1} I(t(\beta(\tau, \alpha)), u(t, u_0, z_0)) \]
\[ + \frac{\Gamma(\alpha + 1)}{T^\alpha} \left[ B^{-1} \int_0^T h_1(z(s, u_0, z_0)) ds - (B^{-1} A + I) u_0 \right] \]

and
\[ \Delta_2(0, u_0, z_0) = -\frac{\gamma}{T^\gamma} \int_0^T (T - s)^{\gamma - 1} \]
\[ g(s, \beta(\tau, \gamma), z(s, u_0, z_0), f(\xi)) - z(\xi, u_0, z_0)) d\xi ds \]
\[ - \frac{\Gamma(\gamma + 1)}{T^\gamma} \sum_{t=1}^{s_1} I(t(\beta(\tau, \gamma)), z(t, u_0, z_0)) \]
\[ + \frac{\Gamma(\gamma + 1)}{T^\gamma} \left[ D^{-1} \int_0^T h_2(u(s, u_0, z_0)) ds - (D^{-1} C + I) u_0 \right] \]

Also, we define the sequences of vector functions \( \Delta_{1m}(0, u_0, z_0) \) and \( \Delta_{2m}(0, u_0, z_0) \) are approximately determined by the following:
\[ \Delta_{1m}(0, u_0, z_0) = -\frac{\alpha}{T^\alpha} \int_0^T (T - s)^{\alpha - 1} \]
\[ f(s, \beta(\tau, \alpha), u_m(s, u_0, z_0), p(\xi)) - z_m(\xi, u_0, z_0)) d\xi ds \]
\[ - \frac{\Gamma(\alpha + 1)}{T^\alpha} \sum_{t=1}^{s_1} I(t(\beta(\tau, \alpha)), u_m(t, u_0, z_0)) \]
\[ + \frac{\Gamma(\alpha + 1)}{T^\alpha} \left[ B^{-1} \int_0^T h_1(z_m(s, u_0, z_0)) ds - (B^{-1} A + I) u_0 \right] \]

and
\[ \Delta_{2m}(0, u_0, z_0) = -\frac{\gamma}{T^\gamma} \int_0^T (T - s)^{\gamma - 1} \]
\[ g(s, \beta(\tau, \gamma), z_m(s, u_0, z_0), f(\xi)) - z_m(\xi, u_0, z_0)) d\xi ds \]
\[ - \frac{\Gamma(\gamma + 1)}{T^\gamma} \sum_{t=1}^{s_1} I(t(\beta(\tau, \gamma)), z_m(t, u_0, z_0)) \]
\[ + \frac{\Gamma(\gamma + 1)}{T^\gamma} \left[ D^{-1} \int_0^T h_2(u_m(s, u_0, z_0)) ds - (D^{-1} C + I) u_0 \right] \]

**Theorem 4.4.** Let the vector functions \( f(t, \beta(\tau, \alpha), u, x) \) and \( g(t, \beta(\tau, \gamma), z, y) \) be defined on the intervals \([a_1, b_1]\) and \([a_2, b_2]\) on \( \mathbb{R}^t \) and periodic in \( t \) of period \( T \), suppose that for all \( m \geq 0 \), then the sequences of the functions \( \Delta_{1m}(0, u_0, z_0) \) and \( \Delta_{2m}(0, u_0, z_0) \) which are defined in (4.20) and (4.21) satisfy the inequalities:

\[ \min_{u_0 \in E_1, z_0 \in E_2} \Delta_{1m}(0, u_0, z_0) \leq - \left( \frac{d_1}{d_2}, A^m(E - \Lambda)^{-1} \eta \right) \]
\[ \max_{u_0 \in E_1, z_0 \in E_2} \Delta_{1m}(0, u_0, z_0) \geq \left( \frac{d_3}{d_4}, A^m(E - \Lambda)^{-1} \eta \right) \]

holds for all \( m \geq 0 \), where
\[ d_1 = \frac{1}{\Gamma(\alpha)} M_{Beta} K_1 + K_2 \]
\[ d_2 = \frac{1}{\Gamma(\gamma)} M_{Beta} K_1 + \frac{\Gamma(\gamma + 1)}{T^\gamma} \|B^{-1}\|T P_1, \]
\[ d_3 = \frac{1}{\Gamma(\gamma)} M_{Beta} L_2 R_1 + \frac{\Gamma(\gamma + 1)}{T^\gamma} \|D^{-1}\|T Q_1, \]
\[ d_4 = \frac{1}{\Gamma(\gamma)} M_{Beta} L_2 R_1 + \frac{\Gamma(\gamma + 1)}{T^\gamma} M_{Beta} L_2 S_2 \]

and \( \eta = \left( \frac{2^{\gamma - 1} \gamma}{2^{\gamma - 1} \gamma + 1} M_{Beta} M_A + M_4 \right) \)

**Proof.** From equations (4.18) to (4.21), we obtain that
\[ \| \Delta_1(0, u_0, z_0) - \Delta_{1m}(0, u_0, z_0) \| \leq \left( \frac{1}{\Gamma(\alpha)} M_{Beta} K_1 + K_2 \right) \|u(t, u_0, z_0)\| \]
\[ + \left( \frac{1}{\Gamma(\gamma)} M_{Beta} L_2 R_1 + \frac{\Gamma(\gamma + 1)}{T^\gamma} M_{Beta} L_2 S_2 \right) \]
\[ \|z(t, u_0, z_0) - z_{m}(t, u_0, z_0)\| \]

and
\[ \| \Delta_2(0, u_0, z_0) - \Delta_{2m}(0, u_0, z_0) \| \leq \left( \frac{1}{\Gamma(\gamma)} M_{Beta} L_2 R_1 + \frac{\Gamma(\gamma + 1)}{T^\gamma} M_{Beta} L_2 S_2 \right) \]
\[ \|z(t, u_0, z_0) - z_{m}(t, u_0, z_0)\| \]

Since \( \|u(t, u_0, z_0) - u_m(t, u_0, z_0)\| \leq A^m(E - \Lambda)^{-1} \)
\[ \left( \frac{2^{\gamma - 1} \gamma}{2^{\gamma - 1} \gamma + 1} M_{Beta} M_A + M_4 \right) \]
\[ \|z(t, u_0, z_0) - z_{m}(t, u_0, z_0)\| \leq A^m(E - \Lambda)^{-1} \]

hence rewrite the equations (4.23) and (4.24) as a vector form. The inequality (4.22) is hold for all \( m \geq 0 \).
\[ \Delta_{2m}(0, u_2, z_2) = \max_{u_2 \in \mathbb{E}_1, z_2 \in \mathbb{E}_2} \Delta_{2m}(0, u_2, z_2) \]

By using inequalities (4.23) to (4.24), the following are obtained

\[ \Delta_1(0, u_2, z_2) = \Delta_{1m}(0, u_2, z_2) = \Delta_{2m}(0, u_2, z_2) \]

and from the continuity of the functions \( \Delta_{1}(0, u_0, z_0) \), \( \Delta_{2}(0, u_0, z_0) \) and the inequalities (4.29) and (4.30), the isolated singular points \( u^0 \in [u_1, u_2] \) and \( z^0 \in [z_1, z_2] \) exist such that \( \Delta_1(0, x_0, y_0) = 0 \) and \( \Delta_2(0, u_0, z_0) = 0 \). This means that (1.1) has a periodic solution \( z(t, u_0, z_0) \).

**Remark 4.5.** Theorem 4.3 is provided when \( R^3 = R^1 \), i.e. \( u_0 \) and \( z_0 \) are scalar singular points and should be isolated. For more details, see Samoilenko and Ronto, 1976.

### 4.4 Stability of Periodic Solution of (1.1) and (1.2)

**Theorem 4.6.** Let the vector functions \( \Delta_{1}(0, u_0, z_0) \) and \( \Delta_{2}(0, u_0, z_0) \) be defined by the equations (4.18) and (4.19) where \( u(t, u_0, z_0) \) is a limit of the sequence of the function (4.1), the function \( z(t, u_0, z_0) \) is a limit of the sequence of the function (4.2), then the following inequalities yield:

\[ \frac{1}{\Gamma(\alpha + 1)} M_{\beta_1} + \frac{\Gamma(\alpha + 1)}{\Gamma(\alpha + 1)} M_{\beta_2} S_1 + \frac{\Gamma(\alpha + 1)}{\Gamma(\alpha + 1)} M_{\beta_3} S_2 + \frac{\Gamma(\alpha + 1)}{\Gamma(\alpha + 1)} M_{\beta_4} L_1 S_2 + \frac{\Gamma(\alpha + 1)}{\Gamma(\alpha + 1)} M_{\beta_5} L_2 S_2 + \frac{\Gamma(\alpha + 1)}{\Gamma(\alpha + 1)} M_{\beta_6} L_3 S_2 + \frac{\Gamma(\alpha + 1)}{\Gamma(\alpha + 1)} M_{\beta_7} L_4 S_2 + \frac{\Gamma(\alpha + 1)}{\Gamma(\alpha + 1)} M_{\beta_8} L_5 S_2 + \frac{\Gamma(\alpha + 1)}{\Gamma(\alpha + 1)} M_{\beta_9} L_6 S_2 \]

and

\[ \frac{1}{\Gamma(\alpha + 1)} M_{\beta_1} + \frac{\Gamma(\alpha + 1)}{\Gamma(\alpha + 1)} M_{\beta_2} S_1 + \frac{\Gamma(\alpha + 1)}{\Gamma(\alpha + 1)} M_{\beta_3} S_2 + \frac{\Gamma(\alpha + 1)}{\Gamma(\alpha + 1)} M_{\beta_4} L_1 S_2 + \frac{\Gamma(\alpha + 1)}{\Gamma(\alpha + 1)} M_{\beta_5} L_2 S_2 \]

where

\[ E_1 = \frac{1}{\Gamma(\alpha + 1)} M_{\beta_1} K_1 + \frac{\Gamma(\alpha + 1)}{\Gamma(\alpha + 1)} M_{\beta_2} S_1 K_9 + \frac{\Gamma(\alpha + 1)}{\Gamma(\alpha + 1)} M_{\beta_3} S_2 K_9 + \frac{\Gamma(\alpha + 1)}{\Gamma(\alpha + 1)} M_{\beta_4} L_1 S_2 K_9 + \frac{\Gamma(\alpha + 1)}{\Gamma(\alpha + 1)} M_{\beta_5} L_2 S_2 K_9 \]

The proof is given by rewriting (4.32) and (4.34) by vector form, we obtain (4.31).

From inequality (4.18), we get

\[ \Delta_{1}(0, u_0, z_0) = \Delta_{1}(0, u_0, z_0) \leq \frac{1}{\Gamma(\alpha + 1)} M_{\beta_1} K_1 + \frac{\Gamma(\alpha + 1)}{\Gamma(\alpha + 1)} M_{\beta_2} S_1 K_9 + \frac{\Gamma(\alpha + 1)}{\Gamma(\alpha + 1)} M_{\beta_3} S_2 K_9 + \frac{\Gamma(\alpha + 1)}{\Gamma(\alpha + 1)} M_{\beta_4} L_1 S_2 K_9 + \frac{\Gamma(\alpha + 1)}{\Gamma(\alpha + 1)} M_{\beta_5} L_2 S_2 K_9 \]

and

\[ \Delta_{2}(0, u_0, z_0) = \Delta_{2}(0, u_0, z_0) \leq \frac{1}{\Gamma(\alpha + 1)} M_{\beta_1} K_1 + \frac{\Gamma(\alpha + 1)}{\Gamma(\alpha + 1)} M_{\beta_2} S_1 K_9 + \frac{\Gamma(\alpha + 1)}{\Gamma(\alpha + 1)} M_{\beta_3} S_2 K_9 + \frac{\Gamma(\alpha + 1)}{\Gamma(\alpha + 1)} M_{\beta_4} L_1 S_2 K_9 + \frac{\Gamma(\alpha + 1)}{\Gamma(\alpha + 1)} M_{\beta_5} L_2 S_2 K_9 \]

where

\[ E_2 = \frac{1}{\Gamma(\alpha + 1)} M_{\beta_1} K_1 + \frac{\Gamma(\alpha + 1)}{\Gamma(\alpha + 1)} M_{\beta_2} S_1 K_9 + \frac{\Gamma(\alpha + 1)}{\Gamma(\alpha + 1)} M_{\beta_3} S_2 K_9 + \frac{\Gamma(\alpha + 1)}{\Gamma(\alpha + 1)} M_{\beta_4} L_1 S_2 K_9 + \frac{\Gamma(\alpha + 1)}{\Gamma(\alpha + 1)} M_{\beta_5} L_2 S_2 K_9 \]

the functions \( u(t, u_0, z_0) \), \( u(t, u_0, z_0) \) and \( z(t, u_0, z_0) \) are solutions of the equation:

\[ u(t, u_0, z_0) = x(t) + \frac{1}{\Gamma(\alpha + 1)} \int_0^t (t-s)^{\alpha-1} f(s, \beta(t, \alpha), \gamma(s, \xi, u_0, z_0)) d\xi ds \]

\[ - \frac{1}{\Gamma(\alpha + 1)} \int_0^t (t-s)^{\alpha-1} f(s, \beta(t, \alpha), u(s, u_0, z_0), \gamma(s, \xi, u_0, z_0)) d\xi ds \]

\[ \frac{1}{\Gamma(\alpha + 1)} \int_0^t (t-s)^{\alpha-1} f(s, \beta(t, \alpha), \gamma(s, \xi, u_0, z_0)) d\xi ds \]

and

\[ z(t, u_0, z_0) = \frac{1}{\Gamma(\alpha + 1)} \int_0^t (t-s)^{\alpha-1} f(s, \beta(t, \alpha), \gamma(s, \xi, u_0, z_0)) d\xi ds \]
Consider the following system of fractional integro-differential equation.

\[ \begin{align*}
g(s, \beta(t, \gamma), z(s, u_0, z_0^k)) & \cdot \int_{0}^{(s)} R(s, \xi)(u(\xi, u_0, z_0^{k})) \\ & - z(\xi, u_0, z_0^{k})) d\xi \quad ds \\
& - \frac{1}{\Gamma(p)} \sum_{i=1}^{(\gamma)} \left( \beta(t, \gamma), z(s, u_0, z_0^{k}) \right)\\
& + \left( \frac{1}{\Gamma(p)} \right) \int_{0}^{T} \left(D^{-1} - 1 \right) \hbar_2(u(s, u_0, z_0^{k})) ds \quad (D^{-1} C + E) z_0^{k} \\
& + \int_{0}^{T} \left[ f_1(t) - f_2(t) \right] + \left( \frac{1}{\Gamma(p)} \right) \int_{0}^{T} \left( D^{-1} C + E \right) z_0^{k} \\
& - \frac{1}{\Gamma(p)} \sum_{i=1}^{(\gamma)} \left( \beta(t, \gamma), z(s, u_0, z_0^{k}) \right)\\
& + \left( \frac{1}{\Gamma(p)} \right) \int_{0}^{T} \left( D^{-1} C + E \right) z_0^{k} \\
& - \frac{1}{\Gamma(p)} \sum_{i=1}^{(\gamma)} \left( \beta(t, \gamma), z(s, u_0, z_0^{k}) \right)\\
& + \left( \frac{1}{\Gamma(p)} \right) \int_{0}^{T} \left( D^{-1} C + E \right) z_0^{k} \\
& - \frac{1}{\Gamma(p)} \sum_{i=1}^{(\gamma)} \left( \beta(t, \gamma), z(s, u_0, z_0^{k}) \right)
\end{align*} \]


