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BOUNDS FOR THE COEFFICIENTS OF TWO NEW SUBCLASSES OF BI-UNIVALENT FUNCTIONS

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ABSTRACT:

This article discusses two new subclasses of the bi-univalent functions category $\sum$ in the open unit disk $\mathbb{U}$. The primary goal of the article is to obtain estimations of the coefficients $|\boldsymbol{a_2}|$ and $|\boldsymbol{a_3}|$ for the functions that are within these two new subclasses.

KEYWORDS: Taylor–Maclaurin Series, Univalent Function, Coefficient Bounds, Bi-univalent Function.

1. INTRODUCTION AND PRELIMINARIES

Assume that $\mathbb{U} = \{z \in \mathbb{C} \text{ such that } |z| < 1\}$ is an open disk which is called unit disk in $\mathbb{C}$ and $\mathcal{A}$ represents the category of functions f that are holomorphic in $\mathbb{U}$ with the Taylor-Maclaurin series:

$$f(z) = z + \sum_{k=2}^{\infty} a_k z^k, \quad z \in \mathbb{U}, \qquad \textbf{(1.1)}$$

and normalization $f(0) = 0,\ f'(0) = 1$. The subclass $\mathcal{S}$ of $\mathcal{A}$ is the category which contains of univalent functions in $\mathbb{U}$. (to learn more, see (Duren, 2001) and also the most recent works (Breaz, Breaz, & Srivastava, 2009; Srivastava & Eker, 2008).) The thoroughly investigation subcategory of univalent functions category $\mathcal{S}$ are the category of starlike functions of the order ξ $(0 \le \xi < 1)$, denoted with the aid of using $\mathcal{S}^*(\xi)$, and the category of convex functions of the order ξ denoted with the aid of using $\mathcal{C}(\xi)$ in $\mathbb{U}$. These two classes are defined as

$$\mathcal{S}^*(\xi) \coloneqq \left\{f \in \mathcal{S};\ \Re e\left(\frac{zf'(z)}{f(z)}\right) > \xi \quad (z \in \mathbb{U};\ \ 0 \le \xi < 1)\right\} \qquad \textbf{(1.2)}$$

and

$$\mathcal{C}(\xi) := \left\{f \in \mathcal{S};\ \Re e\left(1 + \frac{zf''(z)}{f'(z)}\right) > \xi \ \ (z \in \mathbb{U};\ 0 \le \xi < 1)\right\}. \qquad \textbf{(1.3)}$$

From the definitions (1.2) and (1.3) the following can be easily deduced:

$$f(z) \in \mathcal{C}(\xi) \ \Leftrightarrow\ zf'(z) \in \mathcal{S}^*(\xi) \qquad \textbf{(1.4)}$$

This is commonly known that if f is a univalent function, holomorphic from the domain $\mathcal{D}_1$ onto the domain $\mathcal{D}_2$, then the inverse function g which is determined by

$$g\big(f(z)\big) = z\,; \qquad (z \in \mathcal{D}_1)$$

is a univalent and holomorphic function from $\mathcal{D}_2$ to $\mathcal{D}_1$. Additionally, we know that the image of $\mathbb{U}$ under any function $f \in \mathcal{S}$ includes a disk of radius $\frac{1}{4}$ due to the well-known Koebe One-Quarter Theorem (see (Srivastava & Eker, 2008)). So, each univalent function $f \in \mathbb{U}$ has an inverse f^{-1} that follows the following criteria:

$$f^{-1}\big(f(z)\big) = z \qquad (z \in \mathbb{U}),$$

and

$$f\big(f^{-1}(w)\big) = w \quad \left(|w| < r_0(f);\quad r_0(f) \ge \tfrac{1}{4}\right). \qquad \textbf{(1.5)}$$

The series expansion of the inverse function f in a disk about the origin is of the form:

$$f^{-1}(w) = w + \delta_2 w^2 + \delta_3 w^3 + \cdots. \qquad \textbf{(1.6)}$$

The Koebe function's inverse offers the tightest restriction on all $|\delta_k|$ in (1.6) (see (Ding, Ling & Bao, 1995; Lewin, 1967)). According to the previous steps it is clear that the univalent function $f(z)$ and its inverse function $f^{-1}(w)$ satisfy the condition (1.5) in the neighborhood of the origin or equivalently

$$w = f^{-1}(w) + a_2[f^{-1}(w)]^2 + a_3[f^{-1}(w)]^3 + \cdots.$$

Then,

$$f^{-1}(w) = w - a_2[f^{-1}(w)]^2 - a_3[f^{-1}(w)]^3 - \cdots. \qquad \textbf{(1.7)}$$

By using (1.1) and (1.6) in (1.7), a simple computation gives

$$g(w) = f^{-1}(w) = w - a_2 w^2 + (2a_2^2 - a_3)w^3 - (5a_2^3 - 5a_2a_3 + a_4)w^4 + \cdots. \qquad \textbf{(1.8)}$$

A function $f \in \mathcal{A}$ is called bi-univalent in $\mathbb{U}$ if $f(z)$ and $f^{-1}(w)$ are both univalent functions in $\mathbb{U}$. Let $\sum$ denote the category of all bi-univalent functions in $\mathbb{U}$ defined by series (1.1). For a short history, curiously instances of functions which are in (or which do not fall inside) the category $\sum$ and also different other characteristics of the bi-univalent functions category $\sum$ one can make reference the work of Srivastava et al. (Srivastava, Mishra, & Gochhayat, 2010) and its references. In truth, the consideration of the coefficient issues including bi-univalent functions was surveyed as of late by Srivastava et al. (Srivastava, Mishra, & Gochhayat, 2010).

The following are some instances of the functions in the category $\sum$:

$$\frac{z}{1-z}, \qquad -\log(1-z), \qquad \frac{1}{2}\log\left(\frac{1+z}{1-z}\right),$$

and so forth. Although, the famous Koebe function is not in $\sum$. Other common instances of the functions in $\mathcal{S}$ such as

$$z - \frac{z^2}{2} \quad and \quad \frac{z}{1-z^2}$$

are not in the category $\sum$ (see (Xu, Gui, & Srivastava, 2012)). Gao and Zhou (Gao, Zhou, & computation, 2007) demonstrated the functional characteristics for the following subclasses of $\mathcal{A}$:

$\mathcal{R}(\alpha,\beta) = \{f \in \mathcal{A};\ \Re e\big(f'(z) + \beta z f''(z)\big) > \alpha,\ \ \beta > 0,\ \ 0 \le \alpha < 1;\ z \in \mathbb{U}\}.$

Yang and Liu [(Yang & Liu, 2010) Theorem 3.1, p.9], established that the class $\mathcal{R}(\alpha,\beta) \subset \mathcal{S}$ exists if and only if $2(1-\alpha)\sum_{i=1}^{\infty}\frac{(-1)^{i-1}}{\beta i+1} \le 1$.

For $\beta < 1$,let $\mathcal{R}(\beta) = \{f(z) \in \mathcal{A};\ \Re e[f'(z) + zf''(z)] > \beta,\ \ z \in \mathbb{U}\}$.

This category was considered by many authors (see (Chichra, 1977)).

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Ding, Ling & Bao (Ding, Ling & Bao, 1995) presented the following category $Q_\lambda(\beta)$ of holomorphic functions defined as follows:

$Q_\lambda(\beta) = \left\{ f \in \mathcal{A};\ \Re e\left((1-\lambda)\frac{f(z)}{z} + \lambda f'(z)\right) > \beta,\ \beta < 1,\ \lambda \geq 0 \right\}$.

It is obvious $Q_{\lambda_1}(\beta) \subset Q_{\lambda_2}(\beta)$ for $\lambda_1 > \lambda_2 \geq 0$. Thus, for $\lambda \geq 1,\ 0 \leq \beta < 1,\ Q_\lambda(\beta) \subset Q_1(\beta) = \{f \in \mathcal{A};\ \Re e f'(z) > \beta,\ 0 \leq \beta < 1\}$ and hence $Q_\lambda(\beta)$ is univalent category (see (Chichra, 1977)).

Another exciting work is done by Caglar and Deniz (Caglar & Deniz, 2017). They obtained the coefficient estimates for a new subclass $\sum^n(\tau,\gamma,\varphi)$ of analytic and bi-univalent functions in the open unit disk $\mathbb{U}$ defined by Salagean differential operator. Also Srivastava, Bulut, Çağlar and Yağmur (Srivastava, Bulut, Çağlar & Yağmur, 2013) introduced and investigated an interesting subclass $\mathcal{N}_\Sigma^{h,p}(\lambda,\mu)$ of analytic and bi-univalent functions in the open unit disk $\mathbb{U}$. For functions belonging to the class $\mathcal{N}_\Sigma^{h,p}(\lambda,\mu)$, they obtained estimates on the first two Taylor-Maclaurin coefficients $|a_2|$ and $|a_3|$. In addition, Çağlar, Orhan and Yağmur, (Çağlar, Orhan & Yağmur, 2013) considered two new subclasses $\mathcal{N}_\Sigma^\mu\ (\alpha,\lambda)$ and $\mathcal{N}_\Sigma^\mu\ (\beta,\lambda)$ of bi-univalent functions defined in $\mathbb{U}$ and they presented some interesting results.

In 1967, Lewin (Lewin, 1967) investigated the category Σ and demonstrated that $|a_2| \leq 1.51$. Later, Brannan, David & Clunie (Brannan, David & Clunie, 1980) Proved that $|a_2| \leq \sqrt{2}$. Besides that, Netanyahu demonstrated that

$$\max_{f\in\Sigma}|a_2| = \frac{4}{3}$$

in his paper (Netanyahu & analysis, 1969).

Awhile later in 1981, Wright and Styer (Styer & Wright, 1981) showed that there exist a function $f\ (z) \in \Sigma$ with $|a_2| > \frac{4}{3}$. The most excellent known gauge for functions it has been acquired in 1984 by Tan (Tan, 1984), that is $|a_2| \leq 1.485$. The coefficient gauge issue including the bound of $|a_n|\ \ (n \in \mathbb{N}\backslash\{1,2\})$ for every $f \in \Sigma$ given by (1.1) is unresolved. Taha and Brannan (Brannan & Taha, 1988) showed certain subcategory of the bi-univalent functions category Σ comparative to the commonplace subcategory $\mathcal{C}(\alpha)$ and $\mathcal{S}^*(\alpha)$ of convex and starlike function of order $\alpha\ (0 \leq \alpha < 1)$, (see (Brannan, Clunie, & Kirwan, 1970)). Thus, according to Taha and Brannan (Brannan & Taha, 1988), a function $f \in \Sigma$ belongs to the category $\mathcal{S}_\Sigma^*[\alpha]$ of strongly bi-starlike functions of the order $\alpha(0 < \alpha \leq 1)$ if the conditions below are met:

$f \in \Sigma$ and

$$\left|arg\left(\frac{zg'(w)}{g(w)}\right)\right| < \frac{\alpha\pi}{2} \qquad (0 < \alpha \leq 1,\ \ w \in \mathbb{U})$$

and

$$\left|arg\left(\frac{zf'(z)}{f(z)}\right)\right| < \frac{\alpha\pi}{2} \qquad (0 < \alpha \leq 1,\ \ z \in \mathbb{U}),$$

where g is the expansion of f^{-1} to $\mathbb{U}$. The categories $\mathcal{C}_\Sigma^*[\alpha]$ and $\mathcal{S}_\Sigma^*[\alpha]$ of bi-convex function of order α and bi-starlike functions of the order α, comparing (individually) to the work categories $\mathcal{C}(\alpha)$ and $\mathcal{S}^*(\alpha)$, were moreover presented similarly. For every of the function categories $\mathcal{C}_\Sigma^*[\alpha]$ and $\mathcal{S}_\Sigma^*[\alpha]$, they discovered non-sharp gauges on the primary two Taylor–Maclaurin coefficients $|a_2|$ and $|a_3|$ (see (Brannan & Taha, 1988)).

Using the method of convolution on category of holomorphic functions $\mathcal{A}$, Ruscheweyh (Ruscheweyh, 1975) considered the operator $\mathcal{R}$ as follows:

$$\mathcal{R}^\lambda f(z) = f(z) * \frac{z}{(1-z)^{\lambda+1}},$$

where, $z \in \mathbb{U},\ \lambda \in \mathbb{R}$ and $\lambda > -1$.

For $\lambda = n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}$, we get

$$\mathcal{R}^n f(z) = z\frac{\left(z^{n-1}f(z)\right)^{(n)}}{n!}.$$

The expression $\mathcal{R}^n f(z)$ is said to be the Ruscheweyh derivative of order n of f and the sign $*$ is represents the Hadamard product. We can deduce that (Hussain, Khan, Zaighum, Darus, & Shareef, 2017)

$$\mathcal{R}^n f(z) = z + \sum_{k=2}^{\infty} \sigma(n,k)a_k z^k,$$

where

$$\sigma(n,k) = \frac{\Gamma(n+k)}{(k-1)!\,\Gamma(n+1)}.$$

The aim of this article is to define two new categories of functions in the class Σ, depending on the Ruscheweyh operator, and obtain estimations the coefficients $|a_2|$ and $|a_3|$ for all elements of these categories by using techniques that are used previously by Srivastava, Mishra, & Gochhayat. (Srivastava, Mishra, & Gochhayat, 2010), (see also, (Frasin & Aouf, 2011; Xu et al., 2012)).

Definition 1.1. A function f which is given by (1.1) is called in the category $\mathcal{H}_\Sigma(n,\gamma,\alpha)$ if $f \in \Sigma$ and

$$\left|arg\left(\left(\mathcal{R}^n f(z)\right)' + \gamma z\left(\mathcal{R}^n f(z)\right)''\right)\right| < \frac{\alpha\pi}{2};\quad z \in \mathbb{U}, \qquad \textbf{(1.9)}$$

and

$$\left|arg\left(\left(\mathcal{R}^n g(w)\right)' + \gamma w\left(\mathcal{R}^n g(w)\right)''\right)\right| < \frac{\alpha\pi}{2};\ w \in \mathbb{U}, \textbf{(1.10)}$$

where $n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\},\ \gamma > 0,\ \ 0 < \alpha \leq 1$,

$2(1-\alpha)\sum_{i=1}^{\infty}\frac{(-1)^{i-1}}{\gamma i+1} \leq 1$ and the function $g(z)$ is of the form (1.8).

Definition 1.2. A function f which is of the form (1.1) is called in the category $\mathcal{H}_\Sigma(n,\gamma,\beta)$ if $f \in \Sigma$ and

$$\Re e\left(\left(\mathcal{R}^n f(z)\right)' + \gamma z\left(\mathcal{R}^n f(z)\right)''\right) > \beta;\quad z \in \mathbb{U}, \qquad \textbf{(1.11)}$$

and

$$\Re e\left(\left(\mathcal{R}^n g(w)\right)' + \gamma w\left(\mathcal{R}^n g(w)\right)''\right) > \beta;\quad w \in \mathbb{U}, \quad \textbf{(1.12)}$$

where $n \in \mathbb{N}_0 = \mathbb{N} \cup \{0\}, \gamma > 0,\ \ 0 \leq \beta < 1$,

$2(1-\beta)\sum_{i=1}^{\infty}\frac{(-1)^{i-1}}{\gamma i+1} \leq 1$ and the function $g(z)$ is given by (1.8).

In this article, we have two main results that are expressed in the form of two theorems, and to prove these results, we must remind the next lemma.

Lemma 1.3. (Pommerenke & Ruprecht, 1975) Assume that h belongs to $\mathcal{P}$ then $|c_i| \leq 2$ for every $i \geq 1$, in which $\mathcal{P}$ is the collection of all holomorphic functions h in $\mathbb{U}$, $\Re e(h(z)) > 0$ and

$$h(z) = 1 + c_1 z + c_2 z^2 + c_3 z^3 + \cdots;\quad z \in \mathbb{U}.$$

2. BOUNDS OF THE COEFFICIENTS OF THE FUNCTION SUBCLASS $\mathcal{H}_\Sigma(n,\gamma,\alpha)$

In this section, we obtained the estimations of the Taylor coefficients $|a_2|$ and $|a_3|$ of the functions which is belong to this category $\mathcal{H}_\Sigma(n,\gamma,\alpha)$.

Theorem 2.1. If f is given by (1.1) is in the category $\mathcal{H}_\Sigma(n,\gamma,\alpha)$, in which $n \in \mathbb{N}_0,\ \gamma > 0,\ 0 < \alpha < 1$, and

$2(1-\alpha)\sum_{i=1}^{\infty}\frac{(-1)^{i-1}}{\gamma i+1} \leq 1$. Then

$$|a_2| \leq \frac{2\alpha}{\sqrt{|3\alpha(1+2\gamma)(n+1)(n+2) - 4(\alpha-1)(1+\gamma)^2(n+1)^2|}},$$

and

$$|a_3| \leq \frac{4\alpha}{3(1+2\gamma)(n+1)(n+2)} + \frac{\alpha^2}{(1+\gamma)^2(n+1)^2}.$$

Proof. From (1.9) and (1.10), it follows that

$$\left(\mathcal{R}^n f(z)\right)' + \gamma z\left(\mathcal{R}^n f(z)\right)'' = [h_1(z)]^\alpha;\ \ z \in \mathbb{U}, \qquad \textbf{(2.1)}$$

and

$\left(\mathcal{R}^n g(w)\right)' + \gamma w\left(\mathcal{R}^n g(w)\right)'' = [h_2(w)]^\alpha; \;\; w \in \mathbb{U},$ **(2.2)**

where h_1 and h_2 are in $\mathcal{P}$ which have the following forms

$$h_1(z) = 1 + c_1 z + c_2 z^2 + c_3 z^3 + \cdots, \quad \textbf{(2.3)}$$

and

$$h_2(w) = 1 + d_1 w + d_2 w^2 + d_3 w^3 + \cdots. \quad \textbf{(2.4)}$$

By definition,

$$\mathcal{R}^n f(z) = z + \sum_{i=2}^{\infty} \sigma(n,i) a_i z^i$$
$$= z + \sigma(n,2) a_2 z^2 + \sigma(n,3) a_3 z^3 + \cdots,$$
$$\left(\mathcal{R}^n f(z)\right)' = 1 + 2\,\sigma(n,2)\, a_2 z + 3\,\sigma(n,3)\, a_3 z^2 + \cdots,$$

and

$$\gamma z\left(\mathcal{R}^n f(z)\right)'' = 2\,\gamma\sigma(n,2)\, a_2 z + 6\,\gamma\sigma(n,3)\, a_3 z^2 + \cdots.$$

Then

$$\left(\mathcal{R}^n f(z)\right)' + \gamma z\left(\mathcal{R}^n f(z)\right)''$$
$$= 1 + 2(1+\gamma)\,\sigma(n,2) a_2 z + 3(1+2\gamma)\sigma(n,3) a_3 z^2 + \cdots. \quad \textbf{(2.5)}$$

On the other hand,

$$[h_1(z)]^\alpha = 1 + \alpha c_1 z + \left(\alpha c_2 + \tfrac{\alpha(\alpha-1)}{2} c_1^2\right) z^2 + \cdots. \quad \textbf{(2.6)}$$

Again, by using (1.8) we have

$$\mathcal{R}^n g(w) = w - \sigma(n,2) a_2 w^2 + \sigma(n,3)(2a_2^2 - a_3) w^3 + \cdots,$$
$$\left(\mathcal{R}^n g(w)\right)' = 1 - 2\sigma(n,2) a_2 w + 3\sigma(n,3)(2a_2^2 - a_3) w^2 + \cdots,$$

and

$$\gamma w\left(\mathcal{R}^n g(w)\right)'' = -2\gamma\sigma(n,2) a_2 w + 6\gamma\sigma(n,3)(2a_2^2 - a_3) w^2 + \cdots.$$

Then

$$\left(\mathcal{R}^n g(w)\right)' + \gamma w\left(\mathcal{R}^n g(w)\right)''$$
$$= 1 - 2\,(1+\gamma)\sigma(n,2) a_2 w + 3(1+2\gamma)\sigma(n,3)(2a_2^2 - a_3) w^2 + \cdots. \quad \textbf{(2.7)}$$

On the other hand,

$$[h_2(w)]^\alpha = 1 + \alpha d_1 w + \left(\alpha d_2 + \tfrac{\alpha(\alpha-1)}{2} d_1^2\right) w^2 + \cdots. \quad \textbf{(2.8)}$$

Now, equating the equations in (2.5) and (2.6) we get

$$1 + 2(1+\gamma)\sigma(n,2) a_2 z + 3(1+2\gamma)\sigma(n,3) a_3 z^2 + \cdots$$
$$= 1 + \alpha c_1 z + \left(\alpha c_2 + \tfrac{\alpha(\alpha-1)}{2} c_1^2\right) z^2 + \cdots. \quad \textbf{(2.9)}$$

Again, equating the equations in (2.7) and (2.8) gives

$$1 - 2(1+\gamma)\sigma(n,2) a_2 w + 3(1+2\gamma)\sigma(n,3)(2a_2^2 - a_3) w^2 + \cdots$$
$$= 1 + \alpha d_1 w + \left(\alpha d_2 + \tfrac{\alpha(\alpha-1)}{2} d_1^2\right) w^2 + \cdots. \quad \textbf{(2.10)}$$

In this step, by equating the coefficients in (2.9) and (2.10), we get

$$2\,(1+\gamma)\sigma(n,2) a_2 = \alpha c_1, \quad \textbf{(2.11)}$$
$$3(1+2\gamma)\sigma(n,3) a_3 = \alpha c_2 + \tfrac{\alpha(\alpha-1)}{2} c_1^2, \quad \textbf{(2.12)}$$
$$-2\,(1+\gamma)\sigma(n,2) a_2 = \alpha d_1, \quad \textbf{(2.13)}$$

and

$$3(1+2\gamma)\sigma(n,3)(2a_2^2 - a_3) = \alpha d_2 + \tfrac{\alpha(\alpha-1)}{2} d_1^2. \quad \textbf{(2.14)}$$

From (2.11) and (2.13), we get

$$c_1 = -d_1. \quad \textbf{(2.15)}$$

and

$$c_1^2 + d_1^2 = \frac{8(1+\gamma)^2(\sigma(n,2))^2}{\alpha^2} a_2^2. \quad \textbf{(2.16)}$$

From (2.12) and (2.14), we get

$$6(1+2\gamma)\sigma(n,3) a_2^2 = \alpha(c_2 + d_2) + \tfrac{\alpha(\alpha-1)}{2}(c_1^2 + d_1^2).$$

By using (2.16), we get

$$\frac{6\alpha(1+2\gamma)\sigma(n,3) - 4(\alpha-1)(1+\gamma)^2(\sigma(n,2))^2}{\alpha} a_2^2 = \alpha(c_2 + d_2).$$

Therefore,

$$a_2^2 = \frac{\alpha^2(c_2+d_2)}{6\alpha(1+2\gamma)\sigma(n,3) - 4(\alpha-1)(1+\gamma)^2(\sigma(n,2))^2}$$
$$= \frac{\alpha^2(c_2+d_2)}{3\alpha(1+2\gamma)(n+1)(n+2) - 4(\alpha-1)(1+\gamma)^2(n+1)^2}.$$

By using lemma (1.3), we have

$$|a_2|^2 \le \frac{|4\,\alpha^2|}{|3\alpha(1+2\gamma)(n+1)(n+2) - 4(\alpha-1)(1+\gamma)^2(n+1)^2|}$$

or

$$|a_2|^2 \le \frac{4\,\alpha^2}{||3\alpha(1+2\gamma)(n+1)(n+2)| - |4(\alpha-1)(1+\gamma)^2(n+1)^2||}.$$

Hence, we get

$$|a_2| \le \frac{2\alpha}{\sqrt{|3\alpha(1+2\gamma)(n+1)(n+2) - 4(\alpha-1)(1+\gamma)^2(n+1)^2|}}.$$

Finally, we obtained the required bound for $|a_2|$.

Next, we can obtain the bound for $|a_3|$ by simple calculation so,

$$6(1+2\gamma)\sigma(n,3) a_3 - 6(1+2\gamma)\sigma(n,3) a_2^2 = \alpha(c_2 - d_2) + \tfrac{\alpha(\alpha-1)}{2}(c_1^2 - d_1^2).$$

By using (2.15) and (2.16), we obtain

$$6(1+2\gamma)\sigma(n,3) a_3 = \alpha(c_2 - d_2) + 6(1+2\gamma)\sigma(n,3)\left(\frac{\alpha^2 c_1^2}{4(1+\gamma)^2(\sigma(n,2))^2}\right).$$

Then

$$a_3 = \frac{\alpha(c_2 - d_2)}{6(1+2\gamma)\sigma(n,3)} + \frac{\alpha^2 c_1^2}{4(1+\gamma)^2(\sigma(n,2))^2}.$$

By using lemma (1.3), we have

$$|a_3| \le \frac{4\alpha}{3(1+2\gamma)(n+1)(n+2)} + \frac{\alpha^2}{(1+\gamma)^2(n+1)^2}.$$

Obtaining this bound concludes the proof of the theorem. ■

Putting $\gamma = 1$ in the Theorem 2.1, we obtain the next outcome:

Corollary 2.2. If f is given by (1.1) in the category $\mathcal{H}_\Sigma(n,1,\alpha)$, where $n \in \mathbb{N}_0, 0 < \alpha \le 1$, and $2(1-\alpha)\sum_{i=1}^{\infty} \frac{(-1)^{i-1}}{i+1} \le 1$. Then

$$|a_2| \le \frac{2\alpha}{\sqrt{|9\alpha(n+1)(n+2) - 16(\alpha-1)(n+1)^2|}},$$

and

$$|a_3| \le \frac{4\alpha}{9(n+1)(n+2)} + \frac{\alpha^2}{4(n+1)^2}.$$

■

3. COEFFICIENT BOUNDS FOR THE FUNCTION SUBCLASS $\mathcal{H}_\Sigma(n,\gamma,\beta)$

Here, we discussed another subclass of Σ, $\mathcal{H}_\Sigma(n,\gamma,\beta)$, and we obtained the estimation of the Taylor coefficients $|a_2|$ and $|a_3|$ of the functions which is belong to this category.

Theorem 3.1. If f is given by (1.1) in the category $\mathcal{H}_\Sigma(n,\gamma,\beta)$, where $n \in \mathbb{N}_0, \gamma > 0, 0 \le \beta < 1$ *and* $2(1-\beta)\sum_{i=1}^{\infty} \frac{(-1)^{i-1}}{\gamma i+1} \le 1$. Then

$$|a_2| \le \frac{2\sqrt{1-\beta}}{\sqrt{3(1+2\gamma)(n+1)(n+2)}},$$

and

$$|a_3| \le \frac{(1-\beta)^2}{(1+\gamma)^2(n+1)^2} + \frac{4(1-\beta)}{3(1+2\gamma)(n+1)(n+2)}.$$

Proof. From (1.11) and (1.12), it follows that there are two functions h_1 and h_2 in $\mathcal{P}$ such that

$$\left(\mathcal{R}^n f(z)\right)' + \gamma z\left(\mathcal{R}^n f(z)\right)'' = \beta + (1-\beta) h_1(z)\,;\; z \in \mathbb{U}, \quad \textbf{(3.1)}$$

and

$$\left(\mathcal{R}^n g(w)\right)' + \gamma w\left(\mathcal{R}^n g(w)\right)'' = \beta + (1-\beta) h_2(w);\; w \in \mathbb{U}. \quad \textbf{(3.2)}$$

where h_1 and h_2 are in $\mathcal{P}$ which have the forms (2.3) and (2.4) respectively.

So,

$$\beta + (1-\beta) h_1(z) = 1 + (1-\beta) c_1 z + (1-\beta) c_2 z^2 + \cdots, \quad \textbf{(3.3)}$$

and

$$\beta + (1-\beta)h_2(w) = 1 + (1-\beta)d_1 w + (1-\beta)d_2 w^2 + \cdots. \quad \textbf{(3.4)}$$

Now, equating the equations (2.5) and (3.3), we get

$$1 + 2(1+\gamma)\sigma(n,2)a_2 z + 3(1+2\gamma)\sigma(n,3)a_3 z^2 + \cdots = 1 + (1-\beta)c_1 z + (1-\beta)c_2 z^2 + \cdots. \quad \textbf{(3.5)}$$

Again, equating the equations (2.7) and (3.4), we get

$$1 - 2(1+\gamma)\sigma(n,2)a_2 w + 3(1+2\gamma)\sigma(n,3)(2a_2^2 - a_3)w^2 + \cdots = 1 + (1-\beta)d_1 w + (1-\beta)d_2 w^2 + \cdots. \quad \textbf{(3.6)}$$

In this step, we will equate the coefficients in (3.5) and (3.6)

$$2\,(1+\gamma)\sigma(n,2)a_2 = (1-\beta)c_1, \quad \textbf{(3.7)}$$

$$3(1+2\gamma)\sigma(n,3)a_3 = (1-\beta)c_2, \quad \textbf{(3.8)}$$

and

$$-2\,(1+\gamma)\sigma(n,2)a_2 = (1-\beta)d_1 \quad \textbf{(3.9)}$$

$$3(1+2\gamma)\sigma(n,3)(2a_2^2 - a_3) = (1-\beta)d_2, \quad \textbf{(3.10)}$$

From (3.7) and (3.9), we get

$$c_1 = -d_1. \quad \textbf{(3.11)}$$

and

$$c_1^2 + d_1^2 = \frac{8(1+\gamma)^2(\sigma(n,2))^2}{(1-\beta)^2} a_2^2 \quad \textbf{(3.12)}$$

By using (3.8) and (3.10), we get

$$a_2^2 = \frac{(1-\beta)(c_2+d_2)}{6(1+2\gamma)\sigma(n,3)}.$$

By using lemma (1.3), we get

$$|a_2| \le \frac{2\sqrt{1-\beta}}{\sqrt{3(1+2\gamma)(n+1)(n+2)}},$$

which is the bound for $|a_2|$ as given in the theorem.

To obtain the bound for $|a_3|$, subtracting (3.10) from (3.8), we get

$$a_3 = a_2^2 + \frac{(1-\beta)(c_2-d_2)}{6(1+2\gamma)\sigma(n,3)}.$$

Now, using (3.11) and (3.12)

$$a_3 = \frac{(1-\beta)^2 c_1^2}{4(1+\gamma)^2(\sigma(n,2))^2} + \frac{(1-\beta)(c_2-d_2)}{6(1+2\gamma)\sigma(n,3)}.$$

Applying lemma (1.3), we get

$$|a_3| \le \frac{(1-\beta)^2}{(1+\gamma)^2(n+1)^2} + \frac{4(1-\beta)}{3(1+2\gamma)(n+1)(n+2)}.$$

which is the bound for $|a_3|$ as mentioned in the text of the theorem. ∎

Putting $\gamma = 1$ in the Theorem 3.1, we obtain the next outcome:

Corollary 3.2. If f is given by (1.1) in the category $\mathcal{H}_{\Sigma}(n,1,\beta)$, where $n \in \mathbb{N}_0, 0 \le \beta < 1$ *and* $2(1-\beta)\sum_{i=1}^{\infty}\frac{(-1)^{i-1}}{i+1} \le 1$. Then

$$|a_2| \le \frac{2\sqrt{1-\beta}}{3\sqrt{(n+1)(n+2)}},$$

and

$$|a_3| \le \frac{(1-\beta)^2}{4(n+1)^2} + \frac{4(1-\beta)}{9\,(n+1)(n+2)}.$$

∎

4. CONCLUSION

In this article, our investigation is due to the fact that we can find interesting and useful applications of special functions and especially bi-univalent functions. The new bi-univalent function subclasses $\mathcal{H}_{\Sigma}(n,\gamma,\alpha)$ and $\mathcal{H}_{\Sigma}(n,\gamma,\beta)$ in the open disk $\mathbb{U}$, was examined in this paper. We explored the coefficients of $|a_2|$ and $|a_3|$ in the Taylor series of them. Additionally, we discovered some corollaries and implications of the primary findings. Furthermore, the provided bounds improve and extend some previous results.

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REFERENCES

Brannan, D. A., Clunie, J., & Clunie, J. (Eds.). (1980). *Aspects of contemporary complex analysis*. Academic Press.

Brannan, D. A., Clunie, J., & Kirwan, W. E. (1970). Coefficient estimates for a class of star-like functions. *Canadian Journal of Mathematics*, *22*(3), 476-485.

Brannan, D. A., & Taha, T. S. (1988). On some classes of bi-univalent functions. In *Mathematical Analysis and Its Applications* (pp. 53-60). Pergamon.

Breaz, D., Breaz, N., & Srivastava, H. M. (2009). An extension of the univalent condition for a family of integral operators. *Applied Mathematics Letters*, *22*(1), 41-44.

Chichra, P. N. (1977). New subclasses of the class of close-to-convex functions. *Proceedings of the American Mathematical Society*, *62*(1), 37-43.

Caglar, M., & Deniz, E. (2017). Initial coefficients for a subclass of bi-univalent functions defined by Salagean differential operator. *Commun. Fac. Sci. Univ. Ank. Ser. A1 Math. Stat*, *66*, 85-91.

Çağlar, M., Orhan, H., & Yağmur, N. (2013). Coefficient bounds for new subclasses of bi-univalent functions. *Filomat*, *27*(7), 1165-1171.

Ding, S. S., Ling, Y., & Bao, G. J. (1995). Some properties of a class of analytic functions. *Journal of Mathematical Analysis and applications*, *195*(1), 71-81.

Duren, P. L. (2001). *Univalent functions* (Vol. 259). Springer Science & Business Media.

Frasin, B. A., & Aouf, M. K. (2011). New subclasses of bi-univalent functions. *Applied Mathematics Letters*, *24*(9), 1569-1573.

Gao, C. Y., & Zhou, S. Q. (2007). Certain subclass of starlike functions. *Applied mathematics and computation*, *187*(1), 176-182.

Hussain, S., Khan, S., Zaighum, M. A., Darus, M., & Shareef, Z. (2017). Coefficients bounds for certain subclass of biunivalent functions associated with Ruscheweyh-Differential operator. *Journal of Complex Analysis*, *2017*.

Lewin, M. (1967). On a coefficient problem for bi-univalent functions. *Proceedings of the American mathematical society*, *18*(1), 63-68.

Netanyahu, E. (1969). The minimal distance of the image boundary from the origin and the second coefficient of a univalent function in¦ z¦< 1. *Archive for rational mechanics and analysis*, *32*(2), 100-112.

Pommerenke, C. (1975). Univalent functions. *Vandenhoeck and Ruprecht*.

Ruscheweyh, S. (1975). New criteria for univalent functions. *Proceedings of the American Mathematical Society*, *49*(1), 109-115.

Srivastava, H. M., Bulut, S., Çağlar, M., & Yağmur, N. (2013). Coefficient estimates for a general subclass of analytic and bi-univalent functions. *Filomat*, *27*(5), 831-842.

Srivastava, H. M., & Eker, S. S. (2008). Some applications of a subordination theorem for a class of analytic functions. *Applied Mathematics Letters*, *21*(4), 394-399.

Srivastava, H. M., Mishra, A. K., & Gochhayat, P. (2010). Certain subclasses of analytic and bi-univalent functions. *Applied mathematics letters*, *23*(10), 1188-1192.

Styer, D., & Wright, D. J. (1981). Results on bi-univalent functions. *Proceedings of the American Mathematical Society*, *82*(2), 243-248.

Tan, D.L.(1984). Coefficient estimates for bi-univalent functions. *Chinese Ann. Math. Ser. A*, *5*(5), 559-568.

Xu, Q. H., Gui, Y. C., & Srivastava, H. M. (2012). Coefficient estimates for a certain subclass of analytic and bi-univalent functions. *Applied Mathematics Letters*, *25*(6), 990-994.

Yang, D. G., & Liu, J. L. (2010). On a class of analytic functions with missing coefficients. *Applied Mathematics and Computation*, *215*(9), 3473-3481.