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ON THE PARAMETRIZATION OF NONLINEAR IMPULSIVE FRACTIONAL INTEGRO–DIFFERENTIAL SYSTEM WITH NON-SEPARATED INTEGRAL COUPLED BOUNDARY CONDITIONS

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ABSTRACT:
We give a new investigation of periodic solutions of nonlinear impulsive fractional integro-differential system with different orders of fractional derivatives with non-separated integral coupled boundary conditions. Uniformly Converging of the sequence of functions according to the main idea of the Numerical-analytic technique, from creating a sequence of functions. An example of impulsive fractional system is also presented to illustrate the theory.

KEYWORDS: Caputo fractional derivative, fractional integro-differential, integral coupled boundary conditions, Periodic solutions, successive approximation method.

1. INTRODUCTION

Fractional differential equations have been developed in the last decade as good tools to describe the mathematical modeling of systems and processes in the fields of physics, chemistry, aerodynamics, economics, control theory and image processing, etc. (Kilbas *etal*.,2006, Sabatier *etal*.,2007, Lakshmikantham *etal*., 2009). On the other hand, we observe periodic motions in every field of science and everywhere in real life (Farkas,1994). The applications and theory of the fractional differential equations have recently been collected by several researchers for different problems, we refer the reader to (Ahmad and Nieto,2014, Henderson *etal*.,2015). The theory of impulsive differential equations is a new and important field of differential equation theory, many authors see (Agarwal *etal*.,2010, Wang, *etal*.,2012, Feckan *etal*., 2012, Wang *etal*.,2014), which has been an object of intensive investigation in recent years, and applications to different areas have been considered. However, the concept of solutions for impulsive fractional differential equations (Zhou and Chu,2012, Bai *etal*.,2016, Bai and, Zhang,2016, Dong *etal*.,2017) has been argued extensively,

Also an extended this method by (Feckan and Marynets,2017) to nonlinear system of integro-differential equation (Butris *etal*.,2017, Butris, and Taher,2019) and Caputo type differential equations with periodic boundary value problems (Mahmudov *etal*.,2019). Furthermore, the investigate of a coupled system of fractional order is also very significant because this type of system can often occur in applications. The reader is referred to read (Su,2009, Wang *etal*.,2010), and the references cited therein.

Caputo-type fractional integro-differential system with nonlinear coupled integral boundary conditions has been considered in this paper. We apply picard approximations technique proposed by (Ronto and Samoilenko,2000, Ronto *etal*., 2015, Marynets,2016, Feckan and Marynets, 2018) for investigation the existence, uniqueness and approximation of periodic solutions of nonlinear fractional integro-differential system with nonlinear coupled integral boundary conditions. Motivated by the works mentioned and many known results, we use the numerical analytic method to investigate the existence and uniqueness of periodic solutions and define our problem

$$\left.\begin{aligned} {}^{c}D_{0+}^{\alpha}u(t) &= f\left(t,\beta(\tau,\alpha),u(t),\int_0^{a(t)} K(t,s)\big(u(s)-z(s)\big)ds\right) \\ & \qquad t\neq t_i \\ {}^{c}D_{0+}^{\gamma}z(t) &= g\left(t,\beta(\tau,\gamma),z(t),\int_0^{b(t)} R(t,s)\big(u(s)-z(s)\big)ds\right) \\ & \qquad t\neq t_i \\ \Delta u|_{t=t_i} &= I_i\big(\beta(\tau,\alpha),u(t)\big) \quad , \quad \Delta z|_{t=t_i} = J_i\big(\beta(\tau,\alpha),z(t)\big) \\ & \qquad i,j=1,2,\dots,n \end{aligned}\right\} \quad \dots(1.1)$$

with non-separated integral coupled boundary conditions

$$\left.\begin{aligned} Au(0)+Bu(T) &= \int_0^T h_1\big(z(s)\big)ds, \quad with \ \det(B)\neq 0 \\ Cz(0)+Dz(T) &= \int_0^T h_2\big(u(s)\big)ds, \quad with \ \det(D)\neq 0 \end{aligned}\right\} \quad \dots(1.2)$$

for all $t\in[0,T]$, A,B,C *and* D are $n\times n$ matrices where ${}^{c}D_{0+}^{\alpha}$, ${}^{c}D_{0+}^{\gamma}$, denote the Caputo fractional derivatives, $0<\alpha,\gamma\le 1$, also $\beta(\tau,\alpha)$ and $\beta(\tau,\gamma)$ are said to be special functions provided that (Beta function), the function $f, g \in C([0,T]\times\Omega\times D_1,D_2,R)$, $\Omega=(0,T]\times(0,1]$, D_1 *and* D_2 are compact subset of, also a, b, h_1 and h_2 are continuous functions on $[0,T]$

2. BACKGROOUND METERIAL

In this section, some definitions of fraction calculus and lemmas are presented which are used for the statement of the problem (1.1) and (1.2).

Definition 2.1 (Kilbas *etal*.,2006) For a function g given on the interval [a, b], the Caputo fractional order derivative of g is defined by

$${}_{0}^{t}D^{\alpha}g(t)=\frac{1}{\Gamma(n-\alpha)}\int_0^t (t-s)^{n-\alpha-1}g^{(n)}(s)ds \quad \dots(2.1)$$

where $n=[\alpha]+1$ and $[\alpha]$ denotes the integer part of α, and $\Gamma(.)$ denotes the Gamma function.

Definition 2.2 (Kilbas *etal.*,2006) Let g be a function which is defined almost everywhere (a.e) on [a, b], for $\alpha > 0$, we define

$$ {}_a^bD^{-\alpha} f = \frac{1}{\Gamma(\alpha)} \int_a^b (b-t)^{\alpha-1} g(t)dt \quad \dots (2.2) $$

provided that the integral (Lebesgue)exists.

Lemma 2.3 Let g(t) be a continuous function for $t \in [0, \mathrm{T}]$, then the following estimate is hold

$$ \left| \frac{1}{\Gamma(\alpha)} \left[\int_0^t (t-s)^{\alpha-1} g(s)ds - \left(\frac{t}{T}\right)^{\alpha} \int_0^T (T-s)^{\alpha-1} g(s)ds \right] \right| \le \omega(t) \max_{t\in[\tau,\tau+T]} |g(t)| , $$

where $\omega(t) = \frac{2t^{\alpha}}{\Gamma(\alpha+1)} \left(1-\frac{t}{T}\right)^{\alpha}$.

Proof. It is obvious that

$$ \left| \frac{1}{\Gamma(\alpha)} \left[\int_0^t (t-s)^{\alpha-1} g(s)ds - \left(\frac{t}{T}\right)^{\alpha} \int_0^T (T-s)^{\alpha-1} g(s)ds \right] \right| $$

$$ \le \frac{1}{\Gamma(\alpha)} \left[\int_0^t \left((t-s)^{\alpha-1} - \left(\frac{t}{T}\right)^{\alpha} (T-s)^{\alpha-1} \right) |g(s)|ds - \left(\frac{t}{T}\right)^{\alpha} \int_t^T (T-s)^{\alpha-1} |g(s)|ds \right] $$

Form (Feckan and Marynets,2017), the terms becomes

$$ (t-s)^{\alpha-1} - \left(\frac{t}{T}\right)^{\alpha} (T-s)^{\alpha-1} = (t-s)^{\alpha-1} \left(1 - \left(\frac{t}{T}\right)^{\alpha} \left(\frac{t-s}{T-s}\right)^{1-\alpha}\right) $$

$$ \ge (t-s)^{\alpha-1} \left(1 - \left(\frac{t}{T}\right)^{\alpha} \left(\frac{t}{T}\right)^{1-\alpha}\right) = (t-s)^{\alpha-1} \left(\frac{T-t}{T}\right) \ge 0 $$

For any $s \in [0, t)$, we obtain that

$$ \left| \frac{1}{\Gamma(\alpha)} \left[\int_0^t (t-s)^{\alpha-1} g(s)ds - \left(\frac{t}{T}\right)^{\alpha} \int_0^T (T-s)^{\alpha-1} g(s)ds \right] \right| \le \frac{2t^{\alpha}}{\Gamma(\alpha+1)} \left(1-\frac{t}{T}\right)^{\alpha} \max_{t\in[\tau,\tau+T]} |g(t)| \le \omega(t) \max_{t\in[\tau,\tau+T]} |g(t)| $$

The proof of lemma is complete

Definition 2.4 The solutions of the system of fractional integro-differential equations (1.1) and integral boundary conditions (1.2) are defining the following integral equations:-

$$ \begin{aligned} u(t,x_0,y_0) = u_0 &+ \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f\Big(s, \beta(\tau,\alpha), u(s,u_0,z_0), \int_0^{a(s)} K(s,\xi)\big(u(\xi,u_0,z_0) - z(\xi,u_0,z_0)\big)d\xi\Big)\, ds \\ &- \frac{1}{\Gamma(\alpha)} \left(\frac{t}{T}\right)^{\alpha} \int_0^T (T-s)^{\alpha-1} f\Big(s, \beta(\tau,\alpha), u(s,u_0,z_0), \int_0^{a(s)} K(s,\xi)\big(u(\xi,u_0,z_0) - z(\xi,u_0,z_0)\big)d\xi\Big)\, ds \\ &- \left(\frac{t}{T}\right)^{\alpha} \sum_{t=1}^{s_1} I_i\big(\beta(\tau,\alpha), u(s)\big) \\ &+ \left(\frac{t}{T}\right)^{\alpha} \left[B^{-1} \int_0^T h_1\big(z(s,u_0,z_0)\big)ds - (B^{-1}A + I)u_0 \right] \\ &+ \sum_{0<t_i<t} I_i\big(\beta(\tau,\alpha), u(t)\big) \qquad \dots (2.3) \end{aligned} $$

$$ \begin{aligned} z(t,u_0,z_0) = z_0 &+ \frac{1}{\Gamma(\gamma)} \int_0^t (t-s)^{\gamma-1} g\Big(s, \beta(\tau,\gamma), z(s,u_0,z_0), \int_0^{b(s)} R(s,\xi)\big(u(\xi,u_0,z_0) - z(\xi,u_0,z_0)\big)d\xi\Big)\, ds \\ &- \frac{1}{\Gamma(\gamma)} \left(\frac{t}{T}\right)^{\gamma} \int_0^T (T-s)^{\gamma-1} g\Big(s, \beta(\tau,\gamma), z(s,u_0,z_0), \int_0^{b(s)} R(s,\xi)\big(u(\xi,u_0,z_0) - z(\xi,u_0,z_0)\big)d\xi\Big)\, ds \\ &- \left(\frac{t}{T}\right)^{\gamma} \sum_{t=1}^{s_2} J_i\big(\beta(\tau,\gamma), z(t)\big) \\ &+ \left(\frac{t}{T}\right)^{\gamma} \left[D^{-1} \int_0^T h_2\big(u(s,u_0,z_0)\big)ds - (D^{-1}C + I)z_0 \right] \\ &+ \sum_{0<t_i<t} J_i\big(\beta(\tau,\gamma), z(t)\big) \qquad \dots (2.4) \end{aligned} $$

3. CONDITIONS FOR CONVERGENCE OF SUCCESSIVE APPROXIMATION

For investigate of the successive approximation for periodic solution of the problem (1.1) and (2.2), we need the some conditions, suppose that the vector functions $f, g \in C([0,T] \times \Omega \times D_1 \times D_2, R)$, $I_i, J_i \in C(\Omega \times D_1, R)$ $\Omega = (0, \mathrm{T}] \times (0,1]$, D_1 *and* D_2 are compact subset of R^1 , also , φ , h_1 and h_2 are continuous functions on $[0,T]$, and satisfies the following hypothesis.

H_1: There exist positive constants $M, L, K_1, K_2, L_1, L_2, M_{\beta\alpha}$ *and* $M_{\beta\gamma}$ such that

$$ \left. \begin{aligned} \|f(t,\beta(\tau,\alpha),u,x)\| &\le M_{\beta\alpha} M , \\ \|I_i(\beta(\tau,\alpha),u(t))\| &\le M_{\beta\alpha} M_1 \end{aligned} \right\} \quad \dots (3.1) $$

$$ \left. \begin{aligned} \|g(t,\beta(\tau,\gamma),z,y)\| &\le M_{\beta\gamma} L , \\ \|J_i(\beta(\tau,\gamma),z(t))\| &\le M_{\beta\gamma} L_1 \end{aligned} \right\} \quad \dots (3.2) $$

$$ \left. \begin{aligned} &\|f(t,\beta(\tau,\alpha),u_1,x_1) - f(t,\beta(\tau,\alpha),u_2,x_2)\| \le M_{\beta\alpha} (K_1\|u_1-u_2\| + K_2\|x_1-x_2\|) \\ &\|I_i(\beta(\tau,\alpha),u_1) - I_i(\beta(\tau,\alpha),u_2)\| \le M_{\beta\alpha} K_3 \|u_1-u_2\| \end{aligned} \right\} \dots (3.3) $$

$$ \left. \begin{aligned} &\|g(t,\beta(\tau,\gamma),z_1,y_1) - g(t,\beta(\tau,\gamma),z_2,y_2)\| \le M_{\beta\gamma} (L_1\|z_1-z_2\| + L_2\|y_1-y_2\|) \\ &\|I_i(\beta(\tau,\gamma),z_1) - I_i(\beta(\tau,\gamma),z_2)\| \le M_{\beta\gamma} L_3 \|z_1-z_2\| \end{aligned} \right\} \dots (3.4) $$

where

$$ M_{\beta\alpha} = \max_{\tau\in(0,T]} \frac{1}{\beta(\tau,\alpha)} , M_{\beta\gamma} = \max_{\tau\in(0,T]} \frac{1}{\beta(\tau,\gamma)} , $$

$$ x_i = \int_0^{a(t)} K(t,s)\big(u_i(s) - z_i(s)\big)ds , $$

$$ y_i = \int_0^{b(t)} R(t,s)\big(u_i(s) - z_i(s)\big)ds $$

for all $t \in [0,T]$, $\tau \in (0,T]$, $u_1, u_2, z_1, z_2 \in D_1$ and $x_i, y_i \in D_2$, $i = 1,2$.

H_2: There exist positive constants p_0 , p_1 , q_0 and q_1 such that

$$\|h_1(z)\| \le p_0 \ , \ \|h_2(u)\| \le q_0 \quad \dots (3.5)$$
$$\|h_1(z_1) - h_1(z_1)\| \le p_1\|z_1 - z_2\| \quad \dots (3.6)$$
$$\|h_2(u_1) - h_2(u_2)\| \le q_1\|u_1 - u_2\| \quad \dots (3.7)$$

for all $t \in [0,T]$, $u_1, u_2, z_1, z_2 \in D_1$.

H_3: The kernels $K(t,s)$ and $R(t,s)$ satisfy the following conditions, when there exist positive constants Ks and Rs such that

$$\int_0^{a(t)} \|K(t,s)\|ds \le Ks \ \ and \int_0^{b(t)} \|R(t,s)\|ds \le Rs \quad \text{for all} \quad s,t \in [0,T] \quad \dots (3.8)$$

Let ω (t) $= \frac{2t^\alpha}{\Gamma(\alpha+1)}\left(1 - \frac{t}{T}\right)^\alpha$ and $\varphi(t) = \frac{2t^\gamma}{\Gamma(\alpha+1)}\left(1 - \frac{t}{T}\right)^\gamma$, then ω (t) and $\varphi(t)$ take these maximum value at t= $\frac{T}{2}$, and $\|\omega\|_\infty = \frac{T^\alpha}{2^{2\alpha-1}\Gamma(\alpha+1)}$, $\|\varphi\|_\infty = \frac{T^\gamma}{2^{2\gamma-1}\Gamma(\gamma+1)}$, with $\|.\|_\infty = \max_{t\in[0,T]}\{|.|\}$

Define the non-empty set

$$\left.\begin{aligned} D_f &= D_1 - \frac{T^\alpha}{2^{2\alpha-1}\Gamma(\alpha+1)} M_{\beta\alpha} M + M_4 \\ D_g &= D_2 - \frac{T^\gamma}{2^{2\gamma-1}\Gamma(\gamma+1)} M_{\beta\gamma} L + M_5 \end{aligned}\right\} \quad \dots (3.9)$$

where

$M_4 = M_2 + 2S_1M_1$, $M_5 = M_3 + 2S_2L_1$,
$M_2 = \|B^{-1}\|Tp_0 + (\|B^{-1}\|\|A\| + 1)\|u_0\|$
and $M_3 = \|D^{-1}\|Tq_0 + (\|D^{-1}\|\|C\| + 1)\|z_0\|$

Furthermore, we suppose that the largest eigen-value of the matrix

$$\Lambda = \begin{pmatrix} H_1 & H_2 \\ H_3 & H_4 \end{pmatrix}$$

Where

$$H_1 = \frac{T^\alpha}{2^{2\alpha-1}\Gamma(\alpha+1)} M_{\beta\alpha}(K_1 + K_2K_s) + 2M_{\beta\alpha}S_1K_3$$
$$H_2 = \frac{T^\alpha}{2^{2\alpha-1}\Gamma(\alpha+1)} M_{\beta\alpha}K_2K_s + \|B^{-1}\|Tp_1$$
$$H_3 = \frac{T^\gamma}{2^{2\gamma-1}\Gamma(\gamma+1)} M_{\beta\gamma}L_2R_s + \|D^{-1}\|Tq_1$$
$$H_4 = \frac{T^\gamma}{2^{2\gamma-1}\Gamma(\gamma+1)} M_{\beta\gamma}(L_1 + L_2R_s) + 2M_{\beta\gamma}S_2L_3$$

does not exceed unity, i. e.

$$\lambda_{max}(\Lambda) = \frac{m_2 + \sqrt{m_2^2 + 4(m_1 - m_3)}}{2} < 1 \quad \dots (3.10)$$

Where

$m_1 = H_1H_4$, $m_2 = H_1 + H_4$, $m_3 = H_2H_3$

4. MAIN RESULTS

Our main results separate to the following four parts:

4.1 Approximation of Periodic Solution of (1.1) and (1.2)

In this section, we study the periodic approximation solutions of the system of nonlinear fractional integro-differential equations (1.1) with coupled integral boundary conditions (1.2) will be introduced by the following theorem.

In the beginning, we define the following sequence of functions $\{u_m\}_{m=0}^\infty$ and $\{z_m\}_{m=0}^\infty$ given by the iterative formulas

$$\begin{aligned} u_{m+1}(t,u_0,z_0) &= u_0 + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} \\ & f(s,\beta(\tau,\alpha),u_m(s,u_0,z_0),\int_0^{a(s)} K(s,\xi)\big(u_m(\xi,u_0,z_0) - z_m(\xi,u_0,z_0)\big)d\xi)\,ds \\ &- \frac{1}{\Gamma(\alpha)}\left(\frac{t}{T}\right)^\alpha \int_0^T (T-s)^{\alpha-1} \\ & f(s,\beta(\tau,\alpha),u_m(s,u_0,z_0),\int_0^{a(s)} K(s,\xi)\big(u_m(\xi,u_0,z_0) - z_m(\xi,u_0,z_0)\big)d\xi)\,ds \\ &- \left(\frac{t}{T}\right)^\alpha \sum_{t=1}^{S_1} I_i\big(\beta(\tau,\alpha),u_m(t,u_0,z_0)\big) \\ &+ \left(\frac{t}{T}\right)^\alpha \left[B^{-1}\int_0^T h_1\big(z_m(s,u_0,z_0)\big)ds - (B^{-1}A + I)u_0\right] \\ &+ \sum_{0<t_i<t} I_i\big(\beta(\tau,\alpha),u_m(t,u_0,z_0)\big) \quad \dots (4.1) \end{aligned}$$

$$\begin{aligned} z_{m+1}(t,u_0,z_0) &= z_0 + \frac{1}{\Gamma(\gamma)}\int_0^t (t-s)^{\gamma-1} \\ & g(s,\beta(\tau,\gamma),z_m(s,u_0,z_0),\int_0^{b(s)} R(s,\xi)\big(u_m(\xi,u_0,z_0) - z_m(\xi,u_0,z_0)\big)d\xi)\,ds \\ &- \frac{1}{\Gamma(\gamma)}\left(\frac{t}{T}\right)^\gamma \int_0^T (T-s)^{\gamma-1} \\ & g(s,\beta(\tau,\gamma),z_m(s,u_0,z_0),\int_0^{b(s)} R(s,\xi)\big(u_m(\xi,u_0,z_0) - z_m(\xi,u_0,z_0)\big)d\xi)\,ds \\ &- \left(\frac{t}{T}\right)^\gamma \sum_{t=1}^{S_2} J_i\big(\beta(\tau,\gamma),z_m(t,u_0,z_0)\big) \\ &+ \left(\frac{t}{T}\right)^\gamma \left[D^{-1}\int_0^T h_2\big(u_m(s,u_0,z_0)\big)ds - (D^{-1}C + I)z_0\right] \\ &+ \sum_{0<t_i<t} J_i\big(\beta(\tau,\gamma),y_m(t,u_0,z_0)\big) \quad \dots (4.2) \end{aligned}$$

with
$u_0(t,u_0,z_0) = u_0$, $z_0(t,u_0,z_0) = z_0$, $m = 0,1,2,\dots$.

Theorem 4.1. If the system (1.1) with boundary conditions (1.2) satisfy the conditions H_1, H_2 , H_3 and (4.10) , then sequences of functions (4.1) and (4.2), which are periodic in t of period T, converges uniformly as $m \to \infty$ on the domain:-

$$(t,u_0,z_0) \in [0,T] \times D_1 \times D_2 \quad \dots (4.3)$$

to the limit functions $\begin{pmatrix} u(t,u_0,z_0) \\ z(t,u_0,z_0) \end{pmatrix}$ defined on the domain (4.3) which is periodic in t of period T and satisfies the following integral equations:-

$$u(t,u_0,z_0) = u_0 + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1}$$

$$f(s,\beta(\tau,\alpha),u(s,u_0,z_0),\int_0^{a(s)} K(s,\xi)\big(u(\xi,u_0,z_0) - z(\xi,u_0,z_0)\big)d\xi)\,ds$$
$$-\frac{1}{\Gamma(\alpha)}\left(\frac{t}{T}\right)^{\alpha}\int_0^T (T-s)^{\alpha-1}$$
$$f(s,\beta(\tau,\alpha),u(s,u_0,z_0),\int_0^{a(s)} K(s,\xi)\big(u(\xi,u_0,z_0) - z(\xi,u_0,z_0)\big)d\xi)\,ds$$
$$-\left(\frac{t}{T}\right)^{\alpha}\sum_{t=1}^{S_1} I_i\big(\beta(\tau,\alpha),u(t,u_0,z_0)\big)$$
$$+\left(\frac{t}{T}\right)^{\alpha}\left[B^{-1}\int_0^T h_1\big(z(s,u_0,z_0)\big)ds-(B^{-1}A+I)u_0\right]$$
$$+\sum_{0<t_i<t} I_i\big(\beta(\tau,\alpha),u(t,u_0,z_0)\big) \quad \dots (4.4)$$

$$z(t,u_0,z_0)=z_0+\frac{1}{\Gamma(\gamma)}\int_0^t (t-s)^{\gamma-1}$$
$$g(s,\beta(\tau,\gamma),z(s,u_0,z_0),\int_0^{b(s)} R(s,\xi)\big(u(\xi,u_0,z_0) - z(\xi,u_0,z_0)\big)d\xi)\,ds$$
$$-\frac{1}{\Gamma(\gamma)}\left(\frac{t}{T}\right)^{\gamma}\int_0^T (T-s)^{\gamma-1}$$
$$g(s,\beta(\tau,\gamma),z(s,u_0,z_0),\int_0^{b(s)} R(s,\xi)\big(u(\xi,u_0,z_0) - z(\xi,u_0,z_0)\big)d\xi)\,ds$$
$$-\left(\frac{t}{T}\right)^{\gamma}\sum_{t=1}^{S_2} J_i\big(\beta(\tau,\gamma),z(t,u_0,z_0)\big)$$
$$+\left(\frac{t}{T}\right)^{\gamma}\left[D^{-1}\int_0^T h_2\big(u(s,u_0,z_0)\big)ds-(D^{-1}C+I)z_0\right]$$
$$+\sum_{0<t_i<t} J_i\big(\beta(\tau,\gamma),z(t,u_0,z_0)\big) \quad \dots (4.5)$$

on the domain (4.3), provided that

$$\begin{pmatrix}\| u(t,u_0,z_0)-u_{m+1}(t,u_0,z_0)\| \\ \| z(t,u_0,z_0)-z_{m+1}(t,u_0,z_0)\|\end{pmatrix} \le \Lambda^m(E-\Lambda)^{-1} \begin{pmatrix}\frac{T^{\alpha}}{2^{2\alpha-1}\Gamma(\alpha+1)}M_{\beta\alpha}M+M_4 \\ \frac{T^{\gamma}}{2^{2\gamma-1}\Gamma(\gamma+1)}M_{\beta\gamma}L+M_5\end{pmatrix} \quad \dots (4.6)$$

for all $m\ge 0$, $u_0\in D_1$, $z_0\in D_2$ and $t\in R^1$, where E is an identity matrix.

Proof. Setting $m=0$ in the sequence of functions (4.1),(4.2) and by using Lemma 2.3 , we have

$$\|u_1(t,u_0,z_0)-u_0\|\le \omega(t)\left\|f(t,\beta(\tau,\alpha),u_0,\int_0^{a(t)} K(t,s)(u_0-z_0)ds)\right\|$$
$$+\left(\frac{t}{T}\right)^{\alpha}\sum_{t=1}^{S_1}\|I_i(\beta(\tau,\alpha),u_0)\|$$
$$+\left(\frac{t}{T}\right)^{\alpha}\left\|B^{-1}\int_0^T h_1(z_0)ds-(B^{-1}A+I)u_0\right\|$$
$$+\sum_{0<t_i<t}\|I_i(\beta(\tau,\alpha),u_0)\|$$
$$\le\frac{T^{\alpha}}{2^{2\alpha-1}\Gamma(\alpha+1)}M_{\beta\alpha}M+M_4$$

for all $t\in[0,T]$, $u_0\in D_f$ we get $u_1(t,u_0,z_0)\in D_1$, similarly

$$\|z_1(t,x_0,y_0)-z_0\|\le\frac{T^{\gamma}}{2^{2\gamma-1}\Gamma(\gamma+1)}M_{\beta\gamma}L+M_5$$

for all $t\in[0,T]$, $z_0\in D_g$ we obtain that $z_1(t,u_0,z_0)\in D_2$.

Thus by the mathematical induction, we find that

$$\left.\begin{aligned}\| u_m(t,u_0,z_0)-u_0\| &\le\frac{T^{\alpha}}{2^{2\alpha-1}\Gamma(\alpha+1)}M_{\beta\alpha}M+M_4\\ \| z_m(t,x_0,y_0)-z_0\| &\le\frac{T^{\gamma}}{2^{2\gamma-1}\Gamma(\gamma+1)}M_{\beta\gamma}L+M_5\end{aligned}\right\}\dots (4.7)$$

means that $u_m(t,u_0,z_0)\in D_1$, $z_m(t,u_0,z_0)\in D_2$, $u_0\in D_f$, $z_0\in D_g$, for all $t\in[0,T]$, $m=0,1,2,\dots$.

Now, we claim that the sequences of functions (4.1) and (4.2) are uniformly convergent on the domain (4.3).

$$\|u_{m+1}(t,u_0,z_0)-u_m(t,u_0,z_0)\|$$
$$\le\omega(t)\Big\|f(t,\beta(\tau,\alpha),u_m(t,u_0,z_0),\int_0^{a(t)}K(t,s)\big(u_m(s,u_0,z_0)-z_m(s,u_0,z_0)\big)ds)-$$
$$f(t,\beta(\tau,\alpha),u_{m-1}(t,u_0,z_0),\int_0^{a(t)}K(t,s)\big(u_{m-1}(s,u_0,z_0)-z_{m-1}(s,u_0,z_0)\big)ds)\Big\|$$
$$+\sum_{t=1}^{S_1}\big\|I_i\big(\beta(\tau,\alpha),u_m(t,u_0,z_0)\big)-I_i\big(\beta(\tau,\alpha),u_{m-1}(t,x_0,y_0)\big)\big\|$$
$$+\left(\frac{t}{T}\right)^{\alpha}\|B^{-1}\|T\big\|h_1\big(z_m(t,u_0,z_0)-z_{m-1}(t,u_0,z_0)\big)\big\|$$
$$+\sum_{0<t_i<t}\big\|I_i\big(\beta(\tau,\alpha),u_m(t,u_0,z_0)\big)-I_i\big(\beta(\tau,\alpha),u_{m-1}(t,u_0,z_0)\big)\big\|$$

$$\le\big(\omega(t)M_{\beta\alpha}(K_1+K_2K_s)+2M_{\beta\alpha}S_1K_3\big)\times\|u_m(t,u_0,z_0)-u_{m-1}(t,u_0,z_0)\|+\omega(t)M_{\beta\alpha}K_2K_s\times\|u_m(t,u_0,z_0)-u_{m-1}(t,u_0,z_0)\|+\|B^{-1}\|T\,p_1\|z_m(t,u_0,z_0)-z_{m-1}(t,u_0,z_0)\|$$

Therefore, we get

$$\|u_{m+1}(t,u_0,z_0)-u_m(t,u_0,z_0)\|\le(\omega(t)M_{\beta\alpha}(K_1+K_2K_s)+2M_{\beta\alpha}S_1K_3)\|u_m(t,u_0,z_0)-u_{m-1}(t,u_0,z_0)\|+\big(\omega(t)M_{\beta\alpha}K_2K_s+\|B^{-1}\|Tp_1\big)\|z_m(t,u_0,z_0)-z_{m-1}(t,u_0,z_0)\| \quad \dots (4.8)$$

In the same way, we obtain that

$$\|z_{m+1}(t,u_0,z_0)-z_m(t,u_0,z_0)\|\le\big(\varphi(t)M_{\beta\gamma}L_2R_s+\|D^{-1}\|Tq_1\big)\|u_m(t,u_0,z_0)-u_{m-1}(t,u_0,z_0)\|+\big(\varphi(t)M_{\beta\gamma}(L_1+L_2R_s)+2M_{\beta\gamma}S_2L_3\big)\|z_m(t,u_0,z_0)-z_{m-1}(t,u_0,z_0)\| \quad \dots (4.9)$$

We rewrite inequalities (4.8) and (4.9) in vector form to gain

$$\begin{pmatrix}\| u_{m+1}(t,u_0,z_0)-u_m(t,u_0,z_0)\| \\ \| z_{m+1}(t,u_0,z_0)-z_m(t,u_0,z_0)\|\end{pmatrix}\le\Lambda\begin{pmatrix}\| u_m(t,u_0,z_0)-u_{m-1}(t,u_0,z_0)\| \\ \| z_m(t,u_0,z_0)-z_{m-1}(t,u_0,z_0)\|\end{pmatrix} \quad \dots (4.10)$$

By mathematical induction, we obtain that

$$\begin{pmatrix} \| u_{m+1}(t,u_0,z_0) - u_m(t,u_0,z_0) \| \\ \| z_{m+1}(t,u_0,z_0) - z_m(t,u_0,z_0) \| \end{pmatrix} \le \Lambda^m \begin{pmatrix} \| u_1(t,u_0,z_0) - u_0 \| \\ \| z_1(t,u_0,z_0) - z_0 \| \end{pmatrix}$$

Now from $m = 1,2, \ldots$ and $p \ge 1$, we find that

$$\begin{pmatrix} \| u_{m+p}(t,u_0,z_0) - u_m(t,u_0,z_0) \| \\ \| z_{m+p}(t,u_0,z_0) - z_m(t,u_0,z_0) \| \end{pmatrix} \le \Lambda^m (E-\Lambda)^{-1} \begin{pmatrix} \frac{T^\alpha}{2^{2\alpha-1}\Gamma(\alpha+1)} M_{\beta\alpha} + M_4 \\ \frac{T^\gamma}{2^{2\gamma-1}\Gamma(\gamma+1)} M_{\beta\gamma} + M_5 \end{pmatrix} \quad \ldots (4.11)$$

for all $t \in [0,T]$, $u_0 \in D_f$ and $z_0 \in D_g$.

Since $\frac{m_2 + \sqrt{m_2^2 + 4(m_1 - m_3)}}{2} < 1$ *and* $\lim_{m\to\infty} \Lambda^m = 0$, so that the right side of (4.11) tends to zero. Therefore the sequence of function $\begin{pmatrix} u_m(t,u_0,z_0) \\ z_m(t,u_0,z_0) \end{pmatrix}$ is converges uniformly on the domain (4.3) to the limit function $\begin{pmatrix} u(t,u_0,z_0) \\ z(t,u_0,z_0) \end{pmatrix}$ which is defined on the same domain.

Let

$$\lim_{m\to\infty} \begin{pmatrix} u_m(t,u_0,z_0) \\ z_m(t,u_0,z_0) \end{pmatrix} = \begin{pmatrix} u(t,u_0,z_0) \\ z(t,u_0,z_0) \end{pmatrix} \quad \ldots (4.12)$$

Since the sequence of functions (4.1) and (4.2) are periodic in t of period T, then the limiting function $\begin{pmatrix} u(t,u_0,z_0) \\ z(t,u_0,z_0) \end{pmatrix}$ is also periodic in t of period T. Also by the Lemma 2.3 and the inequality (4.11) the inequality (4.6)hold for all $m \ge 0$.

By using the relation (4.12) and proceeding in (4.1) and (4.2) to limit, when $m \to \infty$, it is convincing that the limiting function $\begin{pmatrix} u(t,u_0,z_0) \\ z(t,u_0,z_0) \end{pmatrix}$ is the periodic solution of the integral equations (4.4) and (4.4).

4.2 Uniqueness of Periodic Solution of (1.1) and (1.2)

Theorem 4.2 All assumptions of the Theorem 4.1 are satisfy, then $\begin{pmatrix} u(t,u_0,z_0) \\ z(t,u_0,z_0) \end{pmatrix}$ is a unique solution of the system (1.1) with boundary conditions (1.2) .

Proof. Assume that $\begin{pmatrix} \hat{u}(t,u_0,z_0) \\ \hat{z}(t,u_0,z_0) \end{pmatrix}$ is another solution for the system (1.1) with boundary conditions (1.2), means that

$$\hat{u}(t,u_0,z_0) = u_0 + \frac{1}{\Gamma(\alpha)} \int_0^t (t-s)^{\alpha-1} f\left(s, \beta(\tau,\alpha), \hat{u}(s,u_0,z_0), \int_0^{a(s)} K(s,\xi)\big(\hat{u}(\xi,u_0,z_0) - \hat{z}(\xi,u_0,z_0)\big)d\xi\right) ds$$
$$- \frac{1}{\Gamma(\alpha)}\left(\frac{t}{T}\right)^\alpha \int_0^T (T-s)^{\alpha-1} f\left(s,\beta(\tau,\alpha),\hat{u}(s,u_0,z_0), \int_0^{a(s)} K(s,\xi)\big(\hat{u}(\xi,u_0,z_0) - \hat{z}(\xi,u_0,z_0)\big)d\xi\right) ds$$
$$- \left(\frac{t}{T}\right)^\alpha \sum_{t=1}^{S_1} I_i\big(\beta(\tau,\alpha), \hat{u}(t,u_0,z_0)\big) + \left(\frac{t}{T}\right)^\alpha \left[B^{-1} \int_0^T h_1\big(\hat{z}(s,u_0,z_0)\big)ds - (B^{-1}A + I)u_0\right]$$
$$+ \sum_{0<t_i<t} I_i\big(\beta(\tau,\alpha), \hat{x}(t,u_0,z_0)\big) \quad \ldots (4.13)$$

and

$$\hat{z}(t,u_0,z_0) = z_0 + \frac{1}{\Gamma(\gamma)} \int_0^t (t-s)^{\gamma-1} g\left(s,\beta(\tau,\gamma),\hat{z}(s,u_0,z_0), \int_0^{b(s)} R(s,\xi)\big(\hat{u}(\xi,u_0,z_0) - \hat{z}(\xi,u_0,z_0)\big)d\xi\right) ds$$
$$- \frac{1}{\Gamma(\gamma)}\left(\frac{t}{T}\right)^\gamma \int_0^T (T-s)^{\gamma-1} g\left(s,\beta(\tau,\gamma),\hat{z}(s,u_0,z_0), \int_0^{b(s)} R(s,\xi)\big(\hat{u}(\xi,u_0,z_0) - \hat{z}(\xi,u_0,z_0)\big)d\xi\right) ds$$
$$- \left(\frac{t}{T}\right)^\gamma \sum_{t=1}^{S_2} J_i\big(\beta(\tau,\gamma), \hat{z}(t,u_0,z_0)\big) + \left(\frac{t}{T}\right)^\gamma \left[D^{-1} \int_0^T h_2\big(\hat{u}(s,u_0,z_0)\big)ds - (D^{-1}C + I)z_0\right]$$
$$+ \sum_{0<t_i<t} J_i\big(\beta(\tau,\gamma), \hat{z}(t,u_0,z_0)\big) \quad \ldots (4.14)$$

Now, the difference between the toe solutions $\begin{pmatrix} u(t,u_0,z_0) \\ z(t,u_0,z_0) \end{pmatrix}$ and $\begin{pmatrix} \hat{u}(t,u_0,z_0) \\ \hat{z}(t,u_0,z_0) \end{pmatrix}$, for all $t \in [0,T]$ and $u_0 \in D_f$, $z_0 \in D_g$, we have

$$\| \hat{u}(t,u_0,z_0) - u(t,u_0,z_0) \| \le \big(\omega(t) M_{\beta\alpha}(K_1 + K_2 K_s) + 2M_{\beta\alpha} S_1 K_3\big) \|\hat{u}(t,u_0,z_0) - u(t,u_0,z_0)\| + \big(\omega(t) M_{\beta\alpha} K_2 K_s + \|B^{-1}\| T p_1\big) \|\hat{z}(t,u_0,z_0) - z(t,u_0,z_0)\| \quad \ldots (4.15)$$

and

$$\|\hat{z}(t,u_0,z_0) - z(t,u_0,z_0)\| \le \big(\varphi(t) M_{\beta\gamma} L_2 R_s + \|D^{-1}\| T q_1\big) \|u(t,u_0,z_0) - u(t,u_0,z_0)\| + \big(\varphi(t) M_{\beta\gamma}(L_1 + L_2 R_s) + 2M_{\beta\gamma} S_2 L_3\big) \|\hat{z}(t,u_0,z_0) - z(t,u_0,z_0)\| \quad \ldots (4.16)$$

We rewrite inequalities (4.15) and (4.16) in vector form to gain

$$\begin{pmatrix} \| \hat{u}(t,u_0,z_0) - u(t,u_0,z_0) \| \\ \|\hat{z}(t,u_0,z_0) - z(t,u_0,z_0)\| \end{pmatrix} \le \Lambda \begin{pmatrix} \| \hat{u}(t,u_0,z_0) - u(t,u_0,z_0) \| \\ \|\hat{z}(t,u_0,z_0) - z(t,u_0,z_0)\| \end{pmatrix} \quad \ldots (4.17)$$

By mathematical induction, we obtain that

$$\begin{pmatrix} \| \hat{u}(t,u_0,z_0) - u(t,u_0,z_0) \| \\ \|\hat{z}(t,u_0,z_0) - z(t,u_0,z_0)\| \end{pmatrix} \le \Lambda^m \begin{pmatrix} \| \hat{u}(t,u_0,z_0) - u(t,u_0,z_0) \| \\ \|\hat{z}(t,u_0,z_0) - z(t,u_0,z_0)\| \end{pmatrix} \quad \ldots (4.10)$$

From the condition (4.10), shows that the solution $\begin{pmatrix} u(t,u_0,z_0) \\ z(t,u_0,z_0) \end{pmatrix} = \begin{pmatrix} \hat{u}(t,u_0,z_0) \\ \hat{z}(t,u_0,z_0) \end{pmatrix}$, thus $\begin{pmatrix} u(t,u_0,z_0) \\ z(t,u_0,z_0) \end{pmatrix}$ is a unique periodic solution on the domain (4.3).

4.3 Existence of Periodic Solutions of (1.1) and (1.2)

The problem of the existence of the periodic solution for the system (1.1) with boundary condition (1.2) is uniquely connected with the existence of the zeros of the vector functions:-

$$\Delta_1(0,u_0,z_0) = -\frac{\alpha}{T^\alpha}\int_0^T (T-s)^{\alpha-1}$$
$$f\left(s,\beta(\tau,\alpha),u(s,u_0,z_0),\int_0^{a(s)} K(s,\xi)\big(u(\xi,u_0,z_0) - z(\xi,u_0,z_0)\big)d\xi\right) ds$$
$$-\frac{\Gamma(\alpha+1)}{T^\alpha}\sum_{t=1}^{S_1} I_i\big(\beta(\tau,\alpha),u(t,u_0,z_0)\big)$$
$$+\frac{\Gamma(\alpha+1)}{T^\alpha}\left[B^{-1}\int_0^T h_1\big(z(s,u_0,z_0)\big)ds-(B^{-1}A+I)u_0\right] \quad \text{... (4.18)}$$

and

$$\Delta_2(0,u_0,z_0) = -\frac{\gamma}{T^\gamma}\int_0^T (T-s)^{\gamma-1}$$
$$g\left(s,\beta(\tau,\gamma),z(s,u_0,z_0),\int_0^{b(s)} R(s,\xi)\big(u(\xi,u_0,z_0) - z(\xi,u_0,z_0)\big)d\xi\right) ds$$
$$-\frac{\Gamma(\gamma+1)}{T^\gamma}\sum_{t=1}^{S_2} J_i\big(\beta(\tau,\gamma),z(t,u_0,z_0)\big)$$
$$+\frac{\Gamma(\gamma+1)}{T^\gamma}\left[D^{-1}\int_0^T h_2\big(u(s,u_0,z_0)\big)ds-(D^{-1}C+I)z_0\right] \quad \text{... (4.19)}$$

Also, we define the sequences of vector functions $\Delta_{1m}(0,u_0,z_0)$ and $\Delta_{2m}(0,u_0,z_0)$ are approximately determined by the following:-

$$\Delta_{1m}(0,u_0,z_0) = -\frac{\alpha}{T^\alpha}\int_0^T (T-s)^{\alpha-1}$$
$$f\left(s,\beta(\tau,\alpha),u_m(s,u_0,z_0),\int_0^{a(s)} K(s,\xi)\big(u_m(\xi,u_0,z_0) - z_m(\xi,u_0,z_0)\big)d\xi\right) ds$$
$$-\frac{\Gamma(\alpha+1)}{T^\alpha}\sum_{t=1}^{S_1} I_i\big(\beta(\tau,\alpha),u_m(t,u_0,z_0)\big)$$
$$+\frac{\Gamma(\alpha+1)}{T^\alpha}\left[B^{-1}\int_0^T h_1\big(z_m(s,u_0,z_0)\big)ds-(B^{-1}A+I)u_0\right] \quad \text{... (4.20)}$$

and

$$\Delta_{2m}(0,u_0,z_0) = -\frac{\gamma}{T^\gamma}\int_0^T (T-s)^{\gamma-1}$$
$$g\left(s,\beta(\tau,\gamma),z_m(s,u_0,z_0),\int_0^{b(s)} R(s,\xi)\big(u_m(\xi,u_0,z_0) - z_m(\xi,u_0,z_0)\big)d\xi\right) ds$$
$$-\frac{\Gamma(\gamma+1)}{T^\gamma}\sum_{t=1}^{S_2} J_i\big(\beta(\tau,\gamma),z(t,u_0,z_0)\big)$$
$$+\frac{\Gamma(\gamma+1)}{T^\gamma}\left[D^{-1}\int_0^T h_2\big(u_m(s,u_0,z_0)\big)ds-(D^{-1}C+I)z_0\right] \quad \text{... (4.21)}$$

Theorem 4.3. If the hypotheses and all the conditions of the theorem 4.1 are given, the following inequalities are satisfied:-

$$\begin{pmatrix} \| \Delta_1(0,u_0,z_0) - \Delta_{1m}(0,u_0,z_0) \| \\ \| \Delta_2(0,u_0,z_0) - \Delta_{2m}(0,u_0,z_0) \| \end{pmatrix} \le \begin{pmatrix} \langle \begin{pmatrix} d_1 \\ d_2 \end{pmatrix}, \Lambda^m (E-\Lambda)^{-1}\eta \rangle \\ \langle \begin{pmatrix} d_3 \\ d_4 \end{pmatrix}, \Lambda^m (E-\Lambda)^{-1}\eta \rangle \end{pmatrix} \quad \text{... (4.22)}$$

holds for all $m \ge 0$, where

$$d_1 = \frac{1}{\Gamma(\alpha)}M_{\beta\alpha}(K_1+K_2K_s) + + \frac{\Gamma(\alpha+1)}{T^\alpha}M_{\beta\alpha}K_3S_1\ ,$$
$$d_2 = \frac{1}{\Gamma(\alpha)}M_{\beta\alpha}K_2K_s + \frac{\Gamma(\alpha+1)}{T^\alpha}\|B^{-1}\|Tp_1\ ,$$
$$d_3 = \frac{1}{\Gamma(\gamma)}M_{\beta\gamma}L_2R_s + \frac{\Gamma(\gamma+1)}{T^\gamma}\|D^{-1}\|Tq_1\ ,$$
$$d_4 = \frac{1}{\Gamma(\gamma)}M_{\beta\gamma}(L_1+L_2R_s) + \frac{\Gamma(\gamma+1)}{T^\gamma}M_{\beta\gamma}L_3S_2$$

and $\eta = \begin{pmatrix} \frac{T^\alpha}{2^{2\alpha-1}\Gamma(\alpha+1)}M_{\beta\alpha} + M_4 \\ \frac{T^\gamma}{2^{2\gamma-1}\Gamma(\gamma+1)}M_{\beta\gamma} + M_5 \end{pmatrix}$

Proof . From equations (4.18) to (4.21), we obtain that

$$\| \Delta_1(0,u_0,z_0) - \Delta_{1m}(0,u_0,z_0) \| \le \left(\frac{1}{\Gamma(\alpha)}M_{\beta\alpha}(K_1+K_2K_s) + \frac{\Gamma(\alpha+1)}{T^\alpha}M_{\beta\alpha}K_3S_1\right)\|u(t,u_0,z_0) - u_m(t,u_0,z_0)\|$$
$$+\left(\frac{1}{\Gamma(\alpha)}M_{\beta\alpha}K_2K_s + \frac{\Gamma(\alpha+1)}{T^\alpha}\|B^{-1}\|Tp_1\right)\|z(t,u_0,z_0) - z_m(t,u_0,z_0)\| \quad \text{... (4.23)}$$

and

$$\| \Delta_2(0,u_0,z_0) - \Delta_{2m}(0,u_0,z_0) \| \le \left(\frac{1}{\Gamma(\gamma)}M_{\beta\gamma}L_2R_s + \frac{\Gamma(\gamma+1)}{T^\gamma}\|D^{-1}\|Tq_1\right)\|u(t,u_0,z_0) - u_m(t,u_0,z_0)\|$$
$$+\left(\frac{1}{\Gamma(\gamma)}M_{\beta\gamma}(L_1+L_2R_s) + \frac{\Gamma(\gamma+1)}{T^\gamma}M_{\beta\gamma}L_3S_2\right)\|z(t,u_0,z_0) - z_m(t,u_0,z_0)\| \quad \text{... (4.24)}$$

Since $\|u(t,u_0,z_0) - u_m(t,u_0,z_0)\| \le \Lambda^m(E-\Lambda)^{-1}\left(\frac{T^\alpha}{2^{2\alpha-1}\Gamma(\alpha+1)}M_{\beta\alpha} + M_4\right)$,

and $\|z(t,u_0,z_0) - z_m(t,u_0,z_0)\| \le \Lambda^m(E-\Lambda)^{-1}\left(\frac{T^\gamma}{2^{2\gamma-1}\Gamma(\gamma+1)}M_{\beta\gamma} + M_5\right)$,

hence rewrite the equations (4.23) and (4.24) as a vector form. The inequality (4.22) is hold for all $m \ge 0$.

Theorem 4.4. Let the vector functions $f(t,\beta(\tau,\alpha),u,x)$ and $g(t,\beta(\tau,\gamma),z,y)$ be defined on the intervals $[a_1,b_1]$ and $[a_2,b_2]$ on R^1 and periodic in t of period T, suppose that for all $m \ge 0$, then the sequences of the functions $\Delta_{1m}(0,u_0,z_0)$ and $\Delta_{2m}(0,u_0,z_0)$ which are defined in (4.20)and (4.21) satisfy the inequalities:-

$$\left.\begin{aligned} \min_{u_0\in I_1, z_0\in I_2} \Delta_{1m}(0,u_0,z_0) &\le -\langle \begin{pmatrix} d_1 \\ d_2 \end{pmatrix}, \Lambda^m(E-\Lambda)^{-1}\eta \rangle \\ \max_{u_0\in I_1, z_0\in I_2} \Delta_{1m}(0,u_0,z_0) &\ge \langle \begin{pmatrix} d_1 \\ d_2 \end{pmatrix}, \Lambda^m(E-\Lambda)^{-1}\eta \rangle \end{aligned}\right\} \text{... (4.25)}$$

$$\left.\begin{aligned} \min_{u_0\in I_1, z_0\in I_2} \Delta_{2m}(0u_0,z_0) &\le -\langle \begin{pmatrix} d_3 \\ d_4 \end{pmatrix}, \Lambda^m(E-\Lambda)^{-1}\eta \rangle \\ \max_{u_0\in I_1, z_0\in I_2} \Delta_{2m}(0,u_0,z_0) &\ge \langle \begin{pmatrix} d_3 \\ d_4 \end{pmatrix}, \Lambda^m(E-\Lambda)^{-1}\eta \rangle \end{aligned}\right\} \text{... (4.26)}$$

Then (1.1) and (1.2) has a periodic solution $\begin{pmatrix} u(t,u_0,z_0) \\ z(t,u_0,z_0) \end{pmatrix}$ such that $u_0 \in I_1 = \left[a + \frac{T^\alpha}{2^{2\alpha-1}\Gamma(\alpha+1)}M_{\beta\alpha} + M_4\ , b - \frac{T^\alpha}{2^{2\alpha-1}\Gamma(\alpha+1)}M_{\beta\alpha} + M_4\right]$ and $z_0 \in I_2 = \left[c + \frac{T^\gamma}{2^{2\gamma-1}\Gamma(\gamma+1)}M_{\beta\gamma} + M_5\ , d - \frac{T^\gamma}{2^{2\gamma-1}\Gamma(\gamma+1)}M_{\beta\gamma} + M_5\right]$

Proof . Let u_1, u_2 and z_1, z_2 be any points belonging to the intervals I_1 and I_2 respectively, such that

$$\left.\begin{aligned} \Delta_{1m}(0,u_1,z_1) &= \min_{u_0\in I_1, z_0\in I_2} \Delta_{1m}(0,u_0,z_0) \\ \Delta_{1m}(0,u_2,z_2) &= \max_{u_0\in I_1, z_0\in I_2} \Delta_{1m}(0,u_0,z_0) \end{aligned}\right\} \quad \text{... (4.27)}$$

$$\left.\begin{aligned}\Delta_{2m}(0,u_1,z_1) &= \min_{u_0\in I_1, z_0\in I_2} \Delta_{2m}(0,u_0,z_0) \\ \Delta_{2m}(0,u_2,z_2) &= \max_{u_0\in I_1, z_0\in I_2} \Delta_{2m}(0,u_0,z_0)\end{aligned}\right\} \quad \dots (4.28)$$

By using inequalities (4.23) to (4.24)., the following are obtained

$$\left.\begin{aligned}&\Delta_1(0,u_1,z_1) = \Delta_{1m}(0,u_1,z_1) + (\Delta_1(0,u_1,z_1) - \Delta_{1m}(0,u_1,z_1)) < 0 \\ &\Delta_1(0,u_2,z_2) = \Delta_{1m}(0,u_2,z_2) + (\Delta_1(0,u_2,z_2) - \Delta_{1m}(0,u_2,z_2)) > 0\end{aligned}\right\} \quad \dots (4.29)$$

$$\left.\begin{aligned}&\Delta_2(0,u_1,z_1) = \Delta_{2m}(0,u_1,z_1) + (\Delta_2(0,u_1,z_1) - \Delta_{2m}(0,u_1,z_1)) < 0 \\ &\Delta_2(0,u_2,z_2) = \Delta_{2m}(0,u_2,z_2) + (\Delta_2(0,u_2,z_2) - \Delta_{2m}(0,u_2,z_2)) > 0\end{aligned}\right\} \quad \dots (4.30)$$

and from the continuity of the functions $\Delta_1(0,u_0,z_0)$, $\Delta_2(0,u_0,z_0)$ and the inequalities (4.29) and (4.30), then the isolated singular points $u^0 \in [u_1,u_2]$ and $z^0 \in [z_1,z_2]$ exist such that $\Delta_1(0,x_0,y_0) = 0$ and $\Delta_2(0,u_0,z_0) = 0$. This means that (1.1) has a periodic solution $\begin{pmatrix} u(t,u_0,z_0) \\ z(t,u_0,z_0)\end{pmatrix}$.

Remark 4.5. Theorem 4.3 is provided when $R^n = R^1$, i.e. u_0 and z_0 are scalar singular points and should be isolated. For more details, see (Samoilenko and Ronto, 1976).

4.4 Stability of Periodic Solution of (1.1) and (1.2)

Theorem 4.6. Let the vector functions $\Delta_1(0,u_0,z_0)$ and $\Delta_2(0,u_0,z_0)$ be defined by the equations (4.18) and (4.19) where $u(t,u_0,z_0)$ is a limit of the sequence of the function (4.1), the function $z(t,u_0,z_0)$ is the limit of the sequence of the function (4.2), then the following inequalities yield:-

$$\begin{pmatrix} \|\Delta_1(0,u_0,z_0)\| \\ \|\Delta_2(0,u_0,z_0)\|\end{pmatrix} \le \begin{pmatrix} \frac{1}{\Gamma(\alpha)} M_{\beta\alpha} M + \frac{\Gamma(\alpha+1)}{T^\alpha} M_{\beta\alpha} M_1 S_1 + \frac{\Gamma(\alpha+1)}{T^\alpha}(M_2 - \|u_0\|) \\ \frac{1}{\Gamma(\gamma)} M_{\beta\gamma} L + \frac{\Gamma(\gamma+1)}{T^\gamma} M_{\beta\gamma} L_1 S_2 + \frac{\Gamma(\gamma+1)}{T^\gamma}(M_3 - \|z_0\|)\end{pmatrix} \quad \dots (4.31)$$

and

$$\begin{pmatrix} \|\Delta_1(0,u_0^1,z_0^1) - \Delta_1(0,u_0^2,z_0^2)\| \\ \|\Delta_2(0,u_0^1,z_0^1) - \Delta_2(0,u_0^2,z_0^2)\|\end{pmatrix} \le \begin{pmatrix} (E_1 + E_2F_5F_6)E_5 & (E_1F_3F_4 + E_2)E_6 \\ (E_3 + E_4F_5F_6)E_5 & (E_3F_3F_4 + E_4)E_6\end{pmatrix} \times \begin{pmatrix} \|u_0^1 - u_0^2\| \\ \|z_0^1 - z_0^2\|\end{pmatrix} \quad \dots (4.32)$$

where

$E_1 = \frac{1}{\Gamma(\alpha)} M_{\beta\alpha}(K_1 + K_2K_s) + 2M_{\beta\alpha}S_1K_3,$

$E_2 = \left(\frac{1}{\Gamma(\alpha)} M_{\beta\alpha}K_2K_s + \frac{\Gamma(\alpha+1)}{T^\alpha}\|B^{-1}\|Tp_1\right),$

$E_3 = \left(\frac{1}{\Gamma(\gamma)} M_{\beta\gamma}L_2R_s + \frac{\Gamma(\gamma+1)}{T^\gamma}\|D^{-1}\|Tq_1\right),$

$E_4 = \frac{1}{\Gamma(\gamma)} M_{\beta\gamma}(L_1 + L_2R_s) + 2M_{\beta\gamma}S_2L_3,$

$E_5 = F_1F_3F_7$, $E_6 = F_2F_5F_7$, $F_1 = 1 + \|B^{-1}A + E\|$, $F_2 = 1 + \|D^{-1}C + E\|$,

$F_3 = \left(1 - \frac{T^\alpha}{2^{2\alpha-1}\Gamma(\alpha+1)} M_{\beta\alpha}(K_1 + K_2K_s) + 2M_{\beta\alpha}S_1K_3\right)^{-1},$

$F_4 = \frac{T^\alpha}{2^{2\alpha-1}\Gamma(\alpha+1)} M_{\beta\alpha}K_2K_s + \|B^{-1}\|Tp_1$,

$F_5 = \left(1 - \frac{T^\gamma}{2^{2\gamma-1}\Gamma(\gamma+1)} M_{\beta\gamma}(L_1 + L_2R_s) + 2M_{\beta\gamma}S_2L_3\right)^{-1},$

$F_6 = \frac{T^\gamma}{2^{2\gamma-1}\Gamma(\gamma+1)} M_{\beta\gamma}L_2R_s + \|D^{-1}\|Tq_1$,

$F_7 = (1 - F_3F_4F_5F_6)^{-1}$

Proof . From the properties of the functions $u(t,u_0,z_0)$ and $z(t,u_0,z_0)$ as in the Theorem 4.1, the functions $\Delta_1 = \Delta_1(0,u_0,z_0)$ and $\Delta_2 = \Delta_2(0,u_0,z_0)$, $u_0 \in D_1, z_0 \in D_2$ are continuous and bounded by $M_\gamma M$, $N_\gamma N$ in the domain (4.3).

From (4.18) and (4.19) , we obtained that

$$\|\Delta_1(0,u_0,z_0)\| \le \frac{1}{\Gamma(\alpha)} M_{\beta\alpha} M + \frac{\Gamma(\alpha+1)}{T^\alpha} M_{\beta\alpha} M_1 S_1 + \frac{\Gamma(\alpha+1)}{T^\alpha}(M_1 - \|x_0\|) \quad \dots (4.33)$$

and

$$\|\Delta_2(0,u_0,z_0)\| \le \frac{1}{\Gamma(\gamma)} M_{\beta\gamma} L + \frac{\Gamma(\gamma+1)}{T^\gamma} M_{\beta\gamma} L_1 S_2 + \frac{\Gamma(\gamma+1)}{T^\gamma}(M_2 - \|z_0\|) \quad \dots (4.34)$$

by rewriting (4.33) and (4.34) by vector form, we obtain (4.31).

From inequality (4.18), we get

$$\|\Delta_1(0,u_0^1,z_0^1) - \Delta_1(0,u_0^2,z_0^2)\| \le \left(\frac{1}{\Gamma(\alpha)} M_{\beta\alpha}(K_1 + K_2K_s) + \frac{\Gamma(\alpha+1)}{T^\alpha} M_{\beta\alpha}K_3S_1\right) \|u(t,u_0^1,z_0^1) - z(t,u_0^2,z_0^2)\| + \left(\frac{1}{\Gamma(\alpha)} M_{\beta\alpha}K_2K_s + \frac{\Gamma(\alpha+1)}{T^\alpha}\|B^{-1}\|Tp_1\right) \|z(t,u_0^1,z_0^1) - z(t,u_0^2,z_0^2)\| \quad \dots (4.35)$$

$$\|\Delta_2(0,u_0^1,z_0^1) - \Delta_2(0,u_0^2,z_0^2)\| \le \left(\frac{1}{\Gamma(\gamma)} M_{\beta\gamma}L_2R_s + \frac{\Gamma(\gamma+1)}{T^\gamma}\|D^{-1}\|Tq_1\right) \|u(t,u_0^1,z_0^1) - u(t,u_0^2,z_0^2)\| + \left(\frac{1}{\Gamma(\gamma)} M_{\beta\gamma}(L_1 + L_2R_s) + \frac{\Gamma(\gamma+1)}{T^\gamma} M_{\beta\gamma}L_3S_2\right) \|z(t,u_0^1,z_0^1) - z(t,u_0^2,z_0^2)\| \quad \dots (4.36)$$

where the functions $u(t,u_0^1,z_0^1)$, (t,u_0^1,z_0^1), $u(t,u_0^2,z_0^2)$ and $z(t,u_0^2,z_0^2)$ are solutions of the equation:-

$$\begin{aligned} u(t,u_0^k,z_0^k) &= x_0^k + \frac{1}{\Gamma(\alpha)}\int_0^t (t-s)^{\alpha-1} f\Big(s,\beta(\tau,\alpha), u(s,u_0^k,z_0^k), \int_0^{a(s)} K(s,\xi)\left(u(\xi,u_0^k,z_0^k) - z(\xi,u_0^k,z_0^k)\right)d\xi\Big)\, ds \\ &- \frac{1}{\Gamma(\alpha)}\left(\frac{t}{T}\right)^\alpha \int_0^T (T-s)^{\alpha-1} f\Big(s,\beta(\tau,\alpha), u(s,u_0^k,z_0^k), \int_0^{a(s)} K(s,\xi)\left(u(\xi,u_0^k,z_0^k) - z(\xi,u_0^k,z_0^k)\right)d\xi\Big)\, ds \\ &- \left(\frac{t}{T}\right)^\alpha \sum_{t=1}^{S_1} I_i\left(\beta(\tau,\alpha), u(s,x_0^k,y_0^k)\right) \\ &+ \left(\frac{t}{T}\right)^\alpha \left[B^{-1}\int_0^T h_1\left(z(s,u_0^k,z_0^k)\right)ds - (B^{-1}A + E)u_0^k\right] \\ &+ \sum_{0<t_i<t} I_i\left(\beta(\tau,\alpha), u(s,u_0^k,z_0^k)\right) \quad \dots (4.37)\end{aligned}$$

and

$$z(t,u_0^k,z_0^k) = z_0^k + \frac{1}{\Gamma(\gamma)}\int_0^t (t-s)^{\gamma-1}$$

$$g\left(s,\beta(\tau,\gamma),z\left(s,u_0^k,z_0^k\right),\int_0^{b(s)} R(s,\xi)\left(u\left(\xi,u_0^k,z_0^k\right) - z\left(\xi,u_0^k,z_0^k\right)\right)d\xi\right)ds$$

$$-\frac{1}{\Gamma(\gamma)}\left(\frac{t}{T}\right)^{\gamma}\int_0^T (T-s)^{\gamma-1}$$

$$g\left(s,\beta(\tau,\gamma),z\left(s,u_0^k,z_0^k\right),\int_0^{b(s)} R(s,\xi)\left(u\left(\xi,u_0^k,z_0^k\right) - z\left(\xi,u_0^k,z_0^k\right)\right)d\xi\right)ds$$

$$-\left(\frac{t}{T}\right)^{\gamma}\sum_{t=1}^{S_2} J_i\left(\beta(\tau,\gamma),z\left(s,u_0^k,z_0^k\right)\right) + \left(\frac{t}{T}\right)^{\gamma}\left[D^{-1}\int_0^T h_2\left(u\left(s,u_0^k,z_0^k\right)\right)ds - (D^{-1}C+E)z_0^k\right] + \sum_{0<t_i<t} J_i\left(\beta(\tau,\gamma),z\left(s,u_0^k,z_0^k\right)\right) \quad \ldots (4.38)$$

where $k = 1,2$, From (4.37) and (4.38), we get

$$\| u(t,u_0^1,z_0^1) - u(t,u_0^2,z_0^2) \| \le F_1\|u_0^1 - u_0^2\| + \left(\frac{T^{\alpha}}{2^{2\alpha-1}\Gamma(\alpha+1)} M_{\beta\alpha}(K_1 + K_2K_s) + 2M_{\beta\alpha}S_1K_3\right) \| u(t,u_0^1,z_0^1) - u(t,u_0^2,z_0^2) \| + \left(\frac{T^{\alpha}}{2^{2\alpha-1}\Gamma(\alpha+1)} M_{\beta\alpha}M_{\beta\alpha}K_2K_s + \|B^{-1}\|Tp_1\right) \| z(t,u_0^1,z_0^1) - z(t,u_0^2,z_0^2) \| \quad \ldots (4.39)$$

and

$$\| z(t,u_0^1,z_0^1) - z(t,u_0^2,z_0^2) \| \le F_2\|z_0^1 - z_0^2\| + \left(\frac{T^{\gamma}}{2^{2\gamma-1}\Gamma(\gamma+1)} M_{\beta\gamma}L_2R_s + \|D^{-1}\|Tq_1\right) \| u(t,u_0^1,z_0^1) - u(t,u_0^2,z_0^2) \| + \left(\frac{T^{\gamma}}{2^{2\gamma-1}\Gamma(\gamma+1)} M_{\beta\gamma}(L_1 + L_2R_s) + 2M_{\beta\gamma}S_2L_3\right) \| z(t,u_0^1,z_0^1) - z(t,u_0^2,z_0^2) \| \quad \ldots (4.40)$$

From equations (4.39) and (4.40)

$$\| u(t,u_0^1,z_0^1) - u(t,u_0^2,z_0^2) \| \le F_1F_3\|u_0^1 - u_0^2\| + F_3F_4 \| z(t,u_0^1,z_0^1) - z(t,u_0^2,z_0^2) \| \quad \ldots (4.41)$$

and

$$\| z(t,u_0^1,z_0^1) - z(t,u_0^2,z_0^2) \| \le F_2F_5\|z_0^1 - z_0^2\| + F_5F_6 \| u(t,u_0^1,z_0^1) - u(t,u_0^2,z_0^2) \| \quad \ldots (4.42)$$

Substitutes (4.41) in (4.42) and (4.42) in (4.41), we obtained that

$$\| u(t,u_0^1,z_0^1) - u(t,u_0^2,z_0^2) \| \le F_1F_3F_7\|u_0^1 - u_0^2\| + F_2F_3F_4F_5F_7\|z_0^1 - z_0^2\| \quad \ldots (4.43)$$

and

$$\| z(t,u_0^1,z_0^1) - z(t,u_0^2,z_0^2) \| \le F_1F_3F_5F_6F_7\|u_0^1 - u_0^2\| + F_2F_5F_7\|z_0^1 - z_0^2\| \quad \ldots (4.44)$$

Also, Substitutes (4.43) and (4.44) in (4.35) and (4.36), we get (4.32).

Remark 4.7. (Mitropolsky and Martynyuk, 1979). Theorem 4.6 confirms the stability of the solution of (1.1) and (1.2), that is when a few change happens in the points u_0, z_0, then a few change will happen in the functions $\Delta_1(0,u_0,z_0)$ and $\Delta_2(0,u_0,z_0)$.

5. EXAMPLE

Consider the following system of fractional intrgro-differential equation.

$$\left.\begin{array}{c} {}^cD_{0+}^{0.3}u(t) = \dfrac{1}{100\beta(\tau,\alpha)} \\ \left(e^t + \sin(u(t)) + \displaystyle\int_0^t (t+s)^{\frac{1}{2}}(u(s)-z(s))ds\right) \\ t\in[0,2], t\neq 1 \\ {}^cD_{0+}^{0.6}z(t) = \dfrac{1}{2\beta(\tau,\gamma)} \\ \left(\dfrac{2}{15} + \dfrac{|z(t)|}{9e^t(1+|z(t)|)} + \dfrac{1}{50}\displaystyle\int_0^{t^2} (t+s)^{\frac{1}{3}}(u(s)-z(s))ds\right) \\ t\in[0,2], t\neq 1 \\ \Delta u|_{t=1} = \dfrac{u(t)}{\beta(\tau,\alpha)(4+u(t))}, \\ \Delta z|_{t=1} = \dfrac{z(t)}{\beta(\tau,\gamma)(5+z(t))} \end{array}\right\} \ldots (5.1)$$

with non-separated integral coupled boundary condition

$$\left.\begin{array}{l} 20u(0) + u(2) = \int_0^2 \frac{1}{5(1+|z(s)|)}ds, \\ 30z(0) + z(2) = \int_0^2 \frac{1}{(10+|u(s)|)}ds, \end{array}\right\} \ldots (5.2)$$

where

$$\beta(\tau,\alpha) = \int_0^{\infty} \frac{s^{\tau-1}}{(1+s)^{\tau+0.3}}ds \ , \ \beta(\tau,\gamma) = \int_0^{\infty} \frac{s^{\tau-1}}{(1+s)^{\tau+0.6}}ds$$

means that $M_{\beta\alpha} = \max_{\tau\in(0,2]} \frac{1}{\beta(\tau,\alpha)} = 0.39$, $M_{\beta\gamma} = \max_{\tau\in(0,2]} \frac{1}{\beta(\tau,\gamma)} = 0.96$

Here $T = 2$, $\alpha = 0.3$, $\gamma = 0.6$, $A = 20$, $B = 1$, $C = 30$, $D = 1$, $a(t) = t$, $b(t) = t^2$,

$$K(t,s) = (t+s)^{\frac{1}{2}},\ R(t,s) = (t+s)^{\frac{1}{3}},$$

$$h_1(z(t)) = \frac{1}{5(1+|z(t)|)},\ h_2(u(t)) = \frac{1}{(10+|u(s)|)}$$

we obtain that

$K_s = 5.3333$, $R_s = 12$, $K_1 = K_2 = 0.01$, $K_3 = 0.0625$ $L_1 = 0.0556$, $L_2 = 0.01$, $L_3 = 0.04$, $p_0 = p_1 = 0.2$, $q_0 = 0.1$, $q_1 = 0.01$, $S_1 = S_2 = 1$

Therefore

$$\Lambda = \begin{pmatrix} 0.0935 & 0.4376 \\ 0.3701 & 0.3257 \end{pmatrix},\ \text{hence } \lambda_{\max}(\Lambda) = 0.6284 < 1$$

Therefore, by Theorem 4.1 and Theorem 4.2, the coupled boundary value problem (5.1) and (5.2) has exactly one periodic solution.

6. CONCLUSION

In this article, we studied the existence, uniqueness and approximated of periodic solutions of nonlinear impulsive fractional integro-differential system (1.1) with non-separated integral coupled boundary conditions (1.2). Also, we create a sequence of functions that it proved to be uniformly convergent and satisfied the fractional integro-differential equations and periodic conditions that is the main idea of the Numerical-analytic technique. Finally, an example of fractional system is stated.

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