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ON CUBIC INTERVAL VAGUE SETS AND CUBIC INTERVAL VAGUE TOPOLOGICAL SPACES

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ABSTRACT:
We advance a modern extension formation of cubic vague sets identified as cubic interval vague sets (CIVSs). *We* similarly describe the idea of *internal* cubic interval vague sets (ICIVSs) *and external* cubic interval vague sets (ECIVSs) by examples *and* debate their exciting properties, excluding ICIVSs *and* ECIVSs *under both P and R-Order*. Additionally, *we* show *that the P and R-intersection of* ICIVSs (*or* ECIVSs) necessity not be an ICIVS (or ECIVS). *We* apply the idea of (CIVS) to topological spaces and present the idea of cubic interval vague topological spaces. Further, we introduce P-cubic interval vague topological space and R-cubic interval vague topological space. We discuss basic properties of cubic interval vague topological space.

KEYWORDS: Fuzzy Set, Cubic Vague Set, Cubic Interval Vague Set, Internal Cubic Interval Vague Set, External Interval Cubic Vague Set, Cubic Interval Vague Topological Space.

1. INTRODUCTION

The concept of fuzzy sets presented in *Zadeh in* [5] rested the foundation for the advance of fuzzy Mathematics. This idea has an extensive span of application in numerous twigs of Mathematics for example group theory, logic, measure theory, set theory, group theory, semi-group theory, real analysis, measure theory and topology. As a generalization of fuzzy set, Zadeh in [6] presented the idea of interval valued fuzzy set. Next that, many authors considered the same topic and found some expressive conclusions. Jun *et al in* [3] provided the cubic set (CS) concept and because fuzzy set transactions by *single value membership* and *interval valued fuzzy set* ranges the *membership in the formula of intervals, it was* described with *interval valued fuzzy set and fuzzy set*, that is *a* broader way to detention ambiguity. Jun *et al in* [3], by means of a fuzzy set and an interval valued fuzzy set, they presented a modern idea termed a (*internal, external*) cubic set and evaluated numerous properties. They have also investigated several properties related to P-union, P-intersection, R-union and R-intersection of cubic sets. Akhtar et *al in* [1] introduced the notion of cubic sets applied to topological space and defined P-cubic topological space and R-cubic topological space. [2] The vague set is an extension of fuzzy sets and considered as a specific case of context supported fuzzy set which has the capability to beat the problems confronted after workings fuzzy sets by supply us with an interval-based membership which obviously divides the directory for and versus an element. [4] The cubic vague set (CVS) is a cubic set (CS) generalization by merging the idea of *vague set* to the cubic set meaning. Therefore, the CVS allows the aptitude to deeling with doubts, inaccurate and mysterious info looking at jointly the values of *truth membership and falsity membership*, while cubic set ability only dealing with doubts info without regard the values of truth membership and falsity membership. This is already demonstrating the essential feature of cubic vague set (CVS) opposite that of cubic set (CS). Zadeh in [6], introduced the concept of the interval valued fuzzy set as a generalization of fuzzy set. Another generalization of fuzzy set known as vague set is studied by Gau in [2]. Cubic set was a combination of both fuzzy set and interval valued fuzzy set which was introduced by Jun in [3]. Cubic vague set was a combination of both cubic set and vague set which was introduced by Jun in [4]. Our research flow is as shadows. Initially, *we* study the idea of cubic interval vague set (CIVS), that is a cubic vague set extension. Secondly, *we* state some ideas correlated to the conception of CIVS in addition to some fundamental operations is called *internal cubic interval vague sets* (ICIVSs) and *external cubic interval vague sets* (ECIVSs). Finally, we introduce derived, interior and closure called P-cubic interval vague interior, P-cubic interval vague closure and P-cubic interval vague derived are introduced in cubic interval vague topological spaces.

2. PRELIMINARIES

Definition 2.1. [2] *A vague set A* (VS) in the discourse universe U is a considered through two functions of membership assumed by:

1. Function of truth membership

$$t_A : U \rightarrow [0,1],$$

and

2. Function of false membership

$$f_A : U \rightarrow [0,1],$$

where $t_A(u)$ *is a lower bound of the grade of membership of u derived from the* "evidence for u" *and* $f_A(u)$ *is a lower bound of the negation of u derived from the* "evidence against u" *and*

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$$t_A(u) + f_A(u) \leq 1.$$

So, membership grade of u in the $vague\ set$ A is limited through a $sub\ interval$ $[t_A(u), 1 - f_A(u)]$ of $[0,1]$. This specifies that if the real membership grade is $\mu(u)$, as a consequence

$$t_A(u) \leq \mu(u) \leq 1 - f_A(u).$$

The vague set A is written as

$$A = \{(u, [t_A(u), 1 - f_A(u)])|u \in U\},$$

where the interval $[t_A(u), 1 - f_A(u)]$ *is called* "*vague value*" *of* u *in* A *and denoted by* $V_A(u)$.

Definition 2.2. [3] Suppose that X is to be a non-empty set. A structure

$$A = \{ x, < A(x), \lambda(x) > |x \in X \},$$

is a cubic set in X *in which* A *is an interval value fuzzy set and* λ *is a fuzzy set in* X.

Definition 2.3. [6] Assume that X is a set of universal. And A^V is a cubic vague set demarcated more the universal set X is an ordered pair that is demarcated as shadows:

$$A^V = \{< (x), A_V(x), \lambda_V(x) >: x \in X\},$$

where $A_V =< t_{A_V}, 1 - f_{A_V} > = \{< x, [t^-_{A_V}(x), t^+_{A_V}(x)], [1 - f^+_{A_V}(x), 1 - f^-_{A_V}(x)] >: x \in X, v \in V\}$ represents IVVS defined on X while $\lambda_V = \{< x, t_{\lambda V}, 1 - f_{\lambda V} >\}$ represent VS defined on X such that $t^+_{AV}(x) + f^+_{AV}(x) \leq 1$ and $t^+_{\lambda V}(x) + f^+_{\lambda V}(x) \leq 1$. We denote the pairs as $A^V = < A_V, \lambda_V >$, where $A_V =< [t^-_{AV}(x), t^+_{AV}(x)], [1 - f^+_{AV}(x), 1 - f^-_{AV}(x)] >$ *and* $\lambda_V = (t_{\lambda V}(x), 1 - f_{\lambda V}(x))$. C^X_V *Denoted the sets of all cubic vague sets in X.*

3. CUBIC INTERVAL VAGUE SET

In this part of research, *we* present the cubic interval vague set (CIVS) idea and *internal/external cubic interval vague sets.*

Definition 3.1. *Let X be a universal set. A cubic interval vague set* A^V *defined over the universal set X is an ordered pair which is defined as follows*:

$$A^V = \{< x, A_V(x), B_V(x) > |x \in X\},$$

where

$$A_V =< T_{A_V}, 1 - F_{A_V} > = \{< x, [T^-_{A_V}(x), T^+_{A_V}(x)], [1 - F^+_{A_V}(x), 1 - F^-_{A_V}(x)] >: x \in X, v \in V\},$$

and

$$B_V =< T_{B_V}, 1 - F_{B_V} > = \{< x, [T^-_{B_V}(x), T^+_{B_V}(x)], [1 - F^+_{B_V}(x), 1 - F^-_{B_V}(x)] >: x \in X, v \in V\},$$

represent IVVSs defined on X such that $T^+_{A_V}(x) + F^+_{A_V}(x) \leq 1$ and $T^+_{B_V}(x) + F^+_{B_V}(x) \leq 1$. *We* represent the *pairs as* $A^V =< A_V, B_V >$, *where* $A_V =< [T^-_{A_V}(x), T^+_{A_V}(x)], [1 - F^+_{A_V}(x), 1 - F^-_{A_V}(x)] >$ *and* $B_V =< [T^-_{B_V}(x), T^+_{B_V}(x)], [1 - F^+_{B_V}(x), 1 - F^-_{B_V}(x)] >$. CI^X_V denoted the sets of all cubic interval vague sets in *X*.

Example 3.1. Let $X = \{a, b\}$ *be* a universe *set*. Suppose a IVVSs A_V *and* B_V in *X* are defined by:

$$A_V = \left\{\frac{<[0.1,0.3],[0.3,0.7]>}{a}, \frac{<[0.3,0.4],[0.5,0.6]>}{b}\right\},$$

and

$$B_V = \left\{\frac{<[0.2,0.4],[0.6,0.9]>}{a}, \frac{<[0.1,0.6],[0.6,0.8]>}{b}\right\}.$$

Then the *cubic interval vague set* $A^V = < A_V, B_V >$ is shows by the *tabular* illustration *as in Table* 1.

Table 1: *cubic interval vague set* $A^V = < A_V, B_V >$

X	A_V	B_V
a	< [0.1,0.3], [0.3,0.7] >	< [0.2,0.4], [0.6,0.9] >
b	< [0.3,0.4], [0.5,0.6] >	< [0.1,0.6], [0.6,0.8] >

For each single value in the intervals of B_V, we get a cubic vague set [1] as it is seen in Table 2.

Table 2: cubic vague set $A^V = < A_V, \lambda_V >$

X	A_V	λ_V
a	< [0.1,0.3], [0.3,0.7] >	< [0.2,0.6] >
b	< [0.3,0.4], [0.5,0.6] >	< [0.1,0.7] >

Definition 3.2. *Let X be a universal set and V be a nonempty vague set. A cubic interval vague set* $A^V = < A_V, B_V >$ is named *an internal cubic interval vague set* (Brief. ICIVS) if $B_V(x) \subseteq (A^-_V(x), A^+_V(x))$ for all $x \in X$.

Example 3.2. Let $A^V = \{< (x), A_V(x), B_V(x) >: x \in X\}$ be cubic interval vague set and $A_V(x) = [0.2,0.4], [0.1,0.5]$ *and* $B_V(x) = [0.2,0.3], [0.4,0.5]$ for all $x \in X$, then A^V is an ICIVS.

Definition 3.3. *Let X be a universal set and V be a nonempty vague set. A cubic interval vague set* $A^V = < A_V, B_V >$ is named *an* ***e****xternal cubic interval vague set* (Brief. ECIVS) if $B_V(x) \nsubseteq (A^-_V(x), A^+_V(x))$ for all $x \in X$.

Example 3.3. Let A^V be cubic interval vague set and $A_V(x) = [0.2,0.3], [0.4,0.5]$ *and* $B_V(x) = [0.2,0.4], [0.1,0.5]$ for all $x \in X$, then A^V is an ECIVS.

Theorem 3.1. Let $A^V = < A_V, B_V >$ be *a* CIVS in X which is not an ECIVS. Then $\exists\ x \in X\ s.t\ B_V(x) \subseteq (A^-_V(x), A^+_V(x))$.

Proof. By definition of ECIVS, $B_V(x) \nsubseteq A_V(x)$ for all $x \in X$, if not ECIVS then $\exists\ x \in X\ s.t\ B_V(x) \subseteq A_V(x)$.

Definition 3.4. Let $A^V = < A_V, D_V >$ and $B^V = < B_V, O_V >$ *be two cubic interval vague sets in X. Then we have*

1. *(Equality)* $A^V = B^V \Leftrightarrow A_V(x) = B_V(x)$ *and* $D_V(x) = O_V(x)$.
2. *(P-order)* $A^V \subseteq_P B^V \Leftrightarrow A_V(x) \subseteq B_V(x)$ *and* $D_V(x) \subseteq O_V(x)$.
3. *(R-order)* $A^V \subseteq_R B^V \Leftrightarrow A_V(x) \subseteq B_V(x)$ *and* $D_V(x) \supseteq O_V(x)$.

Example 3.4. Let $A^V = < A_V, D_V >$ and $B^V = < B_V, O_V >$ be two cubic interval vague sets in *X*.

(1) Let $A_V(x) = [0.2,0.3], [0.4,0.5]$ & $D_V = [0.2,0.4], [0.1,0.5]$ & $B_V(x) = [0.2,0.3], [0.4,0.5]$ & $O_V = [0.2,0.4], [0.1,0.5]$, since $A_V = B_V$ *and* $D_V = O_V$ this implies that $A^V = B^V$ (Equality).

(2) Let $A_V(x) = [0.2,0.3], [0.4,0.5]$ & $D_V = [0.2,0.4], [0.1,0.5]$ & $B_V(x) = [0.1,0.3], [0.4,0.6]$ & $O_V = [0.2,0.5], [0.1,0.7]$, since $A_V \subseteq B_V$ *and* $D_V \subseteq O_V$ this implies that $A^V \subseteq_P B^V$ (P-order).

(3) Let $A_V(x) = [0.2,0.3],[0.4,0.5]$ & $D_V = [0.2,0.4],[0.1,0.5]$ & $B_V(x) = [0.1,0.3],[0.4,0.6]$ & $O_V = [0.2,0.3],[0.4,0.5]$, since $A_V \subseteq B_V$ *and* $D_V \supseteq O_V$ this implies that $A^V \subseteq_R B^V$ (R-order).

Definition 3.5. Let $A^V = < A_V, D_V >$ and $B^V = < B_V, O_V >$ *be two cubic interval vague sets in* X. *Then we have*

1. $A^V \cup_P B^V$={$< (x), \sup(A_V(x), B_V(x))$, $\sup(D_V(x), O_V(x)) >: x \in X$} (P-union).
2. $A^V \cap_P B^V$={$< (x), \inf(A_V(x), B_V(x))$, $\inf(D_V(x), O_V(x)) >: x \in X$} (P-intersection).
3. $A^V \cup_R B^V$={$< (x), \sup(A_V(x), B_V(x))$, $\inf(D_V(x), O_V(x)) >: x \in X$} (R-union).
4. $A^V \cap_P B^V$={$< (x), \inf(A_V(x), B_V(x))$, $\sup(D_V(x), O_V(x)) >: x \in X$} (R-intersection).

Example 3.5. Let $A^V = < , A_V, D_V >$ and $B^V = < B_V, O_V >$ *be two cubic interval vague sets in* X *and* V, and let $A_V(x) = [0.2,0.3],[0.4,0.5]$ & $D_V(x) = [0.2,0.4],[0.1,0.5]$ & $B_V(x) = [0.1,0.3],[0.4,0.6]$ & $O_V(x) = [0.2,0.3],[0.4,0.5]$ then

(1). $A^V \cup_P B^V = \{< (x), B_V(x), D_V(x) >\}$.
(2). $A^V \cap_P B^V = \{< (x), A_V(x), O_V(x) >\}$.
(3). $A^V \cup_R B^V = \{< (x), B_V(x), O_V(x) >\}$.
(4). $A^V \cap_R B^V = \{< (x), A_V(x), D_V(x) >\}$.

Definition 3.6. Let $A^V = < A_V, B_V >$ *be a cubic interval vague set*. We let that $\mathbf{0} = [0,0], \mathbf{1} = [1,1]$ and we denote that $\ddot{0} =< \mathbf{0}, \mathbf{0} >$ & $\ddot{1} =< \mathbf{1}, \mathbf{1} >$. We have

1. If $A_V(x) = \ddot{0}, B_V(x) = \ddot{0}$ for all $x \in X$ is denoted by $\hat{0}$.
2. If $A_V(x) = \ddot{0}, B_V(x) = \ddot{1}$ for all $x \in X$ is denoted by $\hat{0}$.
3. If $A_V(x) = \ddot{1}, B_V(x) = \ddot{0}$ for all $x \in X$ is denoted by $\hat{1}$.
4. If $A_V(x) = \ddot{1}, B_V(x) = \ddot{1}$ for all $x \in X$ is denoted by $\hat{1}$.

Definition 3.7. Let $A^V = < A_V, B_V >$ be a cubic interval vague set $(A^V)^c = \{< (x), A_V^c(x), B_V^c(x) >: x \in X\}$, where $A_V^c(x) = [1 - A_V^+(x), 1 - A_V^-(x)]$ *and* $B_V^c(x) = [1 - B_V^+(x), 1 - B_V^-(x)]$.

Theorem 3.2. Let $A^V = \{< (x), A_V(x), B_V(x) >: x \in X\}$ *be* a CIVS *in* X. *If* A^V is ICIVS (*resp*. ECIVS), *then* A^{V^c} *is also an* ICIVS (*resp*. ECIVS).

Proof. *Since* $A^V = \{< (x), A_V(x), B_V(x) >\}$ *is also an* ICIVS (*resp*. ECIVS) *in* X, *we have* $B_V(x) \subseteq A_V(x)$ (*resp*. $B_V(x) \nsubseteq A_V(x)$) $\forall\, x \in X$. *That means*

$$\left(1 - B_V(x)\right) \subseteq (1 - A_V^+(x), 1 - A_V^-(x))$$

(resp. $(1 - B_V(x)) \nsubseteq (1 - A_V^+(x), 1 - A_V^-(x))$) $\forall\, x \in X$.
Thus $A^{V^c} = \{< (x), A_V^c(x), B_V^c(x) >: x \in X, v \in V\}$ *is an* ICIVS (resp. ECIVS) in X.

Theorem 3.3. Assume that $A_i^V =< A_{iV}, B_{iV} | i \in A >$ is a ICIVSs group *in* X. Then the P-union and P-intersection *of* $A_i^V =< A_{iV}, B_{iV} | i \in A >$ *are* ICIVSs in X.

Proof. *Since* A_i^V is an ICIVS in X, *we have*

$$B_{iV}(x) \subseteq (A_{iV}^-(x), A_{iV}^+)\ for\ all\ i \in A.$$

That *means*

$$(\cup_{i\in A} B_{iV})(x) \subseteq \left((\cup_{i\in A} A_{iV})^-(x), (\cup_{i\in A} A_{iV})^+(x)\right),$$

and

$$(\cap_{i\in A} B_{iV})(x) \subseteq ((\cap_{i\in A} A_{iV})^-(x), (\cap_{i\in A} A_{iV})^+(x)).$$

Hence $\cup_p A_i^V$ *and* $\cap_p A_i^V$, $i \in A$ *are ICIVSs in* X.

The next example illustrations that the P-*union* and P-*intersection of two* ECIVSs necessity *not be an* ECIVS.

Example 3.6. Assume that $A^V = < A_V, D_V >$ and $B^V = < B_V, O_V >$ are two ECIVSs in X and V such that $A_V(x) = [0.2,0.3],[0.4,0.6]$, $D_V(x) = [0.7,0.7],[0.8,0.8]$, $B_V(x) = [0.7,0.7],[0.8,0.8]$ and $O_V(x) = [0.2,0.3],[0.4,0.6]$, $\forall x \in X$.

1. Note that $A^V \cup_P B^V = \{< (x), B_V(x), D_V(x) | x \in X >\}$ and $D_V(x) \subseteq B_V(x)\ \forall x \in X$ .Then $A^V \cup_P B^V$ is *not an* ECIVS *in* X.
2. *Note that* $A^V \cap_P B^V = \{< (x), A_V(x), O_V(x) | x \in X >\}$ and $O_V(x) \subseteq A_V(x)\ \forall x \in X$ .Then $A^V \cap_P B^V$ is not an ECIVS in X.

The next example illustrations that the R-*union* and R-*intersection of two* ICIVSs necessity *not be an* ICIVS.

Example 3.7. Assume that $A^V = < A_V, D_V >$ and $B^V = < B_V, O_V >$ are two ICIVSs in $X = [0,1] \times [0,1]$ in which that $A_V(x) = [0.2,0.4],[0.3,0.6]$ and $D_V(x) = [0.3,0.4],[0.4,0.5]$, $B_V(x) = [0.6,0.8],[0.7,0.9]$ and $O_V(x) = [0.7,0.7],[0.8,0.8]\ \forall x \in X$.

1. Note that $A^V \cup_R B^V = \{< (x), B_V(x), D_V(x) | x \in X >\}$ and $D_V(x) \nsubseteq B_V(x)\ \forall x \in X$ .Then $A^V \cup_R B^V$ is not an ICIVS in X.

2. Note that $A^V \cap_R B^V = \{< (x), A_V(x), O_V(x) | x \in X >\}$ and $O_V(x) \nsubseteq A_V(x)\ \forall x \in X$.*Then* $A^V \cap_R B^V$ is *not an* ICIVS *in* X.

The next example illustrations that the R-*union* and R-*intersection of two* ECIVSs necessity *not be an* ECIVS.

Example 3.8. Let $A^V = < A_V, D_V >$ and $B^V = < B_V, O_V >$ are two ECIVSs in $X = [0,1] \times [0,1]$ in which that $A_V(x) = [0.2,0.4],[0.3,0.5]$ and $D_V(x) = [0.1,0.1],[0.2,0.2]$, $B_V(x) = [0.3,0.6],[0.4,0.7]$ and $O_V(x) = [0.2,0.2],[0.3,0.3]\ \forall x \in X$.

1. Note that $A^V \cup_R B^V = \{< (x), B_V(x), D_V(x) | x \in X >\}$ and $D_V(x) \subseteq B_V(x)\ \forall x \in X$ .Then $A^V \cup_R B^V$ is not an ECIVS in X.

2. Note that $A^V \cap_R B^V = \{< (x), A_V(x), O_V(x) | x \in X >\}$ and $O_V(x) \subseteq A_V(x)\ \forall x \in X$ .Then $A^V \cap_R B^V$ is not an ECIVS in X.

4. CUBIC INTERVAL VAGUE TOPOLOGICAL SPACES

Definition 4.1. A *P-cubic interval vague topology* is the cubic interval vague sets family I_p *in* X that holds the subsequent conditions;

1. $\hat{0}, \hat{1} \in I_p$.
2. Let $A_i^V \in I_p$, then $\cup_P A_i^V \in I_p, i \in N$.
3. Let $A^V, B^V \in I_p$, then $A^V \cap_P B^V \in I_p$.

The pair (X, I_p) is called a *P-cubic interval vague topological space.*

Example 4.1. Assume that $X = \{a, b\}$ *and* I_p *are* the collection of cubic interval vague sets in X i.e.
$I_p = \{\hat{0}, \hat{1}, \{< [0.1,0.3],[0.2,0.4] >, < [0.3,0.6],[0.5,0.6] >\}, \{< [0.3,0.5],[0.4,0.5], [0.4,0.7],[0.5,0.7] >\}\}$.
Then I_p is *P-cubic interval vague topology* on X.

Definition 4.2. P-discrete cubic interval vague topology is the collection of all cubic interval vague sets *in* X i.e.
$\boldsymbol{V^X}$ Denotes P-discrete cubic interval vague topology in X.

Example 4.2. Let $X \neq \hat{0}$, and $I_P = \{V^X\}$ where V^X all cubic interval vague subsets of X, Then I_P is obviously a *P-cubic interval vague topology* on X, which is the largest *P-cubic interval vague topology* on X. This *P-cubic interval vague topology* is *P-discrete cubic interval vague topology*.

Definition 4.3. *P-indiscrete cubic interval vague topology* is the collection of cubic interval vague sets $\hat{0}$ and $\hat{1}$ only.

Example 4.3. Let $X \neq \hat{0}$, and $I_P = \{\hat{0}, \hat{1}\}$ be the collection of cubic interval vague sets, then I_P is obviously a *P-cubic interval vague topology* on X, which is the smallest *P-cubic interval vague topology* on X. This *P-cubic interval vague topology* is *P-indiscrete cubic interval vague topology*.

Definition 4.4. The members of *P-cubic interval vague topology* (I_P) are called *P-cubic interval vague open sets* in (X, I_P).

Example 4.4. Let $X = \{a, b\}$, and $I_p = \{\hat{0}, \hat{1}, \{<[0.1,0.3],[0.2,0.4]>, <[0.3,0.6],[0.5,0.6]>\}, \{<[0.3,0.5],[0.4,0.5]\ \ , [0.4,0.7],[0.5,0.7]>\}\}$ be a *P-cubic interval vague topology* on X, then $\hat{0}, \hat{1}, \{<[0.1,0.3],[0.2,0.4]>, <[0.3,0.6],[0.5,0.6]>\}, \{<[0.3,0.5],[0.4,0.5]\ \ and\ [0.4,0.7],[0.5,0.7]>\}$ are *P-cubic interval vague open sets* in (X, I_P).

Definition 4.5. The complement of *P-cubic interval vague open set* is called *P-cubic interval vague closed set* in (X, I_P).

Example 4.5. Let $X = \{a, b\}$, and $I_p = \{\hat{0}, \hat{1}, \{<[0.1,0.3],[0.2,0.4]>, <[0.3,0.6],[0.5,0.6]>\}, \{<[0.3,0.5],[0.4,0.5]\ \ , [0.4,0.7],[0.5,0.7]>\}\}$ be a *P-cubic interval vague topology* on X, then $\hat{0}, \hat{1}, \{<[0.6,0.8],[0.7,0.9]>, <[0.4,0.5],[0.4,0.7]>\}, \{<[0.5,0.6],[0.5,0.7]\ \ , [0.3,0.5],[0.3,0.6]>\}$ are *P-cubic interval vague closed sets* in (X, I_P).

Proposition 4.1. If (X, I_P) is any *P-cubic interval vague topological space*, then

1. $\hat{0}$ and $\hat{1}$ are *P-cubic interval vague closed sets.*
2. The P-intersection of any (finite or infinite) number of *P-cubic interval vague closed sets* is a *P-cubic interval vague closed set.*
3. The P-union of any finite *P-cubic interval vague closed sets is a P-cubic interval vague closed set.*

Definition 4.6. Those cubic interval vague sets which are both *P-cubic interval vague open* and *P-cubic interval vague closed* are called *P-cubic interval vague clopen sets* in (X, I_P).

Example 4.6. Let $X = \{a, b\}$, and $I_P = \{\hat{0}, \hat{1}, \{<[0.4,0.6],[0.5,0.5]>, <[0.3,0.7],[0.2,0.8]>\}\}$ be a *P-cubic interval vague topology* on X, then $\hat{0}, \hat{1}, \{<[0.4,0.6],[0.5,0.5]>, <[0.3,0.7],[0.2,0.8]>\}$ are *P-cubic interval vague clopen sets* in (X, I_P).

Definition 4.7. A cubic interval vague set A^V $in\ (X, I_P)\ is$ a *P-cubic interval vague neighbourhood* $of\ a\ cubic\ interval\ vague\ set\ B^V$ if $there\ exists\ a$ P-$cubic\ interval\ vague\ open\ set\ G_P$ in (X, I_P) such that $B^V \subseteq_P G_P \subseteq_P A^V$.

Remark 4.1. For any two cubic interval vague sets $A^V = <A_V, D_V>$ and $B^V = <B_V, O_V>$, we have

$i. (A^V \cup_P B^V)^c = (A^V)^c \cap_P (B^V)^c$.
$ii. (A^V \cap_P B^V)^c = (A^V)^c \cup_P (B^V)^c$.

Proof. (***i***) Since $A^V \subseteq_P (A^V \cup_P B^V)$ & $B^V \subseteq_P (A^V \cup_P B^V)$ this implies that
$(A^V \cup_P B^V)^c \subseteq_P (A^V)^c$ & $(A^V \cup_P B^V)^c \subseteq_P (B^V)^c \implies$
$(A^V \cup_P B^V)^c \subseteq_P (A^V)^c \cap_P (B^V)^c \quad \rightarrow (1)$
$(A^V)^c \cap_P (B^V)^c \subseteq_P (A^V)^c$ & $(A^V)^c \cap_P (B^V)^c \subseteq_P (B^V)^c$ $and\ hence$
$A^V \subseteq_P ((A^V)^c \cap_P (B^V)^c)^c$ & $B^V \subseteq_P ((A^V)^c \cap_P (B^V)^c)^c$ $this\ implies\ that$
$(A^V \cup_P B^V) \subseteq_P ((A^V)^c \cap_P (B^V)^c)^c \implies$
$(A^V)^c \cap_P (B^V)^c \subseteq_P (A^V \cup_P B^V)^c \rightarrow (2)$.
From (1) & (2) we get $(A^V \cup_P B^V)^c = (A^V)^c \cap_P (B^V)^c$.

Proof. (ii) Since $(A^V \cap_P B^V) \subseteq_P A^V$ & $(A^V \cap_P B^V) \subseteq_P B^V \implies$
$(A^V)^c \subseteq_P (A^V \cap_P B^V)^c$ & $(B^V)^c \subseteq_P (A^V \cap_P B^V)^c$ $and\ therefore$
$(A^V)^c \cup_P (B^V)^c \subseteq_P (A^V \cap_P B^V)^c \quad \longrightarrow (1)$.
$(A^V)^c \subseteq_P (A^V)^c \cup_P (B^V)^c$ & $(B^V)^c \subseteq_P (A^V)^c \cup_P (B^V)^c$ $this\ implies\ that$
$((A^V)^c \cup_P (B^V)^c)^c \subseteq_P A^V$ & $((A^V)^c \cup_P (B^V)^c)^c \subseteq_P B^V$ $and\ therefore$
$((A^V)^c \cup_P (B^V)^c)^c \subseteq_P (A^V \cap_P B^V) \implies$
$(A^V \cap_P B^V)^c \subseteq_P (A^V)^c \cup_P (B^V)^c \longrightarrow (2)$.
From (1) & (2) we get $(A^V \cap_P B^V)^c = (A^V)^c \cup_P (B^V)^c$.

Definition 4.8. Let X be a set of elements and $A^V \in CI_V^x$**,** a cubic interval vague point is a cubic interval vague set in X defined by $\lambda_x > \hat{0}$ $at\ x \in X,\ and\ otherwise\ is\ zero.$ This cubic interval vague point is denoted by λ_x.

Definition 4.9. For $A^V \in CI_V^x$, a cubic interval vague point λ_x is said to belong to a cubic interval vague set A^V $(denoted\ by\ \lambda_x \in A^V)$ if $T_{V\lambda}^+(x) = T_{VA}^+(x), T_{V\lambda}^-(x) = T_{VA}^-(x), F_{V\lambda}^+(x) = F_{VA}^+(x)\ and\ F_{V\lambda}^-(x) = F_{VA}^-(x)\ \ at\ x \in X.$

Definition 4.10. Let X be a set of elements and let $A^V = <A_V, B_V>$ be a cubic interval vague set, a two cubic interval vague points $£_y, \lambda_x$ $such\ that\ \lambda_x \in A^V$ & $£_y \notin A^V$ are said to be a limit cubic interval vague points of A^V if for each *P-cubic interval vague open sets* G_P, H_P containing $\lambda_x, £_y$ respectively such that

$$A^V \cap_P (G_P \backslash \{\lambda_x\}) \neq \hat{0}\ \ and\ \ (A^V \cap_P H_P) \neq \hat{0}.$$

And the set of all limit cubic interval vague point of A^V denoted by $d_P(A^V)$ and called P-cubic interval vague derived set of A^V.

Example 4.7. Let $X = \{x, y\}$ and $I_P = \{\hat{0}, \hat{1}, A_1^V, A_2^V\}$ be a P-cubic interval vague topology on X where $A_1^V = \{<[0.2,0.4],[0.3.0.4]>, <[0.1,0.3],[0.5,0.6]>\}$ and $A_2^V = \{<[0.3,0.5],[0.4,0.6]>, <[0.3,0.5],[0.6,0.8]>\}$ then (X, I_P) is a P-cubic interval vague topological space. Let $A^V = \{<[0.1,0.2],[0.2,0.2]>, <[0.2,0.3],[0.3,0.3]>\}$, now
Let $\lambda_{x1} = \{<[0.2,0.4],[0.3,0.4]>, <\ddot{0}>\}$, then $\lambda_{x1} \in \{\hat{1}, A_1^V\} \implies A^V \cap_P \hat{1} \backslash \{\lambda_{x1}\} \neq \hat{0}$ and $A^V \cap_P A_1^V \backslash \{\lambda_{x1}\} \neq \hat{0}$. This implies that $\lambda_{x1} \in d_P(A^V)$.
And let $\lambda_{x2} = \{<[0.2,0.4],[0.3.0.4]>, <\ddot{0}>\}$, then $\lambda_{x2} \in \{\hat{1}, A_1^V\} \implies A^V \cap_P \hat{1} \backslash \{\lambda_{x2}\} \neq \hat{0}$ and $A^V \cap_P A_1^V \backslash \{\lambda_{x2}\} \neq \hat{0}$ this implies that $\lambda_{x2} \in d_P(A^V)$.
And let $\lambda_{x3} = \{<[0.1,0.2],[0.2.0.2]>, <\ddot{0}>\}$, then $\lambda_{x3} \in \{\hat{1}\} \implies A^V \cap_P \hat{1} \backslash \{\lambda_{x3}\} \neq \hat{0}$ $this\ implies\ that\ \lambda_{x3} \in d_P(A^V)$.
And let $\lambda_{x4} = \{<\ddot{0}>, <[0.3,0.5],[0.6,0.8]>\}$, then $\lambda_{x4} \in \{\hat{1}, A_2^V\} \implies A^V \cap_P \hat{1} \backslash \{\lambda_{x4}\} \neq \hat{0}$ and $A^V \cap_P A_2^V \backslash \{\lambda_{x4}\} = \hat{0}$ this implies that $\lambda_{x4} \notin d_P(A^V)$. And so on therefore,
$d_P(A^V) = \hat{1} \backslash \{\lambda_{x4}\}$.

Definition 4.11. Let (X, I_P) be a cubic interval vague topological space and $A^V = \{<A_V, B_V>\}$ be a cubic interval vague set in X. Then *P-cubic interval vague interior* and *P-cubic interval vague closure* is described by

$int_P(A^V) = \cup_P \{B_P/B_P \text{ is a } P-\text{cubic interval vague open set in } X \text{ and } B_P \subseteq_P A^V\}$.

$cl_P(A^V) = \cap_P \{D_P/D_P \text{ is a } P-\text{cubic interval vague closed set in } X \text{ and } A^V \subseteq_P D_P\}$.

Example 4.8. Let $X = \{a, b\}$ and $I_P = \{\hat{0}, \hat{1}, A_1^V, A_2^V\}$ be a P-cubic interval vague topology on X where $A_1^V = \{<[0.2,0.4],[0.3.0.4]>, <[0.1,0.3],[0.5,0.6]>\}$ and $A_2^V = \{<[0.3,0.5],[0.4,0.6]>, <[0.2,0.5],[0.6,0.7]>\}$, then (X, I_P) is a P-cubic interval vague topological space. Now $(A_1^V)^c = \{<[0.6,0.8],[0.6,0.7]>, <[0.7,0.9],[0.4,0.5]>\}$ and $(A_2^V)^c = \{<[0.5,0.7],[0.4,0.6]>, <[0.5,0.8],[0.3,0.4]>\}$. Consider a cubic interval vague set $B^V = \{<[0.3,0.6],[0.4,0.5]>, <[0.2,0.7],[0.5,0.8]>\}$ and then
$int_P(B^V) = \hat{0} \cup_P A_1^V = A_1^V$ and
$cl_P(B^V) = \hat{1}$.

Proposition 4.2. Let $A^V = \{<A_V, D_V>$ and $B^V = \{<B_V, O_V>$ be any two cubic interval vague sets in (X, I_P). Then $A^V \subseteq_P B^V$ implies $d_P(A^V) \subseteq_P d_P(B^V)$.

Proof. Let $\lambda_x \in d_P(A^V)$. If $\lambda_x \notin d_P(B^V)$, then $\exists$ a P-cubic interval vague open set G_P *contanging* λ_x *such that* $(B^V \cap_P G_P \backslash \{\lambda_x\}) = \hat{0}$. As $A^V \subseteq_P B^V$ we get $(A^V \cap_P G_P \backslash \{\lambda_x\}) = \hat{0}$. Therefore $\lambda_x \notin d_P(A^V)$; which is a contradiction. Thus $\lambda_x \in d_P(A^V) \Longrightarrow \lambda_x \in d_P(B^V)$.
Hence $A^V \subseteq_P B^V \Longrightarrow d_P(A^V) \subseteq_P d_P(B^V)$.

Proposition 4.3. Let $A^V = \{<A_V, D_V>$ & $B^V = <B_V, O_V>$ be any two cubic interval vague sets in a P-cubic interval vague topological space (X, I_P). Then $d_P(A^V \cup_P B^V) = (d_P(A^V) \cup_P d_P(B^V))$.

Proof. Since $A^V \subseteq_P (A^V \cup_P B^V)$ & $B^V \subseteq_P (A^V \cup_P B^V)$, we have $d_P(A^V) \subseteq_P d_P(A^V \cup_P B^V)$ & $d_P(B^V) \subseteq_P d_P(A^V \cup_P B^V)$.
Hence $\left(d_P(A^V) \cup_P d_P(B^V)\right) \subseteq_P d_P(A^V \cup_P B^V) \to (i)$.
Let $\lambda_x \in d_P(A^V \cup_P B^V)$, if $\lambda_x \notin \left(d_P(A^V) \cup_P d_P(B^V)\right)$. Hence $\lambda_x \notin d_P(A^V)$ & $\lambda_x \notin d_P(B^V)$. If $\lambda_x \notin d_P(A^V) \Longrightarrow \exists\, G_P \in (X, I_P)$ *containing* λ_x *such that* $(A^V \cap_P G_P \backslash \{\lambda_x\}) = \hat{0}$. If $\lambda_x \notin d_P(B^V) \Longrightarrow \exists\, H_P \in (X, I_P)$ *containing* λ_x *such that* $(B^V \cap_P H_P \backslash \{\lambda_x\}) = \hat{0}$. G_P & $H_P \in (X, I_P)$ *and we get* $(A^V \cap_P (H_P \cap_P G_P) \backslash \{\lambda_x\}) = \hat{0}$ *and* $(B^V \cap_P (H_P \cap_P G_P) \backslash \{\lambda_x\}) = \hat{0}$. Combining both we get, $((A^V \cup_P B^V) \cap_P (H_P \cap_P G_P) \backslash \{\lambda_x\}) = \hat{0}$. As $(H_P \cap_P G_P) \in (X, I_P)$ and $\lambda_x \in (H_P \cap_P G_P)$, we get $\lambda_x \notin d_P(A^V \cup_P B^V)$; which is a contradiction. Thus $\lambda_x \in d_P(A^V \cup_P B^V) \Longrightarrow \lambda_x \in \left(d_P(A^V) \cup_P d_P(B^V)\right)$. Therefore $d_P(A^V \cup_P B^V) \subseteq_P \left(d_P(A^V) \cup_P d_P(B^V)\right) \to (ii)$. From (i) & (ii) we get $d_P(A^V \cup_P B^V) = (d_P(A^V) \cup_P d_P(B^V))$.

Proposition 4.4. Let $A^V = <A_V, D_V>$ and $B^V = <B_V, O_V>$ be any two cubic interval vague sets in (X, I_P). Then $d_P(A^V \cap_P B^V) \subseteq_P (d_P(A^V) \cap_P d_P(B^V))$.

Proof. Since $(A^V \cap_P B^V) \subseteq_P A^V$ & $(A^V \cap_P B^V) \subseteq_P B^V$, we have $d_P(A^V \cap_P B^V) \subseteq_P d_P(A^V)$ & $d_P(A^V \cap_P B^V) \subseteq_P d_P(B^V)$. Therefore, we get
$d_P(A^V \cap_P B^V) \subseteq_P (d_P(A^V) \cap_P d_P(B^V))$.

Proposition 4.5. Let (X, I_P) *be a P*-cubic interval vague topological space and $A^V = <A_V, B_V>$ *be a* cubic interval vague set in X. *Then* $int_P(A^V) \subseteq_P A^V$.

Proof. Since $int_P(A^V)$ is the largest P-cubic interval vague open set contained *in* A^V we have $int_P(A^V) \subseteq_P A^V$.

Proposition 4.6. For any P-cubic interval vague set A^V in (X, I_P). Then $int_P(A^V) = A^V$ *if and only if* A^V is a P-cubic interval vague open set.

Proof. Since $int_P(A^V)$ is the P-union of all P-cubic interval vague open sets contained in A^V and since A^V is a P-cubic interval vague open set, we have $A^V = int_P(A^V)$.
Obviously, A^V *is a P*-cubic interval vague open set whenever $int_P(A^V) = A^V$.

Proposition 4.7. Let (X, I_P) be a P-cubic interval vague topological space and $A^V = <A_V, B_V>$ be a cubic interval vague set in X. Then $A^V \subseteq_P cl_P(A^V)$.

Proof. Since $cl_P(A^V)$ is the smallest P-cubic interval vague closed set containing A^V we have $A^V \subseteq_P cl_P(A^V)$.

Proposition 4.8. For any P-cubic interval vague set B^V in (X, I_P). Then $cl_P(B^V) = B^V$ *if and only if* B^V *is* P-cubic interval vague closed set .

Proof. *Since* $cl_P(B^V)$ *is the P*-intersection *of* all *P*-cubic interval vague closed sets containing B^V and since B^V is a P-cubic interval vague closed set, we have $cl_P(B^V) = B^V$. Obviously, B^V *is a P*-cubic interval vague closed set whenever $cl_P(B^V) = B^V$.

Proposition 4.9. Let $A^V = \{<(x), A_V(x), D_V(x)>: x \in X\}$ and $B^V = <B_V, O_V>$ be any two cubic interval vague sets in (X, I_P). Then $A^V \subseteq_P B^V$ implies $int_P(A^V) \subseteq_P int_P(B^V)$.

Proof. Given $A^V \subseteq_P B^V$. Since $int_P(A^V) \subseteq_P A^V$, we have $int_P(A^V) \subseteq_P A^V \subseteq_P B^V$ which implies $int_P(A^V)$ is a P-cubic interval vague open set which contained in B^V. Since $int_P(B^V)$ is the largest P-cubic interval vague open set contained in B^V, we conclude that $int_P(A^V) \subseteq_P int_P(B^V)$.

Proposition 4.10. Let $A^V = <A_V, D_V>$ be *a* cubic interval vague set in (X, I_P). Then $int_P\left(int_P(A^V)\right) = int_P(A^V)$.

Proof. Since $int_P(A^V)$ is union of all P-cubic interval vague open sets contained in A^V, $int_P(A^V)$ is a P-cubic interval vague open set. Also, since $int_P(A^V)$ is the largest P-cubic interval vague open set contained in A^V, we have $int_P\left(int_P(A^V)\right) = int_P(A^V)$.

Proposition 4.11. Let $A^V = <A_V, D_V>$ and $B^V = \{<B_V, O_V>$ be any two cubic interval vague sets in (X, I_P). Then $(int_P(A^V) \cup_P int_P(B^V)) \subseteq_P int_P(A^V \cup_P B^V)$.

Proof. Since $A^V \subseteq_P (A^V \cup_P B^V)$ & $B^V \subseteq_P (A^V \cup_P B^V)$, we have $int_P(A^V) \subseteq_P int_P(A^V \cup_P B^V)$ & $int_P(B^V) \subseteq_P int_P(A^V \cup_P B^V)$.
Therefore $(int_P(A^V) \cup_P int_P(B^V)) \subseteq_P int_P(A^V \cup_P B^V)$.

The following example shows that the equality in the Proposition 4.11 is not true in general.

Example 4.9. Let $X = \{a, b\}$ and $I_P = \{\hat{0}, \hat{1}, A_1^V, A_2^V, A_3^V\}$ be a P-cubic interval vague topology on X where
$A_1^V = \{<\ddot{1}>, <[0.1,0.3],[0.5,0.6]>\}$,
$A_2^V = \{<[0.3,0.5],[0.4,0.6]>, <\ddot{1}>\}$ and
$A_3^V = \{<[0.3,0.5],[0.4,0.6]>, <[0.1,0.3],[0.5,0.6]>\}$.
Then (X, I_P) is a P-cubic interval vague topological space. Suppose that $A^V = \{<\ddot{0}>, <[0.1.0.4],[0.4,0.6]>\}$,
$B^V = \{<[0.3,0.6],[0.3,0.6]>, <\ddot{0}>\}$, $A^V \cup_P B^V =$

$\{< [0.3,0.6], [0.3,0.6] >, < [0.1,0.4], [0.4,0.6] >\}$ and then $int_P(A^V) = \hat{0}, int_P(B^V) = \hat{0}$ & $int_P(A^V \cup_P B^V) = A_3^V$. Thus $(int_P(A^V) \cup_P int_P(B^V)) \subseteq_P int_P(A^V \cup_P B^V)$.

Proposition 4.12. Let $A^V = < A_V, D_V >$ and $B^V = < B_V, O_V >$ be any two cubic interval vague sets in (X, I_P). Then $int_P(A^V \cap_P B^V) = (int_P(A^V) \cap_P int_P(B^V))$.

Proof. Since $(A^V \cap_P B^V) \subseteq_P A^V$ & $(A^V \cap_P B^V) \subseteq_P B^V$, we have $int_P(A^V \cap_P B^V) \subseteq_P int_P(A^V)$ & $int_P(A^V \cap_P B^V) \subseteq_P int_P(B^V)$. Therefore $int_P(A^V \cap_P B^V) \subseteq_P (int_P(A^V) \cap_P int_P(B^V)) \rightarrow (i)$.
Now, $int_P(A^V) \subseteq_P A^V$ & $int_P(B^V) \subseteq_P B^V$ which implies that $int_P(A^V) \cap_P int_P(B^V) \subseteq_P A^V \cap_P B^V$. Since $int_P(A^V) \cap_P int_P(B^V)$ is P-cubic interval vague open set, we have $int_P(A^V) \cap_P int_P(B^V) = int_P\big(int_P(A^V) \cap_P int_P(B^V)\big)$
$\subseteq_P int_P(A^V \cap_P B^V) \rightarrow (ii)$.
From (i) & (ii) implies that $int_P(A^V \cap_P B^V) = int_P(A^V) \cap_P int_P(B^V)$.

Proposition 4.13. Let (X, I_P) be a P-cubic interval vague topological space and $A^V = \{< (x), A_V(x), D_V(x) >: x \in X\}$ be a cubic interval vague set in X. Then $cl_P\big(cl_P(A^V)\big) = cl_P(A^V)$.

Proof. Since $cl_P(A^V)$ is intersection of all P-cubic interval vague closed sets containing A^V, $cl_P(A^V)$ is a P-cubic interval vague closed set. Also, since $cl_P(A^V)$ is the smallest P-cubic interval vague closed set containing A^V, we have $cl_P\big(cl_P(A^V)\big) = cl_P(A^V)$.

Proposition 4.14. Let $A^V = < A_V, D_V >$ and $B^V = < B_V, O_V >$ be any two cubic interval vague sets in (X, I_P).Then $A^V \subseteq_P B^V$ implies $cl_P(A^V) \subseteq_P cl_P(B^V)$.

Proof. Since $A^V \subseteq_P B^V$ and $B^V \subseteq_P cl_P(B^V)$, $A^V \subseteq_P B^V \subseteq_P cl_P(B^V)$. Since $cl_P(A^V)$ is the smallest P-cubic interval vague closed set containing A^V, we get $cl_P(A^V) \subseteq_P cl_P(B^V)$.

Proposition 4.15. Let $A^V = < A_V, D_V >$ and $B^V = < B_V, O_V >$ be any two cubic interval vague sets in a P-cubic interval vague topological space (X, I_P). Then $cl_P(A^V \cap_P B^V) \subseteq_P cl_P(A^V) \cap_P cl_P(B^V)$.

Proof. Since $(A^V \cap_P B^V) \subseteq_P A^V$ & $(A^V \cap_P B^V) \subseteq_P B^V$, we have $cl_P(A^V \cap_P B^V) \subseteq_P cl_P(A^V)$ & $cl_P(A^V \cap_P B^V) \subseteq_P cl_P(B^V)$. Hence $cl_P(A^V \cap_P B^V) \subseteq_P cl_P(A^V) \cap_P cl_P(B^V)$.

The following example shows that the equality in the Proposition 4.15 is not true in general.

Example 4.10. Let $X = \{a, b\}$ and $I_P = \{\hat{0}, \hat{1}, A_1^V, A_2^V, A_3^V\}$ be a P-cubic interval vague topology on X *where*
$A_1^V = \{< \ddot{1} >, < [0.1,0.3], [0.5,0.6] >\}$,
$A_2^V = \{< [0.3,0.5], [0.4,0.6] >, < \ddot{1} >\}$ and
$A_3^V = \{< [0.3,0.5], [0.4,0.6] >, < [0.1,0.3], [0.5,0.6] >\}$.
Then (X, I_P) is a P-cubic interval vague topological space. Implies that $(A_1^V)^c = \{< \ddot{0} >, < [0.7,0.9], [0.4,0.5] >\}$, $(A_2^V)^c = \{< [0.5,0.7], [0.4,0.6] >, < \ddot{0} >\}$ and
$(A_3^V)^c = \{< [0.5,0.7], [0.4,0.6] >, < [0.7,0.9], [0.4,0.5] >\}$. Assume that $A^V = \{< \ddot{1} >, < [0.1.0.4], [0.4,0.6] >\}$,
$B^V = \{< [0.3,0.6], [0.3,0.6] >, < \ddot{1} >\}$,
$A^V \cap_P B^V = \{< [0.3,0.6], [0.3,0.6] >, < [0.1,0.4], [0.4,0.6] >\}$ and therefore $cl_P(A^V) = \hat{1}, cl_P(B^V) = \hat{1}$ & $cl_P(A^V \cap_P B^V) = (A_3^V)^c$.
So $cl_P(A^V \cap_P B^V) \subseteq_P cl_P(A^V) \cap_P cl_P(B^V)$.

Proposition 4.16. Let $A^V = < A_V, D_V >$ and $B^V = < B_V, O_V >$ be any two cubic interval vague sets in a P-cubic interval vague topological space (X, I_P). Then $cl_P(A^V \cup_P B^V) = (cl_P(A^V) \cup_P cl_P(B^V))$.

Proof. Since $A^V \subseteq_P (A^V \cup_P B^V)$ & $B^V \subseteq_P (A^V \cup_P B^V)$, we have $cl_P(A^V) \subseteq_P cl_P(A^V \cup_P B^V)$ & $cl_P(B^V) \subseteq_P cl_P(A^V \cup_P B^V)$. Hence $\big(cl_P(A^V) \cup_P cl_P(B^V)\big) \subseteq_P cl_P(A^V \cup_P B^V) \rightarrow (i)$. Now, $A^V \subseteq_P cl_P(A^V)$ & $B^V \subseteq_P cl_P(B^V)$ implies
$(A^V \cup_P B^V) \subseteq_P \big(cl_P(A^V) \cup_P cl_P(B^V)\big)$.
Since $\big(cl_P(A^V) \cup_P cl_P(B^V)\big)$ is a P-cubic interval vague closed set, we have
$cl_P(A^V \cup_P B^V) \subseteq_P \big(cl_P(A^V) \cup_P cl_P(B^V)\big) \rightarrow$(ii).
Therefore from (i) & (ii) we get $cl_P(A^V \cup_P B^V) = (cl_P(A^V) \cup_P cl_P(B^V))$.

Proposition 4.17. For any P-cubic interval vague open set A^V in (X, I_P), we have

1. $(int_P(A^V))^c = cl_P((A^V)^c)$.
2. $(cl_P(A^V))^c = int_P((A^V)^c)$.

Proof. (1). $(int_P(A^V))^c = (\cup_P A_i^V)^c$ where A_i^V are P-cubic interval vague open sets in X and $A_i^V \subseteq_P A^V$, $(int_P(A^V))^c = \cap_P (A_i^V)^c$ where $(A_i^V)^c$ are P-cubic interval vague closed sets and $(A^V)^c \subseteq_P (A_i^V)^c$ which implies $(int_P(A^V))^c = cl_P((A^V)^c)$.
(2). $(cl_P(A^V))^c = (\cap_P A_i^V)^c$ where A_i^V are P-cubic interval vague closed sets in X *and* $A^V \subseteq_P A_i^V$, $(cl_P(A^V))^c = \cup_P (A_i^V)^c$ where $(A_i^V)^c$ are P-cubic interval vague open sets and $(A_i^V)^c \subseteq_P (A^V)^c$ which implies $(cl_P(A^V))^c = int_P((A^V)^c)$.

5. CONCLUSION

In this paper, we introduced the concept of cubic interval vague set as an extension of the concept of cubic vague set which was introduced in [4] and cubic sets which was introduced in [3]. Also, the concepts of external cubic interval vague set (ECIVS) and internal cubic interval vague set (ICIVS) are investigated. Moreover, utilizing the mentioned set, the concept of cubic interval vague topological space is defined and studied. Also, the concepts of derived, interior and closure of cubic interval vague sets in cubic interval vague topological spaces are studied. This type of set can be applied to study more topological concepts such as continuity separation axioms and connectedness.

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